Multiplet bases, recursion relations and full color parton showers

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LUND
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Thesis for the degree of Doctor of Philosophy
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Abstract

The papers in this thesis all concern the treatment of colors in perturbative QCD, both in the context of hard scattering cross sections and for parton showers. The complexity of the color structure of QCD increases quickly with the number of external partons. One way of tackling this issue is by using an orthogonal, group theory based, type of bases, called multiplet bases. This is the topic of papers I, II and IV. In paper III the inclusion of full color treatment in parton showers is addressed.

Paper I concerns the decomposition of QCD color structures into multiplet bases, using Wigner $3j$ and $6j$ coefficients. A strong constraint is put on the required Wigner coefficients for leading order and next-to-leading order QCD amplitudes. The required Wigner coefficients for up to six external gluons were calculated by the method described in the paper.

In paper II the results of paper I is applied to maximally helicity violating amplitude recursion relations, to investigate the viability of multiplet bases in this context. The result is a shift in the computational bottleneck of recursion, from the squaring of amplitudes to the recursion step, but yielding an overall better scaling for the total number of terms encountered.

In paper IV a more general method of constructing multiplet bases is presented, which improves the decomposition of paper I for amplitudes with quarks. New basis vectors are constructed by the presented method and from them Wigner $6j$ cofficients are calculated.

Paper III concerns the implementation of an $N_c = 3$ parton shower in the event generator Herwig. In the implementation, the trace basis has been used, but it could, in a straight-forward way, be extended to other color space bases. The implementation has been used to study the effects of subleading color corrections, for both LEP and LHC events. The effects on observables are comparable to earlier findings for LEP, up to $\sim 10\%$ differences, compared to a leading color shower. For LHC the differences are often of the order of a few percent, but in some cases differences of up to 20% were found.

Key words

QCD, multiplet bases, $SU(N_c)$, recursion relations, BCFW, $N_c = 3$ parton showers

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A doctoral thesis at a university in Sweden takes either the form of a single, cohesive research study (monograph) or a summary of research papers (compilation thesis), which the doctoral student has written alone or together with one or several other author(s).

In the latter case the thesis consists of two parts. An introductory text puts the research work into context and summarizes the main points of the papers. Then, the research publications themselves are reproduced, together with a description of the individual contributions of the authors. The research papers may either have been already published or are manuscripts at various stages (in press, submitted, or in draft).

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Till mormor och farmor

Contents

List of publications

This thesis is based on the following publications, referred to by their Roman numerals:

I **Decomposing color structure into multiplet bases**

M. Sjödahl and **J. Thorén**
JHEP **09** (2015) 055 [1507.03814]

II **Recursion in multiplet bases for tree-level MHV gluon amplitudes**

Y.-J. Du, M. Sjödahl and **J. Thorén**
JHEP **1505** (2015) 119 [1503.00530]

III **Color matrix element corrections for parton showers**

S. Plätzer, M. Sjödahl and **J. Thorén**
[1808.00332], submitted to JHEP

IV **QCD multiplet bases with arbitrary parton ordering**

M. Sjödahl and **J. Thorén**
[1809.05002], submitted to JHEP

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Populärvetenskaplig sammanfattning på svenska

Vad består materia av? Det är en fundamental fråga som mänskligheten försökt besvara i tusentals år. Vårt svar på den frågan har utvecklats enormt de senaste 200 åren. Atomer, från grekiskans ord för odelbar, bygger upp materian vi ser omkring oss. Det antogs länge att atomer verkligen var odelbara, men strax innan sekelskiftet år 1900, kom experimentellt bevis från J. J. Thomson, som påvisade att det fanns en inre struktur i atomer. Thomson hade upptäckt den negativt laddade elektronen, som tillsammans med den positivt laddade kärnan bygger upp atomer. E. Rutherford lyckades ett par årtionden senare visa att även atomkärnan har en inre struktur, den är uppbyggd av positivt laddade protoner och neutrala neutroner. De kommande decennierna resulterade i att fler och fler partiklar upptäcktes. Vid det sena 1960-talet så hade man hundratals, till synes, elementarpartiklar. Vid denna tid framförde M. Gell-Mann och G. Zweig, oberoende av varandra, kvarkmodellen, vilken istället för hundratals partiklar innehåller tre "smaker" (flavor på engelska) av kvarkar. I denna modell består merparten av de hundratals upptäckta partiklarna av bundna tillstånd av tre kvarkar, baryoner, eller en kvark och en antikvark, mesoner. En antikvark är en antipartikel, vilket är något som varje partikel har (om de inte är sin egen antipartikel). En partikel och dess antipartikel delar vissa egenskaper, exempelvis massa, de är lika tunga, medan andra egenskaper skiljer sig, exempelvis elektrisk laddning, om partikeln har positiv laddning, så har dess antipartikel en lika stor, men negativ laddning.

Förståelsen av elementarpartiklar idag har ökat enormt sedan 1960-talet, men kvarkar är fortfarande fundamentala i dagens modell av partikelfysik, standardmodellen. I standardmodellen finns det tre krafter, elektromagnetism, svaga växelverkan och den starka växelverkan. Dessa förmedlas av bosoner. Den masslösa fotonen förmedlar den elektromagnetiska kraften, de massiva, W^+, W^- och Z^0 bosonerna, förmedlar den svaga kraften och de åtta masslösa gluonerna förmedlar den starka kraften. Partiklar som ingår i materia delas upp i två grupper, leptonerna, som inte växelverkar genom den starka kraften, och kvarkar, som växelverkar genom den starka kraften. En till partikel ingår i standardmodellen, den berömda Higgsbosonen, som ger massa åt de andra partiklarna. Så vitt vi vet idag är alla dessa elementarpartiklar, dvs. de har ingen inre struktur. Men som det har visat sig tidigare, så kan det mycket väl finns ytterligare struktur som vi ännu inte haft tillräckligt med energi för att upptäcka.

Den här avhandlingen rör den starka kraften, som beskrivs av teorin kvantkromodynamik (QCD, från engelskans Quantum Chromodynamics). Likt den elektromagnetiska kraften, så har partiklar en laddning, som avgör hur mycket de påverkas av den starka kraften. En elektromagnetisk laddning är antingen positiv eller negativ, olika laddningar attraherar varandra och lika laddningar repellerar varandra. För QCD är det mer komplicerat, en kvark kan ha tre olika laddningar. Laddningarna kallas för färger, och de tre olika möjligheterna är röd, grön och blå. Likt elektromagnetism så har antikvarkar "negativ" färg,

anti-röd, anti-grön och anti-blå. Kvarkar går inte att observera direkt, de kan endast observeras som bundna färglösa tillstånd, baryoner och mesoner. Att QCD laddningar kallas färg kommer av att de bundna tillstånden är de kombinationer av färger som tillsammans ger en "vit" färg, vilket är en färg och dess anti-färg, eller kombinationen röd-grön-blå. Gluon kommer i åtta färger, vilket motsvarar alla kombinationer av en färg och en anti-färg bortsett från den färglösa kombinationen, röd-grön-blå.

För att beräkna tvärsnittet för en process (sannolikheten att den processen sker), behöver ta med bidraget från varje möjlig färgkombination, det vill säga tre möjliga färger för varje kvark och åtta färger för varje gluon i processen. Det blir snabbt ohanterbart, till och med för datorer, när man ökar antalet partiklar i en process. Standard metoden att hantera färgerna, använder att färgerna i QCD kommer från en symmetrigrupp, som kallas $SU(3)$. Genom att använda den matematiska teorin för grupper, så kan beräkningarna av färgernas effekt på tvärsnittet organiseras i så kallad färgbaser. I de flesta fall används så kallade spårbaser, DDM baser eller färgflödesbaser, då dessa baser har flera användbara egenskaper. Artikel I, II och IV i den här avhandlingen utforskar en annan typ av färgbas, multiplettbaser. Denna typ av bas är mer involverad att jobba med än standardbaserna, men den är ortogonal, vilket är en väldigt användbar egenskap, speciellt när antalet partiklar blir stort.

I artikel III behandlas också färger i QCD, men i den artikeln används spårbasen. För att beskriva proton kollisioner på Large Hadron Collider (LHC) vid CERN, behöver man kombinera flera sätt att simulera en krock. Den första delen är att räkna ut tvärsnittet för den så kallade hårda processen. Den hårda processen karaktäriseras av att de involverade partiklarna har hög energi och är väl separerade i vinklar. Dessa tvärsnitt involverar en handfull partiklar, ofta är det två till två, eller två till tre, processer. Men detektorerna vid LHC detekterar upp till hundratals partiklar i kollisioner. Den större delen av dessa partiklar kommer från partonskursdelen av beskrivningen av kollisionen, vilken beskriver strålning som inte klassas som lika hård, det vill säga strålning med liten rörelsemängd jämfört med den hårda processen och strålning som skickas ut kollineärt. Denna strålning kommer också från QCD, och beror på färgerna av de involverade partiklarna. Eftersom partonskuren går från ett tillstånd med en handfull partiklar till ett tillstånd med hundratals partiklar, så är det beräkningsmässigt omöjligt att hantera färgdelen exakt. Därför används en approximation, att det finns oändligt många färger, istället för tre. Detta gör att färgdelen blir lättare att hantera, och partonskuren kan behandla upp till hundratals partiklar i sluttillståndet. I artikel III implementerade vi en algoritm som behandlar dom första utskicken från partonskuren med hela färgstrukturen, och sedan övergår till att använda standardapproximationen med oändligt många färger.

Multiplet bases, recursion relations and full color parton showers

1 Introduction

The main body of this thesis are the four articles in the end, which will be referred to as paper I, paper II, paper III and paper IV. The first part of the thesis is an introduction to the papers.

1.1 The Standard model of particle physics

The standard model of particle physics (SM) is an exceptionally good theory. It explains all current high energy physics measurements (with a few exceptions). But there are pieces that we know are missing from the standard model. The SM does not contain gravity, which would need to be incorporated into a "theory of everything" at some point. At the energies of the LHC, gravity is negligible, it starts becoming relevant at much, much, higher experimental energies. Another experimental problem is the need for "dark matter", seen initially from the orbit of stars around the galactic plane, where the rotational speed is not consistent with the matter that can be seen. While this is an astronomy problem, it can be solved by several extensions of the standard model, usually by introducing a particle (or particles) interacting in the right way to give us this "missing" matter. So far, however, all such extensions have been either ruled out or not yet been tested, due to being out of reach for current experiments.

The SM is a quantum field theory (QFT), which is the theory resulting from the combination of two of the greatest scientific theories of the 20th century, special relativity and quantum mechanics. In the SM there are three types of particles, matter particles (fermions), force carriers (bosons) and the Higgs boson. There are three forces, electromagnetism mediated by the massless photon, the weak force, mediated by the massive $W^{\pm}$ and Z^0 bo-

sons and finally, the strong force, mediated by eight massless gluons. The matter particles can be divided based on their strong interactions. There are six leptons, which do not interact with the strong force, and six quarks which do interact with the strong force. Every particle has an antiparticle (or they are their own antiparticle), for example there is the negatively charged electron, whose antiparticle is the positively charged positron. There is also the Higgs boson, responsible for the mass of the other particles. Curiously enough, the leptons and quarks are organized into families, that are copies of each other, except for the mass. The lightest family consists of up and down quarks, electrons and electron neutrinos (and their antiparticles). The particles of this family makes up almost all matter in the universe.

Of the forces in the SM, electromagnetism is the most familiar one, it is described by the QFT Quantum Electrodynamics (QED). This thesis, however, is mainly concerned with the strong force, responsible for binding together the nuclei of atoms. The strong force is described by the QFT Quantum Chromodynamics (QCD). There are two main differences between QCD and QED. One is that the force carriers, the gluons, have color charges, while the photon is electrically neutral. So the photons do not interact with themselves, as they do not have any electric charge, but the gluons do interact with themselves. The other difference is confinement, which is the fact that quarks and gluons cannot be observed as free particles, only as color neutral bound states. These bound states are called hadrons, and consist of a quark and an antiquark, which is called a meson, or three quarks (antiquarks), which is a baryon (antibaryon).

To describe particle collisions at high energies, several models have to be combined. It used to be the case that newly discovered particles could be directly observed, e.g. the first identified positron (the antiparticle of the electron) was by C. D. Anderson, identified from the track it left in a cloud chamber.[1] This is, however, not the case today, as theorized particles today tend either be short-lived enough that they do not travel far enough to be seen (despite the fact that they are moving close to the speed of light), be weakly interacting enough that they do not leave a visible track in a detector or there are other processes that can look exactly the same, such that they cannot be distinguished. To discover particles despite of this, one can finding the probability of a process, and any other background noise, i.e. any other allowed process that would look indistinguishable. We can then compare the measured number of times the process occurs and compare with the calculated probability, to see if it is more compatible with or without the process of interest. One of the recent achievements of the SM, is the discovery of the Higgs boson. The Higgs boson is so short-lived that it cannot be directly observed. As an example, take the process $gg \to H \to t\bar{t}$, two gluons fuse into a Higgs boson, which then decays into a top-antitop pair. This is then

[1] A cloud chamber is a chamber containing supersaturated water or alcohol. If an ionizing particle (i.e. it has enough energy to knock out electrons from atoms) traverses this chamber, it will leave a visible trail of droplets.

the process we are interested in, but what is actually detected at the LHC is not a couple of particles, but rather up to a hundred of particles, and the detected particles are hadrons, not quarks and gluons. Leaving out a lot of details, and just focusing on three of the main parts in going from a process we are interested in to what is actually seen at the LHC, we have the *hard scattering cross section*, the *parton shower* and *hadronization*.

- *Hard scattering cross section*: the cross section of a process, often denoted σ, is the probability of that process occurring, for example $gg \to H \to t\bar{t}$ that we are considering. For the next steps, we actually want the differential cross section, which is the probability of the process for a specific point in phase space, i.e. a specific set of momenta or angles of the involved particles. This differential cross section can be calculated by a perturbative expansion in the strength of the relevant coupling (here the strength of the Higgs boson coupling to tops), keeping terms up to some order in the expansion (limited by the fast growth of complexity in successive terms).

- *Parton shower (PS)*: there is additional radiation that needs to be included, part of which can be understood as the analog of bremsstrahlung for QED (where an accelerated charge emits radiation). QCD interacts more strongly and gluons carry a color charge themselves, making the bremsstrahlung different from the QED case. The need for the parton shower can be seen from the perturbative calculations in the previous step. The perturbative expansion an expansion in the coupling constant, g, such that higher order terms in the expansion are suppressed by more factors of g. However, for certain momentum configurations (when emissions are collinear or soft) the suppression by g can be compensated for by a phase space enhancement. In this case, the perturbative expansion cannot just be cut off after the first few terms, instead there are contributions to every order in the perturbative expansion, that need to be resummed. At first this seems very worrisome, as calculating more terms in the perturbative expansion becomes extremely computationally expensive quickly. However, the entire perturbative expansion is not required, only part of the expansion is needed. This is handled by a parton shower. Similar contributions can also be included by a method called analytic resummation.

  The PS iteratively adds emissions, taking the initial process into a state with tens or hundreds of particles at the LHC.

- *Hadronization*: as mentioned, we do not detect quarks or gluons, but rather hadrons, which are bound states of three quarks (or antiquarks) or a quark and an antiquark. The hadronization part of a simulated event is the step for going from a set of quarks and gluons into a set of hadrons. The hard scattering cross section and the parton shower can both be derived from first principles, i.e. from QCD, the hadronization cannot however. Two common hadronization models are the Lund string model and

the Herwig cluster model. While they are not derived from QCD, they are of course heavily inspired by it.

In this thesis all four papers are related to QCD. Papers I, II and IV mainly concern the hard scattering part (but the results of the papers can also be applied to parton showers) and paper III concerns the parton shower part. Hence, the introduction is organized as follows: section 1.2 introduces the theory of strong interactions, QCD, then in section 1.3 the mathematical theory of groups is introduced, as it is the common thread in all four papers. In section 1.4, the so-called birdtrack notation is introduced and a few useful relations are derived, which have been heavily used in papers I, II and IV. Then, in section 1.6 the spinor-helicity formalism is introduced and recursion relations are derived. In section 1.7, parton showers are discussed.

1.2 Strong interactions $SU(3)$

The Lagrangian of QCD, describing the interactions of the quark fields, ψ_i, and the gluon fields, G^a, is

$$\mathcal{L}_{QCD} = \overline{\psi}^i (i(\gamma^\mu (D_\mu)_i{}^j - m))\psi_j - \frac{1}{4} F^{a\,\mu\nu} F^a_{\mu\nu}, \tag{1}$$

where the field strength is

$$F^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu, \tag{2}$$

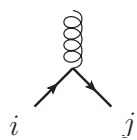
and the covariant derivative is

$$(D_\mu)_i{}^j = \partial_\mu \delta_i{}^j - i g G^a_\mu (t^a)^j{}_i, \tag{3}$$

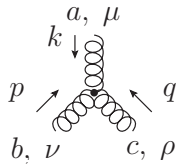
where $i, j = 1, 2, 3$ are quark color indices, $a, b, c = 1, \ldots, 8$ are gluon color indices, and g_s is the coupling strength of QCD. The parts related to the symmetry group of QCD, $SU(3)$, in eqs. (1-3) are the generators, $(t^a)^i{}_j$, and the structure constants, f^{abc}. From a Lagrangian, Feynman rules can be derived, see e.g. [1], for QCD these are (in Feynman gauge):

$$i \xrightarrow[p]{} j \quad = \frac{i(\not{p} + m)\delta^i{}_j}{p^2 - m^2 + i\epsilon}, \tag{4}$$

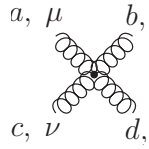
$$a,\ \mu \overset{p}{\text{⌇⌇⌇}} b,\ \nu \ = \frac{-i\eta_{\mu\nu}\delta^a{}_b}{p^2 + i\epsilon}, \tag{5}$$



$$= i g_s \gamma^\mu (t^a)^i{}_j, \tag{6}$$



$$= g_s f^{abc}[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu], \quad (7)$$



$$= \begin{array}{r} -ig_s^2[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + \\ f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + \\ f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})], \end{array} \quad (8)$$

where $\eta^{\mu\nu}$ is the metric tensor. The Feynman rules for external particles are: initial fermions (antifermions) pick up a wave function $u^s(p)$ ($\overline{v}^s(p)$), final fermions (antifermions) pick up a wave function $\overline{u}^s(p)$ ($v^s(p)$) and initial (final) gluons pick up a polarization vector $\epsilon^*_\mu(p)$ ($\epsilon_\mu(p)$).

One way of dealing with the color structure of QCD, is to use a specific basis and calculate with explicit matrices. However, this quickly turns very cumbersome, as it does not take advantage of the group theoretical structure. One common basis choice are the Gell-Mann matrices,

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$t^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ t^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\ t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (9)$$

The structure constants, f^{abc}, are related to the generators, as will be shown in section 1.3, and they are completely antisymmetric in their indices. For the basis eq. (9), the structure constants are

$$f^{123} = 1,$$

$$f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2},$$

$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}. \quad (10)$$

With these explicit generators and structure constants any QCD amplitude could be evaluated. However, there are better ways, avoiding any explicit basis, and even avoiding setting the number of colors to $N_c = 3$, but rather performing the whole calculation for general N_c.

1.3 Group theory

In this section the necessary basic group theory will be reviewed [2, 3, 4].

1.3.1 Definitions

Definition (Group). A set of elements G with an associated operation $*$, is a group if they satisfy the following conditions:

- *Closure:* for every pair of elements, $g_1, g_2 \in G$, $g_3 = g_1 * g_2 \in G$.
- *Associativity:* for every triplet of elements, $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$.
- *Identity:* there exists an identity element, 1, that obeys $1 * g = g * 1 = g$ for all $g \in G$.
- *Inverse:* for every element $g \in G$ there exists an inverse element, g^{-1}, such that $g * g^{-1} = g^{-1} * g = 1$.

A group is *Abelian* if for every pair of elements $g_1, g_2 \in G$ we have $g_1 * g_2 = g_2 * g_1$, and a group is *non-Abelian* if there exists at least one pair of elements $g_1, g_2 \in G$ such that $g_1 * g_2 \neq g_2 * g_1$.

Definition (Representation). A mapping, D, of the elements of a group, G, onto linear operators is a representation if the following conditions are satisfied:

- The identity element of G, 1, is mapped onto the identity operator of the space the operators act on.
- The mapping preserves the group multiplication law, for any two elements $g_1, g_2 \in G$, $D(g_1 * g_2) = D(g_1)D(g_2)$.

A representation is faithful if, for every $g \in G$, $D(g)$ is distinct. Furthermore, a representation is reducible if there exists a change of basis such that $D(g)$ can be put into a block-diagonal form for all g in the group, i.e.

$$S^{-1}D(g)S = \begin{pmatrix} D_1(g) & 0 & \dots \\ 0 & D_2(g) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{11}$$

and it is irreducible if it is not reducible. A powerful theorem for representations is Schur's lemma, which we will make extensive use of.

Theorem (Schur's lemma). If $D_1(g)A = AD_2(g)$ for all $g \in G$, where D_1 and D_2 are inequivalent irreducible representations, then $A = 0$. If $D(g)A = AD(g)$ for all $g \in G$, where D is a finite dimensional irreducible representation, then A must be proportional to the unit matrix.

In physics in general, it is often representations of groups, rather than groups that are of interest, as groups are sets of abstract elements, while representations are sets of linear operators, for which there exists powerful mathematical tools.

A *Lie group* is a continuous, hence infinite, group where the elements $g(\alpha) \in G$ depend smoothly on some parameters α, i.e. there is a notion of closeness in the space of group elements, such that if two elements are close together, then so are their parameters α. Representations of Lie groups can be exponentially parametrized, i.e.

$$D(\alpha) = e^{i\alpha^a t^a}, \tag{12}$$

where D is a representation, α^a are real numbers and t^a are the so-called generators of the group. Close to the unit element for a representation of the group, i.e. when α^a is close to zero for all a, we can Taylor expand eq. (12) and get

$$D(\alpha) = 1 + i\alpha_a t^a + \mathcal{O}(\alpha^2). \tag{13}$$

To preserve the group multiplication law the generators must form a closed commutator algebra, called the *Lie algebra*, i.e.

$$[t^a, t^b] = if^{abc}t^c, \tag{14}$$

where the constants f^{abc} are the structure constants of the group. Since the commutator is antisymmetric in its arguments, the structure constant must also be antisymmetric in its first two indices (we will later show that it is antisymmetric in all indices).

1.3.2 Special Unitary groups, $SU(N_c)$

In this thesis we are mainly concerned with one family of groups, the special unitary groups, $SU(N_c)$, for $N_c = 2, 3, \ldots$. In QCD it is of course well established that $N_c = 3$, but in papers I, II and IV, the number of colors have been left as a free parameter, as the constructions elegantly work for any N_c.

The special unitary group $SU(N_c)$ is the group of rotations in a complex N_c-dimensional space. These transformations leave the scalar product of two vectors in this space invariant, as well as the volume defined by $\epsilon^{\alpha_1\alpha_2\ldots\alpha_{N_c}} v_{\alpha_1} v_{\alpha_2} \ldots v_{\alpha_{N_c}}$, where $\epsilon^{\alpha_1\alpha_2\ldots\alpha_{N_c}}$ is the Levi-Civita tensor and v_{α_i} are vectors in the space. For a representation of the group this is

ensured if the linear operators are unitary, $DD^\dagger = 1$, and have $\det D = 1$. That the determinant is one, is equivalent to the generators being traceless, $\mathrm{tr}(t^a) = 0$, from eq. (12).

The physically important irreducible representations in QCD are the fundamental representation, V, the complex conjugate of the fundamental representation, $\overline{V}$ and the adjoint representation, A, corresponding to the quark, the antiquark and the gluon, respectively. The invariants of the group are also of physical interest, as they show combinations of representations that transform trivially under the group. As QCD has confinement, i.e. only states transforming trivially under $SU(3)$ are observed, we can see what combinations of hadrons are possible from the invariants. A vector in an N_c dimensional space transforms as the fundamental representation, and its complex conjugate as the complex conjugate of the fundamental representation. The scalar product being invariant, means that a quark and an antiquark transforms trivially, and hence correspond to an observable state, these are the so-called mesons. The other invariant is the combination of three quarks (antiquarks), which are the baryons (antibaryons).

1.3.3 Lorentz group

In section 1.6 the Lorentz group will be needed. The Lorentz group is the group of all Lorentz transformations of Minkowski spacetime. In one time and three space dimensions, there are three rotations, whose generators we denote J_i for $i = 1, 2, 3$, and three boosts, with the generators K_i, for $i = 1, 2, 3$. They form the Lorentz commutator algebra

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k, \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k, \\ [J_i, K_j] &= i\epsilon_{ijk} K_k. \end{aligned} \tag{15}$$

By constructing the combinations $J_i^\pm = \frac{1}{2}(J_i \pm iK_i)$ the algebra separates into two pieces,

$$\begin{aligned} [J_i^+, J_j^+] &= i\epsilon_{ijk} J_k^+, \\ [J_i^-, J_j^-] &= i\epsilon_{ijk} J_k^-, \\ [J_i^+, J_j^-] &= 0. \end{aligned} \tag{16}$$

This is in fact two copies of the $SU(2)$ commutator algebra. Due to this, the representations of the Lorentz group can be labeled by the representation in each of the two $SU(2)$ groups. Note that $J_i^\pm$ are related to each other both through complex conjugation and through parity.

1.3.4 Young diagrams

The irreducible representations of $SU(N_c)$ can be labeled by Young diagrams, which are composed of left- and top-justified boxes. A Young diagram of $SU(N_c)$ can have columns that are at most N_c boxes high. A column that is N_c columns high corresponds to the Levi-Civita tensor with N_c indices, which is a singlet for $SU(N_c)$, so the Young diagram with an N_c boxes high column, corresponds to the same representation as the Young diagram with that column removed. As an example, the Young diagrams of the quark, antiquark and gluon representations are

$$V = \square\,, \quad \overline{V} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\!\!\begin{array}{|c|}\hline \\ \hline \cdot \\ \hline \cdot \\ \hline \cdot \\ \hline \\ \hline \end{array}\!\!\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\!\!\begin{array}{c} \\ N_c\text{-}1 \\ \\ \\ \\ \end{array}\,, \quad A = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\!\!\begin{array}{|c|}\hline \\ \hline \cdot \\ \hline \cdot \\ \hline \cdot \\ \hline \\ \hline \end{array}\!\!\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\!\!\begin{array}{c} \\ N_c\text{-}1 \\ \\ \\ \\ \end{array}\!\!\begin{array}{|c|}\hline 1 \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\,, \tag{17}$$

where the height of the first column of $\overline{V}$ and A depends on N_c. The Young diagrams are not only useful as labels of the representations, they can also be used to determine the dimension of the representations and the tensor product of different representations.

The dimension of a representation, α, is given by

$$d_\alpha = \frac{f_Y}{|Y|}, \tag{18}$$

where f_Y and $|Y|$ are numbers calculated from the Young diagram. Both of these numbers are found by filling the boxes of the Young diagram with numbers[2] and then taking the product of the numbers. The first number, f_Y, is calculated by setting the top-left box to N_c and then filling the remaining boxes with numbers fulfilling:

- The numbers increase by one per box going to the right.
- The numbers decrease by one per box going down.

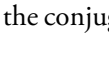
The number f_Y is now given by the product of all of the numbers in the boxes. The second number, $|Y|$, is calculated by using the so-called Hook rule. The number in a box is given by one plus the number of boxes below and to the right of the box.

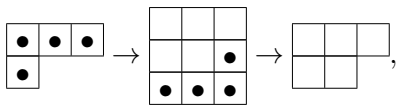
[2]When the boxes are filled with numbers it is called a Young tableau, instead of diagram.

Applying this to the representations in eq. (17) gives

$$\begin{array}{ccccc} & V & \overline{V} & A \\ \hline f_Y & N_c & N_c! & (N_c+1)! \\ |Y| & 1 & (N_c-1)! & (N_c-2)!N_c \\ d & N_c & N_c & N_c^2-1, \end{array} \tag{19}$$

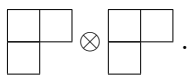
as we should have (for $N_c = 3$ we have 3 quark/antiquark colors and 8 gluon colors).

Representations in $SU(N_c)$ are in general complex, meaning that there is a conjugate representation associated with them. If the conjugate representation is related by a unitary transformation, the representation is real and the representations are the same, otherwise the representations are distinct. Using the Young diagram of a representation it is easy to find the Young diagram of the conjugate representation. The conjugated diagram is found by marking the boxes of the original diagram, rotating them $180°$ degrees, adding boxes on top of each column until they are N_c boxes high and finally removing the marked boxes, the remaining boxes are the conjugated diagram [4]. Using  for $N_c = 3$ as an example, we get

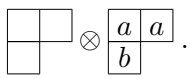
 (20)

i.e. the conjugated diagram of  is  . If using the procedure in eq. (20) on a diagram returns the same diagram, the representation is real.

Finally, we can consider the decomposition of the tensor product of two irreducible representations into a direct sum of irreducible representations [2]. First we draw the two Young diagrams of the representations in the tensor product, for example

 . (21)

One of the diagrams is then filled with letters, a in all boxes of the first row, b in the second row, and so on. For the example tensor product this yields

 . (22)

The labeled boxes will now be added to the unlabeled diagram, one row at a time, with the conditions:

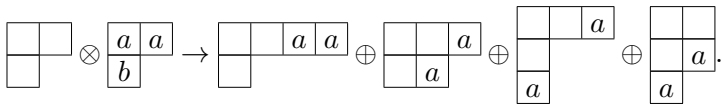
(i) The diagram is still left- and top-justified.

(ii) No column contains more than one a (or b, or c,...).

(iii) No column contains more than N_c boxes.

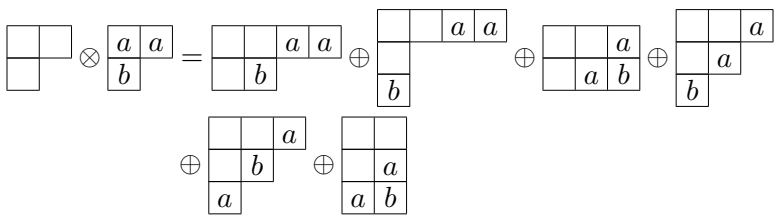
(iv) Reading from right to left, and going downwards, the sequence of letters is admissible, meaning that at no point in the sequence of letters are there more bs than as, or cs than bs, and so on.

After all of the labeled boxes have been added to the unlabeled diagram, the decomposition is finished and the labels can be removed. Columns with N_c boxes can also be removed. The tensor product is equal to the direct sum of all of the representations corresponding to the Young diagrams given by this procedure. Applying this procedure to the example tensor product for the first row gives



(23)

The next step is to repeat the procedure for the second row, filled with bs. If we take $N_c = 3$, there are two admissible ways of adding the box with a b to the first and second diagrams of eq. (23) (adding the box to the first row violates (iv) in both cases) and one way of adding the box to the last two diagrams (since adding it to the fourth row violates (iii)). In total we get



$$= 27 \oplus 10 \oplus \overline{10} \oplus 8 \oplus 8 \oplus 1, \tag{24}$$

where the right-hand-side denote the representations by their dimension (that can be calculated from eq. (18)). Note that we have used that 3 boxes high columns can be removed. The labels are no longer needed, and are only kept in eq. (24) to make it easier to see where each diagram comes from.

1.4 Birdtracks

This section will introduce birdtrack notation [4], and we will derive several useful relations, that are heavily used in papers I, II and IV. Birdtracks is a diagrammatical notation for tensors. It is a great tool for dealing with representation theory, where the group we have in mind is of course the symmetry group of QCD, $SU(3)$. It is, however, not limited to just the $SU(N_c)$ groups, nor even only to group theory. For the applications in this thesis, the number of colors, N_c, is not specified, it is left as a free parameter and hence the results are applicable to any $SU(N_c)$ symmetry group.

1.4.1 Basic notation

In this notation indices correspond to lines with arrows, so a Kronecker delta for two indices is simply

$$\delta^i{}_j = \quad i \longrightarrow j \quad . \tag{25}$$

An upper index corresponds to a line with an arrow pointing away from it, and a lower index has the arrow pointing towards it. A vector is written as

$$v^\mu = \quad v \longleftarrow \mu \tag{26}$$

and a tensor is

$$T^{a\ b\ \ c}_{i\ jk} = \quad \begin{matrix} a \\ i \\ b \end{matrix} \rightarrow \boxed{T} \rightarrow \begin{matrix} c \\ k \\ j \end{matrix} \quad , \tag{27}$$

where the convention is that indices are read counter-clockwise. That there is a circle in eq. (26) and a square eq. (27) is of no significance, however if a tensor is not cyclically symmetric, the shape of it must somehow indicate what index to read first. One of the advantages of the birdtrack notation is that the indices of tensors often do not have to be written out, as they are replaced with lines. This is possible since it is immediately clear how many indices a tensor has by simply counting its legs. If a sum of tensors is considered, the position of the endpoint of the leg is used to determine its index. A contraction of indices is simply connecting lines, for example

$$T^{a\ b\ \ c}_{i\ jk} X_c{}^{kj}_{\ lm}{}^{\ d} = \quad \begin{matrix} a \\ i \\ b \end{matrix} \rightarrow \boxed{T} \rightarrow \boxed{X} \rightarrow \begin{matrix} d \\ m \\ l \end{matrix} \quad . \tag{28}$$

In our applications there will be several types of indices, one for each possible representation. To distinguish the types of indices we will use different types of lines, corresponding to different representations, and in some cases use labels for the lines, denoting what representation it is. The most common representations (quark, antiquark and gluons) have their own types of lines and all other representations share the same type of line, and have a label to distinguish them.

Using this notation for the group theoretical parts of QCD introduced in section 1.2, we can write the generators as

$$(t^a)^i{}_j = \quad \overset{a}{\underset{i \longrightarrow}{\overset{\text{\textcircled{}}}{}}} \ j \tag{29}$$

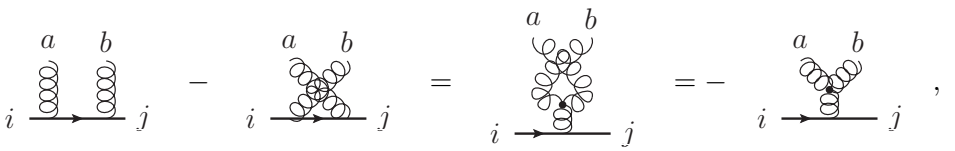
and the structure constants

$$if^{abc} = \quad \overset{a}{\underset{b}{\overset{\text{\textcircled{}}}{}}} \ c \quad , \tag{30}$$

where the plain lines with arrows are quarks (in the fundamental representation) and the curly lines are gluons (in the adjoint representation).

1.4.2 Group theoretical relations

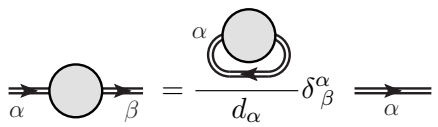
The Lie algebra, eq. (14), is



(31)

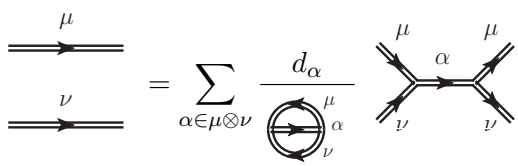
where the second step is using that f^{abc} is antisymmetric in its first two indices. The indices in eq. (31) are superfluous, as the positions of the ends of the legs can be used to determine which legs have the same index in each birdtrack, but they are kept for clarity.

The two group theoretical results that are used the most in papers I, II and IV are Schur's lemma and the completeness relation. Schur's lemma in birdtracks is



(32)

where the double line with an arrow and a representation label (α and β in this case), is the notation we use for representations that are not the quark, antiquark or gluon representation. The blob in eq. (32) is to indicate any kind of color structure with two indices, one in the α representation and one in the β representation. The Kronecker delta makes the entire expression vanish if the two representations are not the same. The completeness relation is a more powerful statement of the decomposition of tensor products into a direct sum of irreducible representations. In birdtracks the relation is

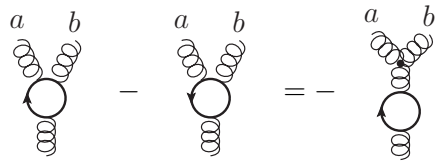


(33)

on the left-hand side we see how tensor products are written in the birdtrack notation, simply two lines. On the right-hand side we see the decomposition of the tensor product into irreducible representations α. The denominator is a so-called Wigner $3j$ coefficient, which is just a normalization factor for the vertices. It is required as the left-hand side does

not contain any vertices and can hence not be dependent on vertex normalizations. The factor for each term in eq. (33) is exactly such that each term in the sum corresponds to a projector, P_α, with the idempotency property, $P_\alpha P_\alpha = P_\alpha$ (as applying Schur's lemma immediately gives the inverse of the factor in eq. (33)).

So far we have only seen how to write different expressions in birdtracks, now we will move on to how to actually perform calculations. The first step is to express the structure constants in terms of the generators. This is easily achieved by contracting the Lie algebra, eq. (31), with a generator,

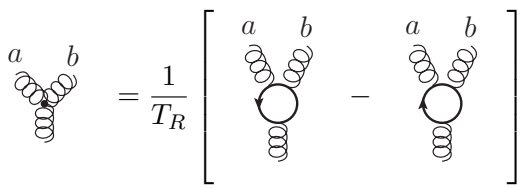


(34)

where in the second term on the left-hand side we have changed the order of all three vertices. In general, changing the order of a vertex could contribute with a sign (as f^{abc} would, since it is antisymmetric in its three indices). The closed quark loop on the right-hand side of eq. (34) can be removed by applying Schur's lemma, eq. (32), giving a factor. This factor is just the generator normalization (and consequently determining the structure constant normalization),

$$= \frac{\mathrm{tr}(t^a t^a)}{d_A}, \tag{35}$$

where we have included the tensor notation for familiarity. Common choices for T_R are $1/2$ and 1, for example in the Gell-Mann matrices, eq. (9), $T_R = 1/2$. Using eq. (35) on eq. (34) and rearranging, gives us

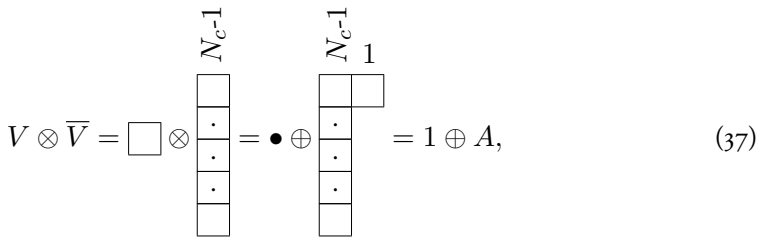


(36)

where it is easy to verify that f^{abc} is indeed antisymmetric in all three of its indices.

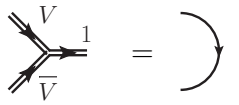
Next we can use the completeness relation, eq. (33), on a quark and an antiquark. For the completeness relation we need to determine what representations α to sum over, the dimensions of those representations and finally we need an expression for the vertices on the right-hand side. The possibilities for α is easy to find from the tensor product of the

Young diagrams. The tensor product of a quark and an antiquark is



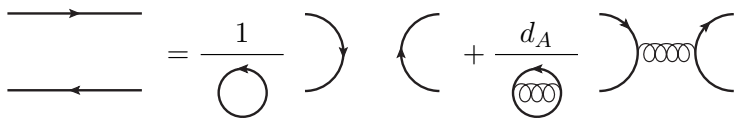
(37)

where • is used as the Young diagram of the singlet representation (which has zero boxes). Hence, we know that for $\mu = V$ and $\nu = \overline{V}$ the representation α can be the singlet and A. The dimension of A is in eq. (19), and the dimension of the singlet is 1. All we are missing now are the vertices. The V-$\overline{V}$-singlet vertex is easy to construct, as a singlet in birdtracks is simply no line at all, i.e. the vertex is



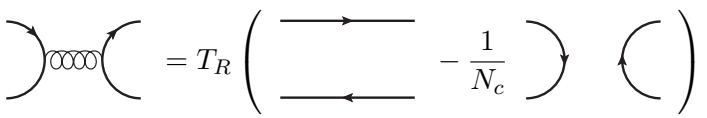
(38)

We note that the normalization of this vertex is such that its Wigner $3j$ coefficient is N_c. The other vertex we already know, from eq. (29). Combining all of these results gives us



(39)

The factor in front of the first term is, if we recall eq. (25), simply $d_V = N_c$ and for the second term the factor is eq. (35). With these factors, we can rearrange eq. (39) into the so-called Fierz' identity,

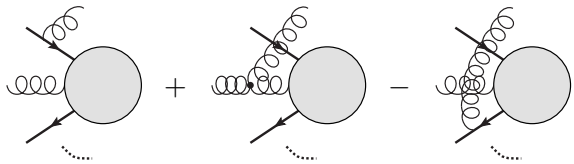


(40)

The Fierz identity, along with eq. (36), can be used to calculate the value of any vacuum bubble, i.e. a color structure with no external legs, consisting solely of quarks and gluons. As a quark loop is equal to the dimension of the fundamental representation, N_c, such an evaluation will in the end give a polynomial in N_c.

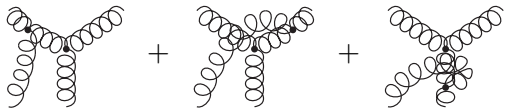
Another useful relation is the invariance condition for color structures. One way of thinking of the invariance condition, is that an infinitesimal group rotation must leave a color

structure unchanged, hence the sum of attaching the generator to each leg of a color structure must vanish. In birdtrack notation the invariance condition is



(41)

where representations flowing in have a positive sign and representations flowing out have a negative sign, and the dots indicate that the gluon should be attached to all external legs. Note that the relative sign of the terms attached to a gluon depends on the sign of the structure constant. We can apply the invariance condition on a structure constant, eq. (30), giving



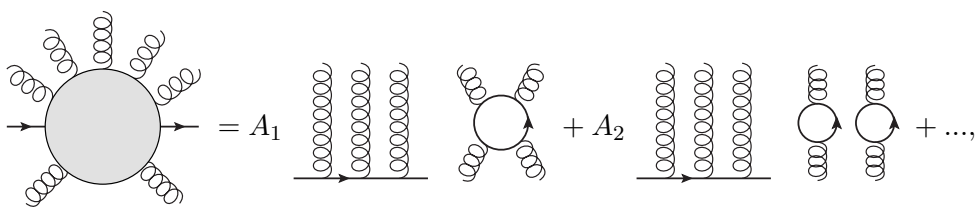
(42)

With these relations we can start deriving different color bases.

1.4.3 Color space bases

Any color structure can be seen as a vector in a vector space, called the color space, and can hence be decomposed into a basis. We can use the relations we have found, in order to rewrite any perturbative QCD amplitude into three different bases.

With the relations eq. (36) and eq. (40), one can remove all triple-gluon vertices and any internal gluon line, this gives a color structure consisting purely of external gluons connected to open (if there are external quarks) or closed quark lines. This is the trace basis[3]. An example of a decomposition into the trace basis is



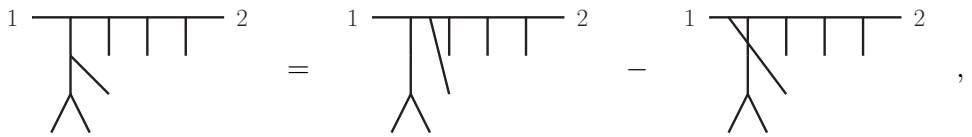
(43)

where the gray blob can be an amplitude to any order in perturbation theory. The basis vectors in this basis are the color structures on the right-hand side of eq. (43), where different permutations of the legs correspond to different basis vectors (with the caveat that closed

[3]It is actually a spanning set if there are enough external partons, as the number of basis vectors is larger than the dimension of the vector space.

quark lines are cyclically symmetric). If we consider a pure gluon color structure at tree-level, the trace basis will consist of a single closed quark line, with all gluons attached to it. If there are N_g external gluons, there are $N_g!$ possible permutations, but due to the cyclic symmetry there are $(N_g - 1)!$ different trace basis vectors. We can easily find the basis vectors that are not related by the cyclic symmetry by fixing one leg, and taking all permutations of the $N_g - 1$ other legs.

Using the example of the invariance condition, eq. (42), another type of color basis can be derived, the Del Duca-Dixon-Maltoni (DDM) basis [5, 6]. If we consider a pure gluon amplitude at tree-level (no loops), we can use eq. (42) to move a structure constant past another structure constant, at the cost of getting two terms. As an example we could have (where, for this and the next equation only, curly gluon lines have been replaced with straight lines without arrows, to make the equation clearer)



(44)

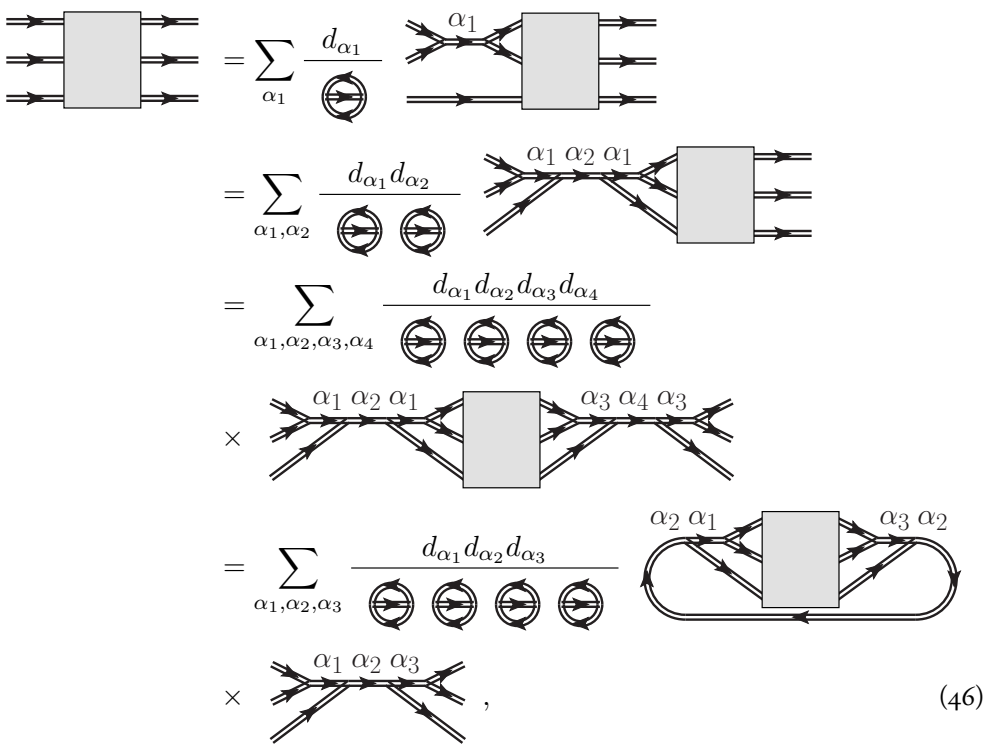
where we, again, note that the structure constants are antisymmetric, meaning that changing the order of a vertex gives a minus sign. The labels, 1 and 2, denote two specific gluons, the goal of rewriting the color structure as in eq. (44), is that we can imagine a gluon connecting 1 and 2, and the above step can be repeated until all other gluons are directly attached to this gluon. The basis vectors are then of form

(45)

where the different basis vectors are all possible permutations of the legs, excluding the legs labeled 1 and 2. Hence, for color structures with N_g gluons, there are $(N_g - 2)!$ basis vectors. The DDM basis is smaller than the trace basis because the position of two gluons are fixed in the DDM basis vectors, while only one gluon has a fixed position for the trace basis vectors.

Another type of basis, the multiplet basis, is the basis used in papers I, II and IV. For any color structure, we can easily find an expression for its decomposition into this basis using the completeness relation, eq. (33), and Schur's lemma, eq. (32). As an example we consider

a color structure with six legs. We apply completeness relations on the legs,



, (46)

where in the last step Schur's lemma was used. In the above, several of the representation labels have been suppressed for clarity. The basis is the color structures of the form of the last line in eq. (46). The advantage of this type of basis, as compared to the trace and DDM bases, is that the basis vectors are orthogonal. This also means that this is a proper basis, and not only a spanning set. The number of basis vectors for a multiplet basis grows as an exponential in the number of external partons, as opposed to a factorial growth for the trace bases and the DDM bases.

1.4.4 Summary

In this section we have seen how to use birdtrack notation for calculating QCD color structures. We have also explored three different types of bases, the trace basis, the DDM basis and multiplet basis. In [7] it was shown how to construct projectors, and with them multiplet basis vectors. With these one can evaluate the vacuum bubbles in eq. (46), which is needed in order to use the multiplet bases. Paper I explores a way of decomposing color structures into the multiplet bases, using so-called Wigner coefficients, paper II applies the result of paper I on amplitude recursion relations. In paper IV, the construction of

projectors is generalized, allowing more freedom in basis choice. Paper III uses the trace basis for the color structure, but the method introduced there could in principle use any color basis.

1.5 Spinor-helicity formalism

1.5.1 Notation

The conventions of this thesis follows [8]. Hence, we are using the mostly-plus metric, $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

In section 1.3.3 it was noted that the representations of the Lorentz group could be labeled by their representation in the two $SU(2)$ groups. The representations for $SU(2)$ are usually labeled by $j = 0, \frac{1}{2}, 1, ...$, where each representation has dimensionality $2j + 1$. From section 1.3.4 we know that we can label the representations with Young diagrams, which in the case of $SU(2)$ is a single row of boxes. The dimension of a single row of boxes, using eq. (18), is $d_n = n + 1$, where n is the number of boxes. Hence, $j = 0$ corresponds to zero boxes and has dimension 1, $j = 1/2$ corresponds to one box with dimension 2, and so on. For the Lorentz group representations this means that we have $(j_1, j_2) = (0, 0),\ (\frac{1}{2}, 0),\ (0, \frac{1}{2}),\ (\frac{1}{2}, \frac{1}{2}), ...$. The first representation is the Lorentz scalar, transforming trivially under the group, the second and third representations are the left-handed and right-handed Weyl spinors, respectively, and the fourth is the four-vector.

We will be dealing with the Weyl spinors. As we saw above, they correspond to the fundamental representations of the two $SU(2)$ groups. Hence, we need the generators for $SU(2)$ in the Lorentz group, namely the Pauli sigma matrices (the analogues of the Gell-Mann matrices, eq. (9), for $SU(2)$)

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{47}$$

Roman indices are used for spatial parts of vectors, i.e. $i, j, ... = 1, 2, 3$ and Greek indices for both time and spatial parts, i.e. $\mu, \nu, ... = 0, 1, 2, 3$. We define $\sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3)$ and $\overline{\sigma}^\mu = (1, -\sigma^1, -\sigma^2, -\sigma^3)$, where 1 is the 2×2 identity matrix. We use the chiral basis for the Dirac matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{48}$$

and

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{49}$$

where 0 and 1 are 2×2 matrices in both equations.

The Weyl spinors will have indices, these will be denoted by Roman letters, but starting at a instead of i (and we will never need enough of them that there would be any confusion). There are two types of such spinor indices, one for each of the $SU(2)$ groups. The two types will be distinguished by the indices of the right-handed spinor coming with a dot on top of them, i.e. $\dot{a}$. As the indices belong to two different $SU(2)$ groups they should never be contracted with each other, so a will only be contracted with another a, and never $\dot{a}$, and vice versa. We will need matrices to raise and lower these types of indices, this is done with two versions of the two index Levi-Civita tensor (one for each type of index, dotted and undotted)

$$\epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{ab} = -\epsilon_{\dot{a}\dot{b}}. \tag{50}$$

The four-component Dirac spinor ψ transforms as $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ under Lorentz transformations. The Dirac conjugate of a four-component spinor ψ is

$$\overline{\psi} = \psi^\dagger \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{b}} \\ \delta_a{}^b & 0 \end{pmatrix}. \tag{51}$$

For the amplitude recursion we will also need Cauchy's residue theorem. Take $f(z)$ to be a complex function that is analytic in a region around z_0, but not necessarily at z_0. Then $f(z)$ has a Laurent expansion around z_0,

$$f(z) = \sum_{i=-\infty}^{\infty} a_i (z - z_0)^i \tag{52}$$

and the residue of $f(z)$ at z_0 is

$$\mathrm{Res}(z_0) = a_{-1}. \tag{53}$$

Theorem (Cauchy's residue theorem). If $f(z)$, for $z \in \mathbb{C}$ is an analytic function at each point within and on a closed contour C in the complex plane, except for a finite number of poles at positions z_j within C, then

$$\oint_C f(z) dz = 2\pi i \sum_j \mathrm{Res}(z_j), \tag{54}$$

where the sum is over all poles within C.

See for example [9] for the proof.

1.5.2 Formalism

We start by considering the Lagrangian for a free massive fermion,

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi, \tag{55}$$

which has the equation of motion

$$(-\not{\partial} + m)\psi = 0. \tag{56}$$

The standard way of solving the equation of motion, for on-shell momenta is by making a plane-wave expansion, i.e.

$$\psi(x) \sim u(p)e^{ip\cdot x} + v(p)e^{-ip\cdot x}, \tag{57}$$

and the general solution is then an integral over such plane-wave solutions. Eq. (57) will be a solution to the Dirac equation, eq. (56), if the four-component spinors $u(p)$ and $v(p)$ solve

$$(\not{p} + m)u(p) = 0, \tag{58}$$

and

$$(-\not{p} + m)v(p) = 0, \tag{59}$$

as u and v are not functions of x. There are two independent solutions for u and for v. We can chose the basis such that the solutions are the eigenstates of the z-component of the spin-matrix and label the solutions with a subscript, $s = \pm$, according to their spin projection onto the z-axis. For massless particles moving in the z-direction, this would correspond to the helicity. The general solution to eq. (55) is then

$$\psi(x) = \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^3 2E_p}\left(b_s(p)u_s(p)e^{ip\cdot x} + d_s^\dagger(p)v_s(p)e^{-ip\cdot x}\right), \tag{60}$$

where $b_s(p)$ is a fermionic annihilation operator and $d_s^\dagger(p)$ is a fermionic creation operator.

We are only concerned with massless particles here, which simplifies eq. (58) to

$$\not{p}u(p) = 0, \tag{61}$$

and eq. (59) simplifies to

$$\not{p}v(p) = 0. \tag{62}$$

As we have crossing symmetry we will only consider outgoing states. An outgoing anti-fermion comes with a wave function v, and an outgoing fermion comes with $\overline{u}$. Crossing

symmetry also relates the wave functions, $u_\pm = v_\mp$ and $\overline{v}_\pm = \overline{u}_\mp$ (it flips helicity). In the chiral basis we can write the two independent solutions as angle and square bras and kets,

$$\overline{u}_+(p) = \begin{pmatrix} [p|^a & 0 \end{pmatrix}, \quad \overline{u}_-(p) = \begin{pmatrix} 0 & \langle p|_{\dot{a}} \end{pmatrix} \tag{63}$$

and

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}, \quad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}. \tag{64}$$

By defining the bi-spinors $p_{\dot{a}b} = p_\mu (\sigma^\mu)_{a\dot{b}}$ and $p^{\dot{a}b} = p_\mu (\overline{\sigma}^\mu)^{\dot{a}b}$, we get

$$\not{p} = \begin{pmatrix} 0 & p_{a\dot{b}} \\ p^{\dot{a}b} & 0 \end{pmatrix}. \tag{65}$$

This can then be used to rewrite the Dirac equation into the massless Weyl equations. So, eq. (61) and eq. (62) becomes

$$\begin{aligned} [p|^a p_{a\dot{b}} = 0, \quad \langle p|_{\dot{a}} p^{\dot{a}b} = 0, \\ p^{\dot{a}b}|p]_b = 0, \quad p_{a\dot{b}}|p\rangle^{\dot{b}} = 0. \end{aligned} \tag{66}$$

Now we can consider the Lorentz invariants we can construct, the angle spinor bracket and the square spinor bracket,

$$\begin{aligned} \langle p_1 p_2 \rangle \equiv \langle p_1|_{\dot{a}}|p_2\rangle^{\dot{a}} = -\langle p_2 p_1 \rangle, \\ [p_1 p_2] \equiv [p_1|^a |p_2]_a = -[p_2 p_1], \end{aligned} \tag{67}$$

where the antisymmetry comes from the fact that the spinor indices are raised and lowered by the Levi-Civita symbol, i.e. $\langle p|_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}|p\rangle^{\dot{b}}$ and $[p|^a = \epsilon^{ab}|p]_b$.

From the spinor completeness relation we can relate the angle and square spinors to the momenta. The spinor completeness relation is

$$u_-(p)\overline{u}_-(p) + u_+(p)\overline{u}_+(p) = -\not{p}, \tag{68}$$

for massless fermions. Using $u_\pm = v_\mp$ this becomes

$$\begin{aligned} -\not{p} &= \begin{pmatrix} |p]_a \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \langle p|_{\dot{a}} \end{pmatrix} + \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix} \begin{pmatrix} [p|^a & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & |p]_a \langle p|_{\dot{a}} \\ |p\rangle^{\dot{a}}[p|^a & 0 \end{pmatrix}, \end{aligned} \tag{69}$$

which, by comparing with eq. (65), gives

$$\begin{aligned} p_{a\dot{b}} = -|p]_a \langle p|_{\dot{b}}, \\ p^{\dot{a}b} = -|p\rangle^{\dot{a}}[p|^b. \end{aligned} \tag{70}$$

Using this and $\mathrm{tr}(\sigma^\mu\overline{\sigma}^\nu) = \mathrm{tr}(\overline{\sigma}^\mu\sigma^\nu) = -2\eta^{\mu\nu}$, one can derive

$$\langle pq\rangle[pq] = 2p\cdot q. \tag{71}$$

The following notation will be used,

$$\langle p|\gamma^\mu|q] \equiv \langle p|_{\dot{a}}(\overline{\sigma}^\mu)^{\dot{a}b}|q]_b, \tag{72}$$

i.e. it should be understood as picking out the component of γ^μ with matching index structure. Using this notation and $(\overline{\sigma}^\mu)^{\dot{a}a}(\overline{\sigma}^\mu)^{\dot{b}b} = -2\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}$ one can derive the Fierz identity for spinors

$$\langle 1|\gamma^\mu|2]\langle 3\gamma_\mu|4] = 2\langle 13\rangle[24], \tag{73}$$

where the short-hand notation $|i] = |p_i]$ and $|i\rangle = |p_i\rangle$ has been used. A very useful relation for spinors is

$$p^\mu = \frac{1}{2}\langle p|\gamma^\mu|p]. \tag{74}$$

We can now state the polarization vectors for massless spin 1 bosons with momentum p,

$$\epsilon_-^\mu(p;q) = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[qp]}, \quad \epsilon_+^\mu(p;q) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle}, \tag{75}$$

where q is an arbitrary reference vector (with the constraint that $q \neq p$). The freedom of choice of q, is due to gauge invariance. By using the Fierz identity, eq. (73), and eq. (74), it is straight-forward to prove that these polarization vectors satisfy the desired properties of polarization vectors, i.e.

$$\epsilon_\pm(p;q)\cdot p = 0, \quad \epsilon_\pm(p;q)\cdot\epsilon_\pm(p;q) = 0 \quad \text{and} \quad \epsilon_+(p;q)\cdot\epsilon_-(p;q) = -1. \tag{76}$$

This freedom can be utilized to simplify calculations, as each external spin 1 particle has their own reference vector. One immediate use of this freedom, is when considering the contraction of two polarization vectors. By using the Fierz identity, eq. (73), on the contraction of two polarization vectors we find

$$\epsilon_-(p_i;q_i)\cdot\epsilon_-(p_j;q_j) \propto \langle p_ip_j\rangle[q_iq_j], \tag{77}$$

$$\epsilon_+(p_i;q_i)\cdot\epsilon_+(p_j;q_j) \propto \langle q_iq_j\rangle[p_ip_j], \tag{78}$$

and

$$\epsilon_-(p_i;q_i)\cdot\epsilon_+(p_j;q_j) \propto \langle p_iq_j\rangle[q_ip_j]. \tag{79}$$

By the antisymmetry of the spinor brackets, we have $\langle pp\rangle = [pp] = 0$. As we are free to choose the reference spinors, we can choose $q_i = q_j$ to make eq. (77) and (78) vanish, and $q_i = p_j$ (or $q_j = p_i$) to make eq. (79) vanish.

We now turn to QCD and consider a pure gluon tree-level amplitude decomposed into the trace basis, as in section 1.4.3,

$$A_n = g^{n-2} \sum_{\text{perms } \sigma,\ \sigma_1=1} A_n[\sigma_1 ... \sigma_n] \, \text{tr}(t^{\sigma_1} ... t^{\sigma_n}), \tag{80}$$

where the sum is over all permutations of the legs 2 to n. The prefactors, $A_n[\sigma_1 ... \sigma_n]$, are called partial amplitudes. Each of the partial amplitudes is gauge invariant, so in the amplitude recursion of the next section they can be treated separately.

The different helicities of the particles come with different spinors. When evaluating amplitudes with the spinor helicity formalism, it is done for each helicity configuration. Helicity is not conserved in QCD, so for a specific helicity configuration, we can calculate how much it "violates helicity conservation", for n external gluons,

$$\sum_{i=1}^{n} h_n, \tag{81}$$

since all particles are outgoing. The maximal value of helicity violation is obtained when all of the gluons have the same helicity. These amplitudes are zero at tree-level, as are the amplitudes with all helicities except one being equal. We can prove that they must vanish, by showing that any Feynman diagram contributing to the amplitude must contain at least one contraction of two polarization vectors which can be made to vanish by an appropriate choice of reference spinors (as mentioned under eq. (79)). As we are considering the n gluon amplitude, there will be n polarization vectors in each Feynman diagram. From the Feynman rules in eq. (5), eq. (7) and eq. (8), we see that the only four-vector contractions that can occur in a diagram are

$$\epsilon_i \cdot \epsilon_j, \quad \epsilon_i \cdot p_j, \quad p_i \cdot p_j. \tag{82}$$

The diagrams with the fewest $\epsilon_i \cdot \epsilon_j$ contractions will be composed purely out of triple-gluon vertices, because the four-gluon vertex does not contain any momentum that could be contracted with a polarization vector. The propagators, eq. (5), do not contribute any factor of momentum to the numerator of the expression. Hence, we will not need to consider the four-gluon vertices or the propagators any further. An n gluon tree-level Feynman diagram, consisting solely of triple-gluon vertices, will contain $n-2$ vertices. From the Feynman rule for triple-gluon vertices eq. (7), we get a factor of a momentum from each vertex. Each external gluon gets a polarization vector. In total we have $n-2$ momenta and n polarization vectors, hence the there must be at least one contraction of the form $\epsilon_i \cdot \epsilon_j$.

For an all-plus (or all-minus) helicity amplitude, $A_n(+, ..., +)$, every diagram contributing to the amplitude will have a factor of $\epsilon_{\pm i} \cdot \epsilon_{\pm j}$. If we pick all reference vectors to be the same, $q_i = q$ for all i, then $\epsilon_{\pm i} \cdot \epsilon_{\pm j} = 0$ for all i and j, from eq. (77) and (78). Hence, all diagrams vanish and the amplitude must vanish.

If we instead consider one of the gluons, say gluon 1, to have negative helicity and the rest positive helicity, i.e. $A_n(-,+,...,+)$, we can pick the reference spinors $q_2 = q_3 = ... = q_n = p_1$. For the same reason as for the all-plus/minus case this means that $\epsilon_{+i} \cdot \epsilon_{+j} = 0$. With this choice of reference spinors $\epsilon_{-1} \cdot \epsilon_{+j} = 0$. As for the all-plus/minus case, all diagrams contributing to the amplitude must vanish.

Note that the amplitude should be independent of the arbitrary reference spinors, and hence this is true for any choice of reference spinors. There is an exception to the argument above for the three gluon amplitudes with one negative helicity gluon and two positive helicity gluons, e.g. $A_3(-,+,+)$. This is due to special three particle kinematics. The momentum is so constrained by momentum conservation that the three gluon amplitude, $A_3(-,+,+)$, is either zero or the choice of reference spinors above, $q_2 = q_3 = p_1$, is not valid. The proof of this can be found in, for example, [8].

With the above argument we can define a so-called Maximally Helicity Violating (MHV) amplitude, an amplitude that violates helicity maximally, while being non-zero at tree-level. Hence, MHV amplitudes are amplitudes with two negative helicity legs and the rest positive helicity (the parity transformed version, two positive and the rest negative helicity are the anti-MHV, or $\overline{\text{MHV}}$, amplitudes).

1.6 Amplitude recursion relations

An alternative to using Feynman diagrams for perturbative calculations in QFTs is amplitude recursion relations. The spinor-helicity formalism is well-suited for recursion relations. In this section A_n can be any tree-level amplitude derived from a local Lagrangian.

If we consider the amplitude for massless particles, then it depends on the momenta and type of the external particles (for example helicity). As in the previous section we consider all particles to be outgoing. Momentum conservation is then

$$\sum_{i=1}^{n} p_i^\mu = 0, \tag{83}$$

for n external particles with momenta p_i^μ. Consider now n complex momenta r_i^μ that fulfill:

(i) Momentum conservation, $\sum_{i=1}^{n} r_i^\mu = 0$.

(ii) Mutual orthogonality, $r_i \cdot r_j = 0$ for all i, j.

(iii) Orthogonality to the momenta p_i, $r_i \cdot p_i = 0$ (without a sum) for all i.

We consider the shifted momenta $\hat{p}_i^\mu = p_i^\mu + z r_i^\mu$ for any complex z. These momenta will have the properties:

1. Momentum conservation,

$$\sum_{i=1}^{n} \hat{p}_i^\mu = \sum_{i=1}^{n} p_i^\mu + z \sum_{i=1}^{n} r_i^\mu = 0 \tag{84}$$

   from eq. (83) and (i).

2. Masslessness,

$$\hat{p}_i^2 = p_i^2 + 2p_i \cdot r_i + r_i^2 = 0 \tag{85}$$

   from the masslessness of p_i, (ii) and (iii).

3. The invariant mass of any set I of the momenta p_i, with at least 2 momenta and at most $n-2$ momenta, will be linearly dependent on z. If we define $\hat{P}_I^\mu = \sum \hat{p}_i^\mu$, $P_I^\mu = \sum p_i^\mu$ and $R_I^\mu = \sum r_i^\mu$, where the sums are over $i \in I$, we have

$$\hat{P}_I^2 = \left(\sum_{i \in I} \hat{p}_i \right)^2 = P_I^2 + 2z P_I \cdot R I_+ z^2 R_I^2 = P_I^2 + z 2 P_I \cdot R_I, \tag{86}$$

   where the z^2 term vanishes due to (ii).

Now we can consider the tree-level color ordered amplitude A_n, as a complex function of z. For general momenta p_i (i.e. no internal propagator is on-shell) $A_n(z)$ is an analytic function, except for poles of the propagators for z such that eq. (86) goes on-shell. Our goal is to evaluate $A_n(0)$, which has real physical momenta. If we then consider $A_n(z)/z$, which has an additional pole at $z = 0$, with residue $A_n(0)$, Cauchy's residue theorem, eq. (54), can be used,

$$\frac{1}{2\pi i} \oint_C \frac{A_n(z)}{z} dz = A_n(0) + \sum_j \mathrm{Res}(z_j), \tag{87}$$

where we assume C to include $z = 0$, and the sum is over any other poles enclosed by C. Here, if $A_n(z)/z$ falls off fast enough[4] as z tends to infinity, the left-hand side vanishes if C is taken be a circle with its radius tending to infinity. For the purposes of this thesis the integrand falls off fast enough, and the left-hand side can hence be set to zero, as we will only consider pure gluon amplitudes with a specific shift (i.e. a specific choice of r_is). Shifts for which the integral vanishes are called valid or good shifts. If we consider the poles that

[4] As the length of the contour, C, grows as $|z|$, which is canceled by the factor $1/z$, $A_n(z) \to 0$ as $|z| \to \infty$ is sufficient for the left-hand side in eq. (87) to vanish.

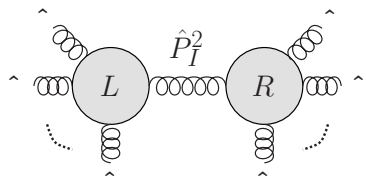
are summed over in eq. (87), they will occur for z such that eq. (86) goes on-shell. This happens when

$$z_I = -\frac{P_I^2}{2P_I \cdot R_I}, \tag{88}$$

and as we are interested in the residue (which can be found from the Laurent expansion, eq. (52) and eq. (53)), we can rewrite eq. (86) as

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I). \tag{89}$$

Now we consider the behavior of $A_n(z)/z$ close to one of these poles. Of the Feynman diagrams contributing to $A_n(z)$, the ones that dominate close to z_I are the ones with the propagator $\hat{P}_I^2$, i.e. any diagram of the form



, (90)

where the carets are to indicate that all of the legs are, possibly[5], shifted, and the external legs connecting to the blob labeled L are $i \in I$ and the external legs connecting to the blob labeled R are then all of the legs that are not in I. The total contribution to the pole at z_I, is then the sum of all diagrams of the form eq. (90). Hence, both the left and right blobs are tree-level amplitudes. The amplitude close to z_I is then

$$\frac{A_n(z)}{z} \underset{z \text{ near } z_I}{\to} \sum_{h_I} A_L(z_I)\frac{1}{z_I \hat{P}_I^2} A_R(z_I) = -\sum_{h_I} \frac{1}{z - z_I} A_L(z_I)\frac{1}{P_I^2} A_R(z_I) \tag{91}$$

where we have used eq. (89). In this form it is easy to read off the residue,

$$\text{Res}_{z_I} = -\sum_{h_I} A_L(z_I)\frac{1}{P_I^2} A_R(z_I). \tag{92}$$

Plugging this expression into eq. (87), defining B_n to be the left-hand side and rearranging, gives

$$A_n(0) = \sum_I \sum_{h_I} A_L(z_I)\frac{1}{P_I^2} A_R(z_I) + B_n, \tag{93}$$

where the sum is over all, so-called, factorization channels I (sets of at least 2 and at most $n-2$ momenta), and A_L and A_R are called subamplitudes. Note that the momentum arguments have been suppressed in eq. (93), but $A_n(0)$ have the unshifted momenta, while the subamplitudes have the shifted momenta.

[5]Some of the shift vectors, r_i, can be trivial, as conditions (i), (ii) and (ii) are still fulfilled.

1.6.1 BCFW recursion

BCFW, Britto-Cachazo-Feng-Witten, recursion relations [10, 11] are special cases of the above relation, where a particular choice of r_i has been made. When considering the shift vectors, r_i, some of them can be trivial, i.e. $r_i = 0$, and still fulfill (i), (ii) and (iii)[6]. For this shift, only two of the r_i will be non-trivial, for any two legs i and j. It is at this point that we will actually make use of the spinor helicity formalism. The shift is conveniently expressed in the angle and square spinors

$$\begin{aligned} |\hat{i}] &= |i] + z|j], \quad |\hat{j}] = |j], \\ |\hat{i}\rangle &= |i\rangle, \quad |\hat{j}\rangle = |j\rangle - z|i\rangle. \end{aligned} \tag{94}$$

To find the corresponding shifts of four-vectors we can use eq. (74),

$$\begin{aligned} \hat{p}_i^\mu &= \frac{1}{2}\langle\hat{i}|\gamma^\mu|\hat{i}] = \frac{1}{2}\langle i|\gamma^\mu|i] + z\frac{1}{2}\langle i|\gamma^\mu|j] \\ \hat{p}_j^\mu &= \frac{1}{2}\langle\hat{j}|\gamma^\mu|\hat{j}] = \frac{1}{2}\langle j|\gamma^\mu|j] - z\frac{1}{2}\langle i|\gamma^\mu|j], \end{aligned} \tag{95}$$

and read off r_i,

$$r_i^\mu = \frac{1}{2}\langle i|\gamma^\mu|j] = -r_j^\mu. \tag{96}$$

The shift should satisfy conditions (i), (ii) and (iii) of the previous section. We immediately see that (i), momentum conservation, is satisfied, and using the Fierz identity, eq. (73), mutual orthogonality and orthogonality to the momenta can be shown.

The recursion relation is still given by eq. (93), but the sum over factorization channels, I, only include terms where i and j are in different subamplitudes.

In [12] it is shown that if i and j are adjacent gluon lines, the BCFW shift is a valid shift for (i, j) having the helicities $(-,-)$, $(-,+)$, $(+,+)$ (but not $(+,-)$)[7]. If i and j are not adjacent, the amplitude falls off at large z with one additional power of $1/z$, hence the same shifts are valid in that case, but the $(+,-)$ shift is still not valid.

If we now consider a color ordered MHV amplitude, $A_n[1^-, 2^-, 3^+, ..., n^+]$, where we shift leg 1 and 2, eq. (93) becomes

$$\begin{aligned} A_n[1^-, 2^-, 3^+, ..., n^+] = \sum_{h=\pm}\sum_{i=3}^{n-1} A_L[\hat{2}^-, 3^+, ..., i^+, -\hat{P}_I^h]\frac{1}{P_I^2} \\ \times A_R[\hat{P}_I^{-h}, (i+1)^+, ..., n^+, \hat{1}^-]. \end{aligned} \tag{97}$$

[6]All r_i being trivial would not be an interesting shift, as $A_n(z)$ would then be a constant w.r.t. z, and B_n would not vanish, unless $A_n(0) = 0$.

[7]There is, of course, no $i \leftrightarrow j$ symmetry for which shifts are valid, since the BCFW shift is not symmetric in i and j, see eq. (94) and eq. (96).

For $h = +$, A_L will only contain one negative helicity leg, and for $h = -$, A_R will only contain one negative helicity leg. Hence, the only terms in the sum that survive will be $i = 3$ and $i = n - 1$, as the three parton amplitudes are special. These two divisions are called $(3, n-1)$ and $(n-1, 3)$, from the number of legs in the left and right subamplitudes. In appendix A of paper II there is a proof of why the $(3, n - 1)$ division vanishes. Then there is only one term left, and we end up with the, remarkably simple, BCFW recursion relation for MHV amplitudes,

$$A_n[1^-, 2^-, 3^+, ..., n^+] = A_L[\hat{2}^-, 3^+, ..., (n-1)^+, \hat{P}^-_{n1}] \frac{1}{P_I^2} A_R[-\hat{P}^+_{n1}, n^+, \hat{1}^-], \quad (98)$$

where $\hat{P}_{n1} = p_n + \hat{p}_1$.

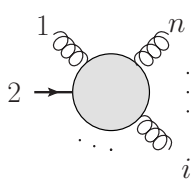
1.6.2 Summary

In this section we have seen the derivation of recursion relations, in particular the application of the spinor-helicity formalism to find the elegant recursion relation for MHV amplitudes, eq. (98). This is the recursion relation considered in paper II. The standard color bases used for recursion relations are the trace and DDM bases, that we saw in section 1.4. The advantage of them is that the recursion step is very simple, but squaring an amplitude is not due to non-orthogonality and over-completeness. In paper II, the viability of the multiplet basis is explored, where the recursion step becomes more complicated, but squaring becomes trivial due to orthogonality.

1.7 Parton showers

So far in this thesis, only the hard part of scattering cross sections, calculated using full amplitudes, has been considered. To motivate the need of parton showers we can consider QCD amplitudes. As we saw in section 1.6, amplitudes have singularities when propagators go on-shell. Let A_n be an amplitude for with some momenta, p_i^n,

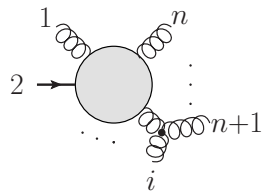
$$A_n(p_1, ..., p_n) = \quad , \quad (99)$$



which gets contributions from some set of Feynman diagrams. We consider the amplitude $A_{n+1}(p'_1, ..., p'_{n+1})$, where the momenta of p'_{n+1} is small compared to all other momenta p_i. The most important Feynman diagrams contributing to A_{n+1} will be when the soft gluon is attached to an external leg. This is because any internal leg will be far off-shell,

such that shifting its momentum by a small amount (by attaching the $(n+1)$th gluon to it) will have a small effect. The external legs, however, are on-shell, and attaching the $(n+1)$th gluon to them will give a propagator that is only slightly off-shell, and hence very large. So the most important contributions to the amplitude, are

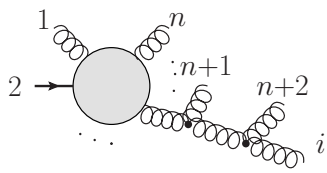
$$A_{n+1}(p'_1,...,p'_n,p'_{n+1}) \approx \sum_i$$



. (100)

The same thing occurs if p_{n+1} is collinear with some momentum p_i, rather than soft, the propagator would go on-shell in the collinear limit. The equation corresponding to eq. (100) would only have one term though, as it is only if the $(n+1)$th gluon is attached to the parton it is collinear with that a propagator would go on-shell. The naive assumption would be that if we had calculated A_n, then A_{n+1} would be a smaller contribution, as the vertex in eq. (100) comes with a coupling constant and is hence suppressed. But, as argued above, the propagator grows large, which will compensate for the smallness of the coupling constant. Hence, A_n will give a poor approximation of the cross section, as A_{n+1} is not negligible in some regions of phase space of the $(n+1)$th particle. But, this argument can be repeated, leading to us having to include an infinite number of emissions.

Next we consider two collinear emissions from the parton i, the emission of the $(n+1)$th and the $(n+2)$th gluon. As the emissions should be collinear, the virtuality of the propagators, $Q^2_{n+1} = 2p_i \cdot p_{n+1}$ and $Q^2_{n+2} = 2p_i \cdot p_{n+2}$, should be small, and we consider which contribution to the amplitude will be largest. The relevant Feynman diagram is

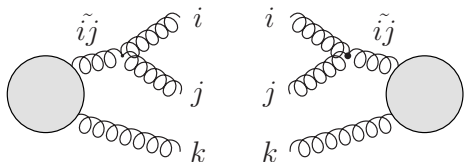


, (101)

where the left propagator (from eq. (5)) will get the factor $\frac{1}{(p_i+p_{n+1}+p_{n+2})^2}$ and the right propagator gets the factor $\frac{1}{(p_i+p_{n+2})^2}$. If $Q^2_{n+1} \gg Q^2_{n+2}$, p_{n+2} in the left propagator can be neglected, and the propagator is $\sim 1/p_i \cdot p_{n+1}$ (since the external partons are massless, i.e. $p^2 = 0$). The right propagator is, similarly $\sim 1/p_i \cdot p_{n+2}$. Hence, the first propagator is a large number, and the second propagator is an even larger number. For the other order, $Q^2_{n+1} \ll Q^2_{n+2}$, both propagators are $\sim 1/p_i \cdot p_{n+2}$, as p_{n+1} can be neglected for the first propagator. So, for this order the amplitude will not be enhanced as much as for the first order, which then means that emissions tend to be ordered. This ordering of the emissions is in their virtuality, but there is a freedom in the choice of ordering variable, as long as

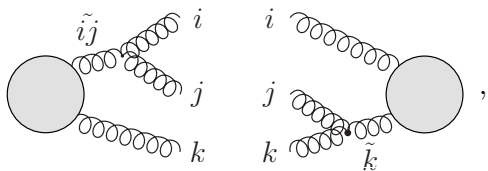
the ordering variables agree in the collinear limit. Common choices are virtuality of the propagator (as we did here), angle of the emission or transverse momentum. The latter two choices are more common, as they preserve the coherence of QCD better, which is the destructive interference of emissions reducing the total emission rate.

So far we have seen that certain amplitudes will become important in the soft and collinear limits. The cross section is proportional to the amplitude square, so just looking at amplitudes is not sufficient. When considering the cross section, $\sigma_{n+1} \propto |A_{n+1}|^2$, in the soft limit, the most important terms will be, so-called, self-energy diagrams



(102)

and interference diagrams



, (103)

where the other legs are not shown. It turns out that in the soft and collinear limits, the cross section factorizes into a so-called splitting kernel and a cross section with fewer particles,

$$d\sigma_{n+1} \propto dz \frac{dQ^2}{Q^2} P_{gg \leftarrow g}(z) \sigma_n. \tag{104}$$

The parton shower then treats the splitting kernels $P_{ij \leftarrow k}$ as probabilities of going from a state with n partons to a state with $n+1$ partons. The calculation of the splitting kernels can be found in [13].

The emissions are thus ordered in some variable (virtuality, angle, transverse momentum), which we now call q. The splitting kernel is interpreted as a probability of an emission occurring. To find the q-value of the next emission, and the additional splitting variables, x, (e.g. energy fraction z and azimuthal angle, ϕ), the probability distribution

$$dS_P(\mu|q,x|Q) = dq d^d x \left[\Delta_P(\mu|Q)\delta(q-\mu) + P(q,x)\Delta_P(q|Q)\right], \tag{105}$$

is used, where $P(q,x)$ is the splitting kernel (which we allow to depend on the ordering variable), μ is a cutoff scale (at which point hadronization kicks in and the parton shower should stop) and

$$\Delta_P(q|Q) = \exp\left(-\int_q^Q dk \int d^d z P(k,z)\right). \tag{106}$$

The interpretation of this is that $\Delta_P(q|Q)$ is the no emission probability, i.e. the probability that no emission occurred for any splitting variables z at any value k of the ordering variable between the q-values, Q and q.

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Scientific publications

Author contributions

Paper I: Decomposing color structure into multiplet bases

M. Sjödahl and **J. Thorén**, *JHEP* **09** (2015) 055 [1507.03814].

This paper builds on the work initiated in my master thesis *Decomposing colour structures into multiplet bases* (supervised by Malin Sjödahl). In the paper we investigate the method of using Wigner $3j$ and $6j$ coefficients for decomposing group invariants in the context of QCD. The result of the paper was a recipe for decomposing color structures into multiplet bases, using a comparatively small set of Wigner $6j$ coefficients. I calculated this set of Wigner coefficients for NLO amplitudes with up to 6 external gluons. For the writing of the manuscript, I wrote a draft for sections 3 and 4 and appendices A and C. Malin Sjödahl wrote the remaining sections and heavily edited my draft. The calculated Wigner coefficients were checked by both me and Malin Sjödahl.

Paper II: Recursion in multiplet bases for tree-level MHV gluon amplitudes

Y.-J. Du, M. Sjödahl and **J. Thorén**, *JHEP* **1505** (2015) 119 [1503.00530].

In this paper we investigate the viability of amplitude recursion in the multiplet basis. This was done by comparing the number of terms one would encounter in the "bottleneck" for the basis in question: the squaring of the amplitude for traditional bases and the recursion step for the multiplet basis. Yi-Jian Du performed the recursion in the trace basis and worked out the kinematical recursion for the paper. I worked out the details of the recursion for the color structure in the multiplet basis. The counting of the number of terms required for the multiplet, trace and DDM bases depending on the number of external gluons was done by both me and Malin Sjödahl. I wrote a draft of section 3.3, this was then edited by Malin Sjödahl, or by me with suggestions from Malin Sjödahl.

Paper III: Color matrix element corrections for parton showers

S. Plätzer, M. Sjödahl and **J. Thorén**, [1808.00332], submitted to JHEP.

This paper builds on previous work by Malin Sjödahl and Simon Plätzer, where they implemented a full color shower in a LEP context. The goal was to include color suppressed dipole emissions for realistic events in the event generator Herwig, for any process, in particular for LHC events. For this project, I implemented the shower in Herwig with help from Simon Plätzer, modified the Rivet analyses and did most of the runs. As the subleading color dipoles can have negative splitting kernels, a weighted veto algorithm was required. Initially we had convergence issues due to large and negative weights. I found a solution to this, described in section 5 and appendix A of the paper. In terms of the writing of the manuscript, I wrote section 5, with helpful comments from Malin Sjödahl, and a small part of sections 7 and 8.

Paper IV: QCD multiplet bases with arbitrary parton ordering

M. Sjödahl and **J. Thorén**, [1809.05002], submitted to JHEP.

The idea for this paper, applying similar ideas as Malin Sjödahl and Stefan Keppeler did in their paper, was conceived by me after suggestions from Malin Sjödahl. This paper builds upon paper I, but generalizes the construction of the basis vectors, allowing for more freedom in the choice of multiplet bases. This freedom of choice can be utilized to choose more suitable multiplet bases for applications of the multiplet bases, for example in recursion relations with fermions or in parton showers. The paper gives a recipe for constructing projectors for multiplet bases with any parton ordering. These can then be used to construct multiplet basis vectors and Wigner $6j$ coefficients. I implemented the method we describe in the paper, and calculated the projectors, and from them I calculated basis vectors and Wigner $6j$ coefficients. The coefficients have been thoroughly tested by both me and Malin Sjödahl. I have written the draft of the paper.