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# A COUPLING SCHEME RELEVANT TO HIGH ANGULAR MOMENTA
# AND INTERMEDIATE NUCLEAR DEFORMATIONS*

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Based on the particle-plus-rotor model, it is shown that a new coupling scheme should occur in nuclei under certain conditions. In this coupling scheme, $j$ is quantized along the direction of $I$ rather than along the symmetry axis. Simple approximate wave functions for this scheme have been developed and compared with the exact solutions of the particle-plus-rotor model.

With decreasing distortion and increasing spin the effects of Coriolis and centrifugal couplings on the nuclear orbital increase until in a transitional region an entirely new coupling scheme develops in which the projection of $j$ on the rotation axis is approximately a good quantum number of the system [1]. This new type of coupling become valid first for particular states of high spin. In these cases the odd single particle follows the rotating deformed core only partially, and orients its orbital plane in such a way as to essentially minimize the Coriolis energy. This is achieved for these states when the orbital plane nearly coincides with the plane of rotation of the nucleus.

To study the situation we consider the rotor-plus-particle Hamiltonian [2]

$$H = A[I^2 + j^2 - I_3^2 - j_3^2] + H_C' + H_{sp} = H_o + H_C' \qquad (1)$$

where

$$H_{sp} = H_{sph} + k\beta Y_{20}(\hat{3}) \qquad (2)$$

and

$$H_C' = -2A(I_1 j_1 + I_2 j_2) = -A(I_+ j_- + I_- j_+) . \qquad (3)$$

The single-particle Hamiltonian is thus associated with a quadrupole field oriented along the core symmetry axis $\hat{3}$. The term $H_C'$ is conventionally called the Coriolis coupling term although it contains parts of both Coriolis and centrifugal energies. Consider first the normal adiabatic wave function with the single-

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particle space limit to one $j$ subshell, which is a satisfactory approximation for the unique-parity case considered here,

$$\psi_{M\Omega}^{Ij} = \chi_\Omega^j D_{M\Omega}^I . \qquad (4)$$

The diagonal energies of $H_o$ in this representation are

$$\langle H_o \rangle = e_j' + A[I(I+1) + j(j+1)] + (K - 2A)\Omega^2 \qquad (5)$$

where the diagonal value of $H_{sp}$ has been obtained as

$$\langle H_{sp} \rangle = e_j + k\beta \left( \frac{3\Omega^2 - j(j+1)}{4j(j+1)} \right) = e_j' + K\Omega^2 . \qquad (6)$$

The neglected and mainly non-diagonal $H_C'$ becomes a very large terms for the small deformations and large rotational frequencies here considered.

The exact solutions of eq. (1) in the truncated $j$-space are of the following type

$$\psi = \sum_\Omega C_\Omega \chi_\Omega^j D_{M\Omega}^I \qquad (7)$$

and can be obtained for the purpose of comparison from exact diagonalization. Mottelson [3] has recently suggested a way to obtain approximate, but analytically calculable, $C_\Omega$. These coefficients are obtained as values of a harmonic oscillator wave function in a space where $\Omega$ is considered a continuous coordinate. The corresponding approximate solutions are surprisingly accurate over a considerable region of deformation. For smaller distortions than those considered here, the scheme developed by Vogel [4] is designed to be applicable. For a further development along this line see Kleinheinz et al., [5].

In the present investigation we propose a different method to obtain $C_\Omega$. It is apparent from eq. (5) that

a particularly favorable region of degeneracy occurs when $K = 2A$, in which case $\langle H_0 \rangle$ is independent of $\Omega$. We can exploit this degeneracy by considering linear combinations of eq. (4) that approximately diagonalize $H'_C$. Let $\chi^j_\alpha$ be an intrinsic wave function so defined that the projection of $j$ on the nuclear $\hat{1}$-axis, is equal to $\alpha$. This wave function is decomposed in terms of intrinsic basis states having $j_3 = \Omega$;

$$\chi^j_\alpha = \sum_\Omega C_\Omega(\alpha) \chi^j_\Omega \tag{8}$$

where

$$C_\Omega(\alpha) = D^j_{\alpha\Omega}(0, \pi/2, 0) = d^j_{\alpha\Omega}(\pi/2) . \tag{9}$$

From this, a new wave function describing the entire system is constructed:

$$\psi^{Ij}_{M\alpha} = \sum_\Omega C_\Omega(\alpha) \chi^j_\Omega D^I_{M\Omega} . \tag{10}$$

The restriction of reflection symmetry in the plane perpendicular to the $\hat{3}$-axis leads to a condition of compatibility between $I, j$ and $\alpha$ such that $I - \alpha$ must be even. This type of wave function insures that $R_3 = 0$ is fulfilled identically and $\alpha$ becomes the projection of $j$ on $I$ rather than on the $\hat{1}$-axis. Observing that

$$I_- D^I_{M\Omega} = \sqrt{I(I+1) - \Omega(\Omega+1)} \, D^I_{M\Omega+1}$$
$$= \sqrt{I(I+1)} \left(1 - \tfrac{1}{2} \frac{\Omega(\Omega+1)}{I(I+1)} \cdots\right) D^I_{M\Omega+1} \tag{11}$$

and retaining only the leading term, one can show that eq. (10) behaves as an approximate eigenfunction of $H'_C$

$$H'_C \psi^{Ij}_{M\alpha} \approx - 2AI\alpha \, \psi^{Ij}_{M\alpha} \tag{12}$$

where, in addition, we have replaced $\sqrt{I(I+1)}$ with $I$. Eq. (11) is particularly accurate for $I > j$, in which case generally $(\Omega/I)^2 \ll 1$ since $|\Omega| \leqslant j$. Note that no similar approximation is required of $\Omega$ relative to $j$.

For the case $K = 2A$ in eq. (5), one thus immediately obtains the following energy spectrum in terms of $I, j$, and $\alpha$;

$$E(I, j, \alpha) = e'_j + A [j(j+1) + I(I+1) - 2I\alpha] . \tag{13}$$

The lowest-lying high-spin states have $\alpha = j$ and the associated band has $I$-values compatible with $I - j$ being even. This $\alpha = j$ band has been called a "de-

coupled" band [1] and its energy experssion may be rewritten

$$E = \text{const} + A(I - j)(I - j + 1) = \text{const} + AR'(R' + 1) \tag{14}$$

where $R' = 0, 2, 4 \ldots$ , reproducing an even-even rotational spectrum as has been found in recent experiments [1].

The above discussion has been restricted to $I > j$. A completely analogous treatment can be carried out for $I < j$, since $H'_C$ exhibits complete symmetry in $I$ and $j$. Starting with $D^I_{M\kappa}$, which is an eigenfunction of $I_1$ with the eigenvalue $\kappa$, one may make a decomposition along the nuclear 3-axis as

$$D^I_{M\kappa} = \sum_\Omega b_\Omega(\kappa) D^I_{M\Omega} \tag{15}$$

where

$$b_\Omega(\kappa) = d^I_{\kappa\Omega}(\pi/2) . \tag{16}$$

In complete analogy with eq. (10) one obtains

$$\psi^{Ij}_{M\kappa} = \sum_\Omega b_\Omega(\kappa) \chi^j_\Omega D^I_{M\Omega} . \tag{17}$$

Since $\Omega \leqslant I < j$ we can use the approximation $\sqrt{j(j+1) - \Omega(\Omega+1)} \approx j$, obtaining

$$H'_C \psi^{Ij}_{M\kappa} \approx - 2Aj\kappa \, \psi^{Ij}_{M\kappa} \tag{18}$$

which leads to a diagonal energy

$$E(I, j, \kappa) = e'_j + A [j(j+1) + I(I+1) - 2j\kappa] . \tag{19}$$

The symmetry condition requires $j - \kappa$ to be even, restricting the permissible $\kappa$-values of eq. (19).

Eqs. (13) and (19) correspond very closely to the diagonalization of eq. (1) under the condition that $K = 2A$ in eq. (5). For the one-particle case considered in fig. 1, with $A = 130$ and $j = 11/2$, this cancellation is found to occur at $\beta = +0.18$. To find the region of $\beta$ over which our approximations apply, we have evaluated numerically the diagonal part of the Hamiltonian (1) with respect to the wave functions of eqs. (10) and (17) for a range of $\beta$ values around $K = 2A$ using the dependence of $K$ and $A$ on $\beta$ as given in ref. [1]. For a few of the low-lying states, fig. 1 contains a comparison of these diagonal energies of $H$ with the exact ones. The large region of applicability in $\beta$ reflects the fact that the value of $K - 2A$ in eq. (5) remains negligibly small for a considerable range of $\beta$ around the point of exact cancellation.
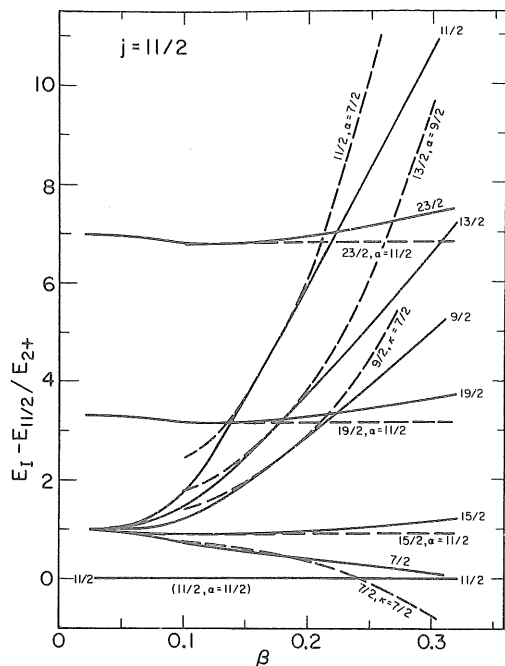
Fig. 1. The exact solutions to eq. (1) (solid lines) are compared with the diagonal energies for the wave functions given by eqs. (10) and (17) (dashed lines). The ordinate is the difference in energy between the state having spin $I$ and the lowest $I = 11/2$ state for the respective type of solution, in units of the even-even $I = 2^+$ energy. The exact solutions shown are for a one-particle case (the Fermi surface is always below the entire $j$-shell), pure $j = 11/2$ wave functions, and no pairing correlations; however, the results are not very sensitive to any of these conditions.

Also, inclusion of the next higher-order term neglected in eq. (11) leads effectively only to a small displacement of the cancellation point in $\beta$.

To illustrate further the region of validity of the present scheme we have chosen to compare in fig. 2 the diagonal energies of eq. (1) for three coupling schemes, namely the "deformation-aligned" or "strong-coupling" scheme appropriate for large $\beta$, the "non-aligned" or "weak-coupling" scheme appropriate near $\beta = 0$ (cf. ref. [4]), and the present "rotation-aligned" coupling scheme. In this figure the state $I = j = 11/2$ is shown for a case with one particle in the $h_{11/2}$ orbital. We have plotted the energy for pure configurations having $R = 0$, $\Omega = 1/2$ and $11/2$, and $\alpha = 1/2$ and $11/2$. For $|\beta| \lesssim 0.1$, the weak coupling $R = 0$ configura-
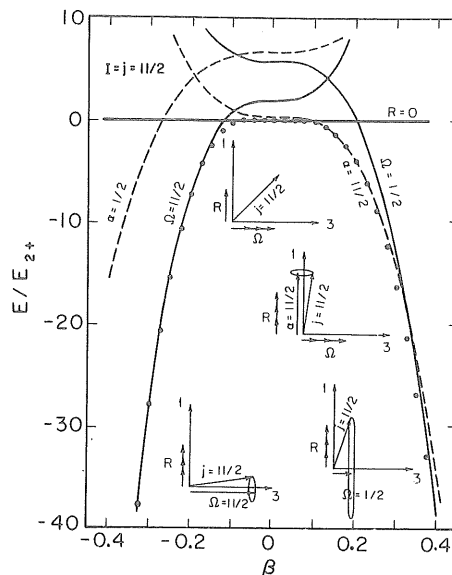


Fig. 2. The energy of the lowest $I = 11/2$ state (in units of the even-even $I = 2^+$ energy) is shown for three different coupling schemes (lines) and for the exact diagonalization of eq. (1) (dots). The conditions are the same as those for fig. 1. The inserts show schematically vector model schemes for the lowest-lying configurations, where the 1-axis is taken to be the rotation axis, and the multiple-pronged arrows indicate mixtures of states.

tion lies lowest. On the oblate side the $\Omega = 11/2$ and $\alpha = 1/2$ states cross below the $R = 0$ state, but $\Omega = 11/2$ always lies well below $\alpha = 1/2$, resulting in nearly normal rotational bands for $\beta < -0.1$. On the prolate side $\Omega = 1/2$ and $\alpha = 11/2$ cross below $R = 0$, and $\alpha = 11/2$ is lowest for $0.1 \leqslant \beta \leqslant 0.3$, and thereafter $\Omega = 1/2$ becomes lowest. The exact energies of $H$ are shown as dots in this figure. The close correspondence of these dots with the lowest lines in fig. 2 indicates the adequacy with which the three coupling schemes represent the exact solution of $H$ in eq. (1). The rotation-aligned coupling scheme here proposed is seen to have rather broad region of applicability connecting the weak-coupling region with that of the deformation-aligned coupling scheme on the side where low-$\Omega$ orbitals occur near the Fermi surface.

It has been shown that a new coupling scheme should occur in nuclei when the deformation is around $+0.2$ $(-0.2)$ and there is one, or a few, particles (holes) in a high-$j$ shell. This is seen to correspond to a situation where $j$ is quantized along the rotation

axis rather than the deformation axis, and simple wave functions have been proposed for such a system.

## References

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