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Optimal Control of Markov Processes with Incomplete State-Information II. The Convexity of the Lossfunction*

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1. INTRODUCTION

A nonlinear adaptive control problem was discussed in [1]. It was shown that by quantization of time and state space the problem could be reduced to a variational problem for a Markov chain with incomplete state-information. To solve the variational problem we introduced a *hyperstate* or an *information state* consisting of a vector $w(t)$ such that $w_i(t)$ is the conditional probability that the Markov process is in state i given all measured variables up to time t. We choose the state space S as the subset of R_n defined by

$$S = \{x;\, x_i \geqslant 0\}.$$

After introducing the lossfunction $V : S \to R_1$, it was shown in [1] that the variational problem could be reduced to the solution of the following functional equation

$$V_t(w) = \max_u \left\{ (g, w) + \sum_j \| A_j w \| \, V_{t+1} \left(\frac{A_j w}{\| A_j w \|} \right) \right\}$$

$$V_N(w) = \max_u (g, w), \qquad w \in S \tag{1.1}$$

where A_j is the linear transformation defined by

$$(A_j w)_i = \sum_s q_{ij} p_{si} w_s$$

$$\| x \| = \sum | x_i | . \tag{1.2}$$

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The notation is that of [1]. The i:th component of the vector g denotes the instantaneous gain achieved by being in state i at time t and choosing the control variable u. P is the transition matrix of the Markov chain and Q is the observation matrix defined in [1], q_{ij} thus denotes the probability that the measuring equipment denotes the process as being in the jth state when it actually is in state i. The matrices P and Q as well as the vector g depend on u and t. As P and Q are probability matrices having nonnegative elements it follows that A_j maps S into S. In [1] it was assumed that Q does not depend on u. All results of [1] will, however, also be valid when Q depends on u. Equation (1.1) admits an analytical solution only in very specific cases. The equation can, however, always be solved numerically. In the example 1 of [1] we found that V was convex in w and in example 2 of [1] where the maximum operation of (1.1) was substituted by a minimum operation, we found that V was concave in w. In this paper we will establish that this observation is true in general. Apart from being an amusing curiosity the result is useful for establishing convergence properties as well as for the simplification of numerical algorithms.

2. Main Result

Before giving the main theorem we will establish some simple properties of convex functions. We have

Lemma 1. *Let $f_1(x)$ and $f_2(x)$ be convex functions. The function*

$$f(x) = \max\{f_1(x), f_2(x)\} \tag{2.1}$$

is then also convex.

Proof. We first show that

$$\max\{a + b, c + d\} \leqslant \max\{a, c\} + \max\{b, d\}. \tag{2.2}$$

Consider four separate cases:

1. If $a > c$ and $b > d$, the right member becomes $a + b$. Further $a + b > c + d$ and the result holds.

2. If $a > c$ and $b \leqslant d$, the right member becomes $a + d$. Further $a + b \leqslant a + d$ and $c + d < a + d$.

3. If $a \leqslant c$ and $b > d$, the right member becomes $b + c$. Further $a + b \leqslant b + c$ and $c + d < b + c$.

4. If $a \leqslant c$ and $b \leqslant d$, the right member becomes $c + d$. Further $a + b \leqslant c + d$ and the result also holds.

Now let $0 \leqslant \lambda \leqslant 1$ and $\mu = 1 - \lambda$. Consider the value of the function f defined by (2.1) for the argument $\lambda x + \mu y$. We have

$$\begin{aligned} f(\lambda x + \mu y) &= \max\{f_1(\lambda x + \mu y), f_2(\lambda x + \mu y)\} \\ &\leqslant \max\{\lambda f_1(x) + \mu f_1(y), \lambda f_2(x) + \mu f_2(y)\} \\ &\leqslant \max\{\lambda f_1(x), \lambda f_2(x)\} + \max\{\mu f_1(y), \mu f_2(y)\} \\ &= \lambda \max\{f_1(x), f_2(x)\} + \mu \max\{f_1(x), f_2(x)\} \\ &= \lambda f(x) + \mu f(y) \end{aligned} \tag{2.3}$$

where the first inequality follows from f_1 and f_2 being convex, the second from equation (2.2) and the last two equalities from λ and μ being nonnegative and equation (2.1). The result is then established.

We have further

LEMMA 2. *Let the function $g : S \to R_1$ be convex and let A be a linear transformation which maps S into S. The function $f : S \to R_1$ defined by*

$$f(x) = \| Ax \| \cdot g\left(\frac{Ax}{\| Ax \|}\right), \qquad x \in S \qquad \| Ax \| \neq 0 \tag{2.4}$$

is then also convex.

PROOF. Let $0 \leqslant \lambda \leqslant 1$ and $\mu = 1 - \lambda$, take $x \in S$ and $y \in S$, then

$$\begin{aligned} f(\lambda x + \mu y) &= \| \lambda Ax + \mu Ay \| \, g\left(\frac{\lambda Ax + \mu Ay}{\| \lambda Ax + \mu Ay \|}\right) \\ &= \| \lambda Ax + \mu Ay \| \, g\left(\lambda_1 \frac{Ax}{\| Ax \|} + \mu_1 \frac{Ay}{\| Ay \|}\right) \end{aligned} \tag{2.5}$$

where

$$\lambda_1 = \frac{\lambda \| Ax \|}{\| \lambda Ax + \mu Ay \|}, \qquad \mu_1 = \frac{\mu \| Ay \|}{\| \lambda Ax + \mu Ay \|}. \tag{2.6}$$

As λ and μ are nonnegative and A maps S into S, $\lambda Ax \in S$, $\lambda Ay \in S$, $\lambda Ax + \mu Ay \in S$. For two elements of u and v of S we have

$$\| u + v \| = \| u \| + \| v \|$$

hence

$$\lambda_1 + \mu_1 = 1.$$

Now using the convexity of g we find

$$g\left(\lambda_1 \frac{Ax}{\| Ax \|} + \mu_1 \frac{Ay}{\| Ay \|}\right) \leqslant \lambda_1 g\left(\frac{Ax}{\| Ax \|}\right) + \mu_1 g\left(\frac{Ay}{\| Ay \|}\right). \tag{2.7}$$

Combining (2.5), (2.6) and (2.7) we find

$$f(\lambda x + \mu y) \leqslant \lambda \| Ax \| \cdot g\left(\frac{Ax}{\| Ax \|}\right) + \mu \| Ay \| \cdot g\left(\frac{Ay}{\| Ay \|}\right)$$
$$= \lambda f(x) + \mu f(y)$$

and the result is established.

We can now state the main result.

THEOREM. *Let A_j be mappings from S into S the functions $V_t : S \to R_1$ defined recursively by* (1.1) *are then convex.*

PROOF. The linear function (g, w) is convex. By repeated application of Lemma 1 we now find that $V_N(w)$ is convex. Now consider $V_{N-1}(w)$. It follows from Lemma 2 that $V_N(A_j w/\| A_j w \|) \| A_j w \|$ is convex. As a sum of convex functions is convex we find that both terms within the brackets of the right member of (1.1) are convex. Application of Lemma 1 now shows that $V_{N-1}(w)$ is convex. Now proceeding by induction we can show that all functions $V_t(w)$ are convex, and the theorem is proved.

REMARK. We can show in a completely analogous way that the solutions $V_t(w)$ of the equation

$$V_{t-1}(w) = \min_u \left\{(g, w) + \sum_j \| A_j w \| \, V_t\left(\frac{A_j w}{\| A_j w \|}\right)\right\}$$

$$V_N(w) = \min_u (g, w)$$

are concave. Compare [1] Fig. 2.

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