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RECURSIVE FORMULAS FOR THE EVALUATION OF CERTAIN COMPLEX INTEGRALS*

By K. J. ÅSTRÖM (*Brown University*)

Abstract. This paper presents recursive formulas which admit simultaneous tests of stability and evaluation of quadratic loss functions for linear discrete-time dynamical systems. The method admits a significant reduction of the number of computations in comparison with previously known methods.

1. Introduction. We shall consider the evaluation of integrals of the type

$$I = \frac{1}{2\pi i} \oint \frac{B(z)B(z^{-1})}{A(z)A(z^{-1})} \cdot \frac{dz}{z} \tag{1}$$

where A and B are polynomials with real coefficients

$$A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \tag{2}$$

$$B(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n \tag{3}$$

and $\oint$ denotes the integral along the unit circle in the positive direction.

Integrals such as (1) occur in many control and communication problems. The sum of squares of the values of the impulse response of a dynamical system with the pulse transfer function $B(z)/A(z)$ is e.g. given by (1). Evaluation of quadratic loss-functions, generation of quadratic Lyapunov functions for linear systems and investigation of the accuracy of parameter estimation in linear systems also lead to integrals of type (1) (see, e.g., [1] and [3]).

Closed form solutions of (1) for polynomials of low order are available in the literature, (see, e.g., Jury [3, p. 298–299]). For large n, say $n \geq 4$, the closed form solutions are, however, very cumbersome to use. It is also well known that $a_0 I$, where I is the integral defined by (1), can be obtained as the first component of the vector x which satisfies the following linear equation:

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$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & a_0 + a_2 & a_1 + a_3 & a_2 + a_4 & \cdots & a_{n-1} \\ a_2 & a_3 & a_0 + a_4 & a_1 + a_5 & \cdots & a_{n-2} \\ \vdots & & & & & \\ a_{n-1} & a_n & 0 & 0 & \cdots & a_1 \\ a_n & 0 & 0 & 0 & \cdots & a_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{n} b_i^2 \\ 2 \sum_{i=0}^{n-1} b_i b_{i+1} \\ 2 \sum_{i=0}^{n-2} b_i b_{i+2} \\ \vdots \\ 2 \sum_{i=0}^{1} b_i b_{i+n-1} \\ 2 b_0 b_n \end{bmatrix}$$

(see, e.g., [3, p. 168–172]). Some simplifications which can be used if x_1 is computed through the evaluation of determinants are discussed in [4].

In this paper we present recursive equations for the integral which lead to considerably fewer computations than a direct solution of the linear system. The recursive equations are derived using elementary results from the theory of analytic functions. The recursive equations can be conveniently used both for hand and machine computation. Analogous results for continuous-time systems have been obtained by Nekolný and Beneš [5].

2. Preliminaries and notations. We first observe that the integral (1) will always exist if the polynomial $A(z)$ has all its zeros inside the unit circle. In such a case we can always find a stable dynamical system with the pulse transfer function $B(z)/A(z)$, and the integral (1) is then simply the sum of squares of the ordinates of the impulse response of the system.

If $A(z)$ has zeros on the unit circle the integral diverges. If $A(z)$ has zeros both inside and outside the unit circle but not on the unit circle, the integral (1) still exists. In such a case we can always find a polynomial $A'(z)$ with all its zeros inside the unit circle such that

$$A(z)A(z^{-1}) = A'(z)A'(z^{-1})$$

and the integral then represents the sum of squares of the impulse response of a stable dynamical system whose pulse transfer function is $B(z)/A'(z)$.

In many practical cases, however, we obtain the integral as a result of an analysis of a dynamical system whose pulse transfer function is $B(z)/A(z)$. In such a case it is naturally of great importance to test that $A(z)$ has all its zeros inside the unit circle because when this is not the case the dynamical system will be unstable although the integral (1) exists.

In order to present the result in a simple form we will first introduce some notation. Let A^* denote the polynomial defined by

$$A^*(z) = z^n A(z^{-1}) = a_0 + a_1 z + \cdots + a_n z^n. \tag{4}$$

Further introduce the polynomials

$$A_k(z) = a_0^k z^k + a_1^k z^{k-1} + \cdots + a_k^k, \tag{5}$$

$$B_k(z) = b_0^k z^k + b_1^k z^{k-1} + \cdots + b_k^k, \tag{6}$$

which are defined recursively by

$$A_{k-1}(z) = z^{-1}\{A_k(z) - \alpha_k A_k^*(z)\}, \tag{7}$$

$$B_{k-1}(z) = z^{-1}\{B_k(z) - \beta_k A_k^*(z)\}, \tag{8}$$

where

$$\alpha_k = a_k^k/a_0^k \tag{9}$$

$$\beta_k = b_k^k/a_0^k \tag{10}$$

and

$$A_n(z) = A(z), \tag{11}$$

$$B_n(z) = B(z). \tag{12}$$

If these equations have a meaning we must naturally require that all a_0^k are different from zero. To see the implications of this we will make use of the following theorem.

THEOREM 1. *The polynomial $A(z)$ has all its zeros inside the unit circle if and only if $a_0^k > 0$ for all k.*

This theorem is essentially the Schur–Cohn stability criterion for linear discrete-time dynamical systems (see, e.g., [2], [3, p. 126], [7]). We also have the following result which will be used in the proof of our main result.

THEOREM 2. *Let the polynomial $A_k(z)$ have all its zeros inside the unit circle; then $A_{k-1}(z)$ also has all its zeros inside the unit circle.*

This theorem is also given by Schur [7]. A simple proof is given by Růžička [6].

We thus find that the polynomials $A_k(z)$ can always be defined if the original polynomial has all its zeros inside the unit circle. If this is not the case we will always get $a_0^k \leq 0$ at some step in the reduction. The equations (7) and (8) can thus be profitably exploited as a stability criterion.

3. The main result. We will now show that the integral (1) can be computed recursively. For this purpose we introduce the integrals I_k defined by

$$I_k = \frac{1}{2\pi i}\oint \frac{B_k(z)B_k(z^{-1})}{A_k(z)A_k(z^{-1})}\cdot\frac{dz}{z}. \tag{13}$$

It follows from (1) that $I = I_n$. We now have

THEOREM 3. *Let the polynomial $A(z)$ have all its zeros inside the unit circle. The integrals I_k defined by* (13) *then satisfy the following recursive equations*:

$$\{1 - \alpha_k^2\}I_{k-1} = I_k - \beta_k^2, \tag{14}$$

$$I_0 = \beta_0^2. \tag{15}$$

Proof. As $A(z)$ has all its zeros inside the unit circle, it follows from Theorem 1 that all a_0^k's are different from zero. It thus follows from (9) and (10) that all polynomials A_k's and B_k's can be defined. Furthermore it follows from Theorem 2 that all polynomials A_k's have all zeros inside the unit circle. All integrals I_k thus exist.

To prove the theorem we will make use of the theory of analytic functions. The integral (13) equals the sum of the residues at the poles of the function $B_k(z)B_k(z^{-1})/$

$\{zA_k(z)A_k(z^{-1})\}$ inside the unit circle. As the integral is invariant under the change of variables $z \to 1/z$, we also find that the integral equals the sum of the residues of the poles outside the unit circle. We will first assume that the coefficients of A are such that its zeros are distinct and different from zero. Now consider

$$I_{k-1} = \frac{1}{2\pi i} \oint \frac{B_{k-1}(z)B_{k-1}(z^{-1})}{A_{k-1}(z)A_{k-1}(z^{-1})} \cdot \frac{dz}{z}.$$

The poles of the integrand inside the unit circle are $z = 0$ and the zeros z_i of the polynomial $A_{k-1}(z)$. It follows from (7) and (4) that

$$A_k(z_i) = \alpha_k A_k^*(z_i) = \alpha_k z_i^k A_k(z_i^{-1}),$$
$$A_{k-1}(z_i^{-1}) = z_i\{A_k(z_i^{-1}) - \alpha_k A_k^*(z_i^{-1})\}.$$

Hence

$$A_{k-1}(z_i^{-1}) = z_i\{A_k(z_i^{-1}) - \alpha_k z_i^{-k} A_k(z_i)\}$$
$$= (1 - \alpha_k^2) z_i A_k(z_i^{-1}).$$

Furthermore it follows from (4) and (7) that

$$A_{k-1}^*(z) = A_k^*(z) - \alpha_k A_k(z).$$

Hence

$$A_{k-1}^*(0) = A_k^*(0) - \alpha_k A_k(0) = a_0^k - \alpha_k a_k^k = a_0^k(1 - \alpha_k^2).$$

The functions

$$\frac{B_{k-1}(z)B_{k-1}(z^{-1})}{A_{k-1}(z)A_{k-1}(z^{-1})} \cdot \frac{1}{z} = \frac{B_{k-1}(z)B_{k-1}^*(z)}{A_{k-1}(z)A_{k-1}^*(z)} \cdot \frac{1}{z}$$
$$\frac{B_{k-1}(z)B_{k-1}(z^{-1})}{A_{k-1}(z)\{z(1 - \alpha_k^2)A_k(z^{-1})\}} \cdot \frac{1}{z} = \frac{B_{k-1}(z)B_{k-1}^*(z)}{A_{k-1}(z)\{(1 - \alpha_k^2)A_k^*(z)\}} \cdot \frac{1}{z}$$

have the same simple poles inside the unit circle and the same residues at these poles. Hence

$$I_{k-1} = \frac{1}{1 - \alpha_k^2} \cdot \frac{1}{2\pi i} \oint \frac{B_{k-1}(z)B_{k-1}(z^{-1})}{A_{k-1}(z)A_k(z^{-1})} \frac{dz}{z^2}$$
$$= \frac{1}{1 - \alpha_k^2} \cdot \frac{1}{2\pi i} \oint \frac{B_{k-1}(z)B_{k-1}(z^{-1})}{A_k(z)A_{k-1}(z^{-1})} dz \tag{16}$$

where the second equality is obtained by making the variable substitution $z \to z^{-1}$. The integrand has poles at the zeros of $A_k(z)$. It follows, however, from (7) that

$$A_{k-1}(z^{-1}) = z\{A_k(z^{-1}) - \alpha_k A_k^*(z^{-1})\}$$
$$= z\{A_k(z^{-1}) - \alpha_k z^{-k} A_k(z)\}.$$

Hence for z_i such that $A_k(z_i) = 0$ we get $A_{k-1}(z_i^{-1}) = z_i A_k(z_i^{-1})$. The functions

$$\frac{B_{k-1}(z)B_{k-1}(z^{-1})}{A_k(z)A_{k-1}(z^{-1})}$$

and

$$\frac{B_{k-1}(z)B_{k-1}(z^{-1})}{A_k(z)A_k(z^{-1})}\cdot\frac{1}{z} = \frac{B_{k-1}(z)B_{k-1}^*(z)}{A_k(z)A_k^*(z)}$$

thus have the same poles inside the unit circle and the same residues at these poles. The integrals of these functions around the unit circle are thus the same. Eq. (16) now gives

$$I_{k-1} = \frac{1}{1-\alpha_k^2}\frac{1}{2\pi i}\oint \frac{B_{k-1}(z)B_{k-1}(z^{-1})}{A_k(z)A_k(z^{-1})}\cdot\frac{dz}{z}.$$

Now introduce (8) and we find

$$\begin{aligned}(1-\alpha_k^2)I_{k-1} &= \frac{1}{2\pi i}\oint \frac{\{B_k(z)-\beta_k A_k^*(z)\}\{B_k(z^{-1})-\beta_k A_k^*(z^{-1})\}}{A_k(z)A_k(z^{-1})}\cdot\frac{dz}{z}\\ &= \frac{1}{2\pi i}\oint \frac{B_k(z)B_k(z^{-1})}{A_k(z)A_k(z^{-1})}\cdot\frac{dz}{z} - \frac{\beta_k}{2\pi i}\oint \frac{B_k(z)A_k^*(z^{-1})}{A_k(z)A_k(z^{-1})}\cdot\frac{dz}{z}\\ &\quad - \frac{\beta_k}{2\pi i}\oint \frac{A_k^*(z)B_k(z^{-1})}{A_k(z)A_k(z^{-1})}\cdot\frac{dz}{z} + \frac{\beta_k^2}{2\pi i}\oint \frac{A_k^*(z)A_k^*(z^{-1})}{A_k(z)A_k(z^{-1})}\cdot\frac{dz}{z}.\end{aligned} \tag{17}$$

The first integral equals I_k. The second integral can be reduced as follows:

$$\begin{aligned}\frac{\beta_k}{2\pi i}\oint \frac{B_k(z)A_k^*(z^{-1})}{A_k(z)A_k(z^{-1})}\cdot\frac{dz}{z} &= \frac{\beta_k}{2\pi i}\oint \frac{B_k(z)A_k(z)}{A_k(z)A_k^*(z)}\cdot\frac{dz}{z}\\ &= \frac{\beta_k}{2\pi i}\oint \frac{B_k(z)}{A_k^*(z)}\cdot\frac{dz}{z} = \beta_k\frac{B_k(0)}{A_k^*(0)} = \beta_k\frac{b_k^k}{a_0^k} = \beta_k^2\end{aligned}$$

where the first equality follows from (4), the third from the residue theorem and the fifth from (10). Similarly we find that the third integral of the right member also equals β_k^2. By using (4) the fourth term of the right member of (17) can be reduced as follows:

$$\frac{\beta_k^2}{2\pi i}\oint \frac{A_k^*(z)A_k^*(z^{-1})}{A_k(z)A_k(z^{-1})}\cdot\frac{dz}{z} = \frac{\beta_k^2}{2\pi i}\oint \frac{dz}{z} = \beta_k^2.$$

Summarizing, we find (14). When $k = 0$ we get from (13)

$$I_0 = \frac{1}{2\pi i}\oint \left(\frac{b_0^0}{a_0^0}\right)^2\frac{dz}{z} = \beta_0^2.$$

Since I_k is a continuous function of the parameters a_n, we also find that (14) and (15) hold even when A has multiple zeros. The proof of the theorem is now completed.

4. Computational aspects. Notice that it follows from (7) that

$$1-\alpha_k^2 = (a_0^k - a_k a_k^k)/a_0^k = a_0^{k-1}/a_0^k \tag{18}$$

The equation (14) can then be written as

$$a_0^k I_k = a_0^{k-1}I_{k-1} + \beta_k b_k^k = a_0^{k-1}I_{k-1} + (b_k^k)^2/a_0^k. \tag{19}$$

It is now a simple matter to evaluate the integrals numerically using the recursive equation (14) or (19).

The number of arithmetic operations required to compute I using the recursive formulas is shown in Table 1. The corresponding number of operations required for a

TABLE 1

Number of arithmetic operations required to compute the integral using the recursive formulas

Order of polynomial	Add/Subtract	Multiply	Divide
2	8	6	5
5	35	30	11
10	120	110	21
n	$n^2 + 2n$	$n^2 + n$	$2n + 1$

TABLE 2

Number of arithmetic operations required to compute the integral using gaussian reduction

Order of polynomial	Add/Subtract	Multiply	Divide
2	12	13	4
5	82	81	16
10	476	461	66
n	$\frac{4n^3 + 15n^2 + 20n + 12}{12}$ (n even) $\frac{4n^3 + 15n^2 + 20n + 9}{12}$ (n odd)	$\frac{2n^3 + 6n^2 + 16n + 6}{6}$	$\frac{n^2 + n + 2}{2}$

straight-forward solution of (3) for x_1 using gaussian reduction is shown in Table 2. A comparison of the tables shows that the recursive formulas give a considerable saving.

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