# A Generalization of the Phragmén-Lindelöf Principle for Elliptic Differential Equations

Wanby, Göran

*Published in:*
Mathematica Scandinavica

1978

MATH. SCAND. 43 (1978), 259–274

# A GENERALIZATION OF THE PHRAGMÉN–LINDELÖF PRINCIPLE FOR ELLIPTIC DIFFERENTIAL EQUATIONS

GÖRAN WANBY

## 1. Preliminaries.

This paper extends a Phragmén–Lindelöf type theorem on subharmonic functions to subsolutions of a linear uniformly elliptic partial differential equation.

Let $\Omega \subset \mathbf{R}^n$, $n \geqq 2$, be an open cone with its vertex at the origin and with boundary $\partial\Omega$, smooth outside the origin. The Phragmén–Lindelöf theorem says that if $u$ is subharmonic in $\Omega$ and $\leqq 0$ at $\partial\Omega$ then, roughly speaking, it is bounded or else $M(u,r) = \max u(y)$ when $|y| = r$ and $y \in \Omega$ grows like $r^k$ as $r \to \infty$. This is the growth of a suitably normalized nonnegative comparison function or gauge $v \not\equiv 0$, harmonic in $\Omega$ and zero on $\partial\Omega$. In particular $k$ is the positive root of the equation $k(k+n-2) = \mu$, where $\mu$ is the least positive eigenvalue of the Beltrami operator in $\Omega \cap \{|x| = 1\}$ for functions vanishing at the boundary of this region. Among the refinements of this result is, e.g., a theorem by Dahlberg [1] cited below, where the gauge $v_\lambda(x)$ is the unique harmonic function in $\Omega$ equal to $|x|^{k\lambda}$, $0 < \lambda < 1$, on $\partial\Omega$. It is homogeneous of degree $k\lambda$ and has the following property.

*Let $u$ be subharmonic in $\Omega$ and put $u(y) = \overline{\lim}\, u(x)$ when $x \to y$ and $y$ is on $\partial\Omega$. Then, if $u(0) \leqq 0$ and $u(y) \leqq CM(u,|y|)$ on $\partial\Omega \setminus \{0\}$, $C = M(v_\lambda, 1)^{-1}$, either $u \leqq 0$ or else $M(u,r)r^{-k\lambda}$ tends to a positive limit as $r \to \infty$.*

Similar problems are studied in Essén–Lewis [3]. Lewis [9] also obtained an analogous result for arbitrary unbounded domains of $\mathbf{R}^2$. When $\Omega$ is a half-space $x_n > 0$, then $k = 1$ and

$$v_\lambda(x) = \int P(x,y)|y|^\lambda \, dy_1 \ldots dy_{n-1} ,$$

where $P(x,y) = \gamma_n x_n |x-y|^{-n}$ is the Poisson kernel normalized so that $\int P(x,y)\, dy_1 \ldots dy_{n-1} = 1$. This gives $M(v_\lambda, 1) = C(\lambda, n)^{-1}$, where

$$(1) \qquad C(\lambda,n) = \pi^{\frac{1}{2}} \Gamma(\tfrac{1}{2}(n-1)) \Gamma(\tfrac{1}{2}(n-1+\lambda))^{-1} \Gamma(\tfrac{1}{2}(1-\lambda))^{-1} .$$

Received September 1, 1976; in revised form October 23, 1977.

We shall give a variant of Dahlberg's result for general uniformly elliptic second order differential operators

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)u''_{ij}(x), \qquad a_{ij}=a_{ji} .$$

Let $\mathscr{L}_\alpha$ denote the class of all such operators for which the matrix $(a_{ij}(x))$ has its eigenvalues in the interval $[\alpha, 1]$, $\alpha>0$. No continuity of the coefficients is required so far. Let $\theta$ be the polar angle, $\theta=\arccos(x_n/r)$, $r=|x|$, and let $\Omega_0$ be the cone $\{x\ ;\ 0\leqq\theta<\psi_0\}$, $\psi_0<\pi$. (If $n=2$ we consider the sector $\{x\ ;\ |\theta|<\psi_0\}$ without explicitly saying so in the sequel.) Miller [10] has obtained the precise growth condition $u(x)=O(r^k)$, $k=k(\psi_0,\alpha)$, to ensure that $Lu\geqq 0$ in $\Omega_0$, $u\leqq 0$ on $\partial\Omega_0$ implies $u\leqq 0$ in $\Omega_0$. The case $\psi_0=\pi/2$, corresponding to $k=1$, was treated by Gilbarg [4] and Hopf [7].)

Our gauge $v$ will be homogeneous of degree $k\lambda$, $0<\lambda<1$, equal to $|x|^{k\lambda}$ on $\partial\Omega_0$ and superharmonic with respect to all $L$ in $\mathscr{L}_\alpha$, that is, $Lv\leqq 0$ in $\Omega_0$. Using Miller's results, we can find such a function when $\alpha\geqq(1-k\lambda)(n-1)^{-1}$. If $\alpha=1$, this is Dahlberg's function for $\Omega_0$. With the same hypothesis as before but with $M^+=\max(M,0)$ and with $C=C(\lambda,n,\psi_0,\alpha)=M(v,1)^{-1}$ we now get an analogous but weaker result.

THEOREM 1. *Let $u$ be a $C^2$ function in $\Omega_0$ satisfying $Lu\geqq 0$ for some $L\in\mathscr{L}_\alpha$. If $u(y)<\infty$ when $y\in\partial\Omega_0$ and*

$$u(y) \leqq CM^+(u,|y|) \quad \text{on } \partial\Omega_0\setminus\{0\} , \tag{2}$$

*then*

$$\varlimsup_{r\to\infty} M^+(u,r)r^{-k\lambda} \leqq C^{-1} \varliminf_{r\to\infty} M^+(u,r)r^{-k\lambda} . \tag{3}$$

*If $\varliminf_{r\to\infty} M^+(u,r)r^{-k\lambda}=0$ then $u\leqq 0$ in $\Omega_0$.*

The proof is carried out in section 2, where we also give an example which shows that (2) cannot be weakened if we want (3) to hold for some constant $C^{-1}$ independent of $u$. Inequalities of type (2) have been studied in classical function theory. For examples we refer to Hellsten, Kjellberg and Norstad [6] and to Drasin and Shea [2].

To improve theorem 1 we shall assume that $L$ tends to a constant coefficient operator for large $x$. We shall also let $\Omega$ be the half-space $x_n>0$ so that $k=1$, $\psi_0=\pi/2$. By a change of variables preserving $\Omega$, we may take the limit of $(a_{ij}(x))$ to be the unit $n\times n$ matrix.

The coefficients $a_{ij}$ are assumed to be uniformly Hölder continuous at $\partial\Omega$, i.e.

$$x\in\Omega,\ y\in\partial\Omega \Rightarrow a_{ij}(x)-a_{ij}(y) = O(|x-y|^{\varepsilon}) \tag{4}$$

for some $\varepsilon > 0$. The approach to $\delta_{ij}$ at infinity is measured by a parameter $\beta > 0$:

$$x \to \infty \Rightarrow a_{ij}(x) - \delta_{ij} = O(|x|^{-\beta}) . \tag{5}$$

Our gauge will be equal to $|x|^\lambda$ at $\partial\Omega$ for large $x$ and the corresponding $C = C(\lambda, n)$ is given by (1). Then we shall prove

THEOREM 2. *Let $u$ be a $C^2$ function in $\Omega$ satisfying $Lu \geqq 0$ and $\lambda$ a given number such that $0 < \lambda < \min(\beta, 1)$. Then, if $u(y) < \infty$ on $\partial\Omega$, $u(y) < \mu M^+(u, |y|)$ on $\partial\Omega \setminus \{0\}$ for some $\mu < 1$ and*

$$u(y) \leqq C(1 - A|y|^{-\delta}) M^+(u, |y|) \quad \text{on } \partial\Omega,\ |y| > \text{some } r_0$$

*where $A > 0$ and $0 < \delta < \min(\lambda, \beta/2)$, either $u \leqq 0$ throughout $\Omega$ or else $M^+(u, r) r^{-\lambda}$ has a positive or infinite limit as $r \to \infty$.*

Thus, apart from the restrictions on $\lambda$, in order to get a precise result, we have to strengthen the boundary condition by inserting a factor which is $< 1$ but tends to 1 for large $y$, not too rapidly depending on $\lambda$ and $\beta$.

Note that if the matrix $(a_{ij}(x))$ has a limit $(b_{ij})$ at $x = \infty$ not normalized to $(\delta_{ij})$, then the distances of the theorem have to be measured in the metric given by $(b_{ij})^{-1}$.

For $\lambda = 1$ a corresponding theorem has been proved by Serrin [12], who generalized the result of Heins [5] for subharmonic functions.

Our proof depends heavily on the construction of a substitute for the Poisson kernel. This is done in section 3. The rest of the proof is then similar to the verification of the Lewis–Essén–Dahlberg result. We also give an example of a function $u$ which satisfies the assumptions of theorem 2 and for which $M^+(u, r) r^{-\lambda}$ has a finite positive limit.

## 2. Proof of theorem 1.

We shall first present the gauge. For the maximizing operator $M_\alpha$, defined on $C^2$ by

$$(M_\alpha u)(x) = \max_{L \in \mathcal{L}_\alpha} Lu(x) ,$$

Miller [10] has established the following theorem:

*If $\alpha$ and $k$ are given numbers, $0 < \alpha \leqq 1$, $k > 0$, there is a unique solution $f = f_{k,\alpha}(\theta)$ of the problem*

$$f(0) = 1,\ r^k f(\theta) \in C^2 \text{ and } M_\alpha(r^k f(\theta)) = 0 \text{ on } \mathbf{R}^n \setminus \{\text{closed negative } x_n\text{-axis}\} .$$

It turns out that $f_{k,\alpha}$ depends continuously on $k$ and $\alpha$ in $[0,\pi)$ and has a first zero $\psi(k,\alpha)$ in $(0,\pi)$. Further $\psi(k,\alpha)$ is strictly decreasing with respect to $k$ and strictly increasing with respect to $\alpha$ except at $k=1$, where $\psi(1,\alpha)=\pi/2$ for all $\alpha$. Let $k(\psi,\alpha)$ denote the inverse of $\psi$ for fixed $\alpha$. If $\Omega_0=\{x\ ;\ 0\leqq\theta<\psi_0\}$ and $k=k(\psi_0,\alpha)$ the function $h=r^k f_{k,\alpha}(\theta)$ satisfies

$$Lh \leqq 0 \text{ in } \Omega_0 \text{ for every } L \in \mathscr{L}_\alpha, \quad h=0 \text{ at } \partial\Omega_0 .$$

Thus $h$ exhibits the Phragmén–Lindelöf growth for solutions of $Lu=0$, $L \in \mathscr{L}_\alpha$ in $\Omega_0$, vanishing at $\partial\Omega$.

With the given $\lambda$, $0<\lambda<1$, let $f=f_{k\lambda,\alpha}$. Then $\psi_0=\psi(k,\alpha)<\psi(k\lambda,\alpha)$ so $f(\theta)>0$ when $0\leqq\theta\leqq\psi_0$. We have $L_0(r^{k\lambda}f(\theta))=0$ for some $L_0 \in \mathscr{L}_\alpha$ (Miller [10, p. 300]). Using a polar representation ([10, p. 303]) we obtain

$$(6)\qquad L_0(r^{k\lambda}f) = r^{k\lambda-2}[cf''(\theta)+(2b(k\lambda-1)+d(n-2)\cot\theta)f'(\theta)+ \\ +k\lambda(a(k\lambda-1)+c+d(n-2))f(\theta)]$$

for some functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$, where the eigenvalues of $\begin{pmatrix} a & b \\ b & c\end{pmatrix}$ and $d(x)$ belong to the interval $[\alpha,1]$. If $k\lambda-1>0$, $a(k\lambda-1)+c+d(n-2)$ is positive. If $k\lambda-1\leqq 0$ we use $a\leqq 1$ and $c,d\geqq\alpha$ to get

$$a(k\lambda-1)+c+d(n-2) \geqq k\lambda-1+(n-1)\alpha .$$

Hence, if $\alpha\geqq(1-k\lambda)/(n-1)$,

$$cf''(\theta)+(2b(k\lambda-1)+d(n-2)\cot\theta)f'(\theta) \leqq 0 ,$$

when $0\leqq\theta\leqq\psi_0$. It follows from the minimum principle that $f$ is a strictly decreasing function of $\theta$. Let

$$v = r^{k\lambda}f(\theta)f(\psi_0)^{-1} .$$

Then $Lv\leqq 0$ in $\Omega_0$ for all $L \in \mathscr{L}_\alpha$, $v=r^{k\lambda}$ at $\partial\Omega_0$ and

$$1 \leqq v(x) \leqq f(0)f(\psi_0)^{-1} = C^{-1} \quad \text{on } |x|=1 .$$

To prove theorem 1 we observe that $Lu\geqq 0$ in $\Omega_0$ and the boundary condition imply that $M^+(r)=M^+(u,r)$ increases with $r>0$, and increases strictly when positive. This follows from a variant of the maximum principle for elliptic equations. See Protter–Weinberger [11, pp. 97–100]. (Here we use the assumption $u(0)<\infty$.)

The function $w=u-av$, where $a$ is a positive constant, satisfies $Lw\geqq 0$ in $\Omega_0$, $w(x)\leqq CM^+(w,r)$ on $\partial\Omega_0\setminus\{0\}$, $w(0)<\infty$. Choose $a=M^+(R)R^{-k\lambda}$. Then

$$M^+(w,R) \leqq M^+(R)-M^+(R)R^{-k\lambda} \inf_{|x|=R,x\in\Omega_0} v(x) = 0 ,$$

so we get

(7) $\quad u(x) \leqq M^{+}(R)R^{-k\lambda}v(x) \leqq M^{+}(R)R^{-k\lambda}C^{-1}|x|^{k\lambda} \quad$ when $|x| \leqq R$ .

Hence

$$M^{+}(r)r^{-k\lambda} \leqq C^{-1}M^{+}(R)R^{-k\lambda} \quad \text{for } r \leqq R .$$

Letting, in order, $R$ and $r$ tend to infinity through suitable sequences we obtain (3). From (7) we also get

$$u(x) \leqq v(x) \varliminf_{R \to \infty} M^{+}(R)R^{-k\lambda} .$$

Thus, either $u \leqq 0$ in $\Omega_0$ or $\varliminf_{R \to \infty} M^{+}(R)R^{-k\lambda} > 0$.

REMARK 1. If the boundary condition (2) is satisfied only when $|y| \geqq r_0 > 0$ but we have

(2′) $$u(y) \leqq \mu M^{+}(|y|), \quad y \in \partial\Omega_0 \setminus \{0\} ,$$

for some $\mu$, $0 < \mu < 1$, the conclusion of the theorem still holds. To see this we replace $u$ by $u - M^{+}(r_0)$. Then (2) and (2′) are still valid and we use the maximum principle on $w$ in $|x| \geqq r_0$, $x \in \Omega_0$. As above we get (3) and conclude

$$u(x) \leqq M^{+}(r_0) + v(x) \varliminf_{r \to \infty} M^{+}(r)r^{-k\lambda} .$$

Since, by (2′) $M^{+}(r)$ is strictly increasing when positive we get

$$u \leqq 0 \text{ in } \Omega_0 \quad \text{or} \quad \varliminf_{r \to \infty} M^{+}(r)r^{-k\lambda} > 0 .$$

If $Lu \geqq 0$ only when $|x| \geqq r_0$ the last sentence of the theorem should be:

$$u(x) \leqq M^{+}(r_0) \quad \text{or} \quad \varliminf_{r \to \infty} M^{+}(r)r^{-k\lambda} > 0 .$$

REMARK 2. If the boundary condition (2) is replaced by

(2″) $$u(y) \leqq (C - \varepsilon)M^{+}(|y|)$$

for some $\varepsilon > 0$, then either $u \leqq 0$ or $M^{+}(r)r^{-k\lambda} \to \infty$, as $r \to \infty$. In fact, (2″) and the continuity of $f_{k\lambda,\alpha}$ with respect to the parameters imply that

$$u(y) \leqq C(\lambda', n, \psi_0, \alpha)M^{+}(|y|) ,$$

if $\lambda'$ is sufficiently close to $\lambda$. Thus $u \leqq 0$ or

$$\varliminf_{r \to \infty} M^{+}(r)r^{-k\lambda'} > 0 \quad \text{for some } \lambda' > \lambda .$$

In the latter case obviously $M^{+}(r)r^{-k\lambda} \to \infty$ when $r \to \infty$.

We now give an example where, with arbitrary given positive $A$ and $B$, $A > B$,

$$\overline{\lim_{r\to\infty}}\, M^+(r)r^{-k\lambda} = A, \qquad \underline{\lim}_{r\to\infty}\, M^+(r)r^{-k\lambda} = B$$

and where the boundary condition is replaced by

$$u(y) \leqq (C+\varepsilon)M^+(|y|), \qquad \varepsilon > 0 .$$

Let $f = f_{k\lambda,\alpha}$ so that $L_0(r^{k\lambda}f) = 0$ for some $L_0 \in \mathscr{L}_\alpha$. Let $g = (1+\varepsilon)^{-1}(f+\varepsilon)$ and put

$$u(x) = g(\theta)r^{k\lambda}h(r), \qquad \text{where } h(r) = [A+B+(A-B)\cos(\log(\log r))]/2 .$$

Since $h'(r) = O((r\log r)^{-1})$ and $h''(r) = O((r^2\log r)^{-1})$, it is easy to see that

$$L_0 u = L_0(gr^{k\lambda}h) = hL_0(gr^{k\lambda}) + r^{k\lambda-2}O((\log r)^{-1}) .$$

From (6) we get

$$L_0(gr^{k\lambda}) = (1+\varepsilon)^{-1}r^{k\lambda-2}k\lambda\varepsilon(a(k\lambda-1)+c+d(n-2)) ,$$

which is positive, as we assume $\alpha > (1-k\lambda)/(n-1)$. Thus $L_0 u \geqq 0$ in $\Omega_0$ when $|x| \geqq$ some $r_0$. Since $M^+(r) = r^{k\lambda}h(r)$ we get

$$u(y) < (C+\varepsilon)M^+(|y|) \qquad \text{when } y \in \partial\Omega_0 ,$$

and

$$\overline{\lim_{r\to\infty}}\, M^+(r)r^{-k\lambda} = A, \quad \underline{\lim}_{r\to\infty}\, M^+(r)r^{-k\lambda} = B .$$

## 3. A substitute for the Poisson kernel.

LEMMA. *Let $L \in \mathscr{L}_\alpha$ satisfy* (4) *and* (5) *and let $\Omega = \{x_n > 0\}$. Then, for any $\gamma < \min(\beta/2, 1)$ there exists a $C^2$ function $K(x,y)$, defined when $x \in \bar{\Omega}$, $y \in \partial\Omega \setminus \{0\}$, $x \neq y$, such that*

(i) *$LK(x,y) \leqq 0$ for fixed $y$, $|x| \geqq 1$, $|y| \geqq$ some $r_0$*

(ii) *If $y^0 \in \partial\Omega$ and $\varphi$ is continuous at $y^0$ with $\int_{\partial\Omega}\varphi(y)(1+|y|)^{-n}dy$ convergent, then*

$$\lim_{x\to y_0} \int_{\partial\Omega\cap\{|y|\geqq 1\}} K(x,y)\varphi(y)\,dy = \varphi(y^0), \qquad |y^0| > 1$$

(iii) *There is a constant $B > 0$ such that*

$$P(x,y)(1-B|y|^{-\beta}) \leqq K(x,y) \leqq P(x,y)(1+B|y|^{-\gamma}), \qquad |y| \geqq 1 ,$$

*where $P(x,y) = \gamma_n x_n |x-y|^{-n}$ is the Poisson kernel of $\Omega$.*

PROOF. A similar construction for the unit ball has been made by Serrin [13].

Let $A_{ij}(x)$ be the elements of the inverse of $(a_{ij}(x))$. The existence of

$$\lim_{x\to\infty} A_{ij}(x) = \delta_{ij}$$

follows from (5), which also implies

$$|A_{ij}(x)-\delta_{ij}| \leqq c_1|x|^{-\beta} . \tag{8}$$

It is no restriction to prescribe $\det(a_{ij}(x))=1$. Let

$$\varrho_1(x,y) = \left(\sum_{i,j=1}^{n} A_{ij}(y)(x_i-y_i)(x_j-y_j)\right)^{\frac{1}{2}}$$

and

$$\varrho_2(x,y) = \left(\sum_{i,j=1}^{n} \delta_{ij}(x_i-y_i)(x_j-y_j)\right)^{\frac{1}{2}} = |x-y| ,$$

when $x \in \Omega$, $y \in \partial\Omega$. We put

$$k_\nu(x,y) = x_n\varrho_\nu(x,y)^{-n}, \quad \nu=1,2 . \tag{9}$$

Then

$$\begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2}{\partial x_i\partial x_j}k_1(x,y) = 0 \quad \text{and} \\ &\sum_{i,j=1}^{n} \delta_{ij}\frac{\partial^2}{\partial x_i\partial x_j}k_2(x,y) = 0 . \end{aligned} \tag{10}$$

By direct calculation we also find

$$|\mathrm{grad}_x\, k_\nu(x,y)| \leqq c_2|x-y|^{-n}, \quad \nu=1,2 , \tag{11}$$

$$\left|\frac{\partial^2}{\partial x_i\partial x_j}k_\nu(x,y)\right| \leqq c_3|x-y|^{-n-1}, \quad \nu=1,2 . \tag{12}$$

The constants $c_1$, $c_2$ and $c_3$, and in the following $c_4, c_5, \ldots$, depend on $n$ and the ellipticy constant. Let $\chi$ be a nonnegative $C^\infty$ function on the interval $[0,\infty)$ such that

$$\chi(t) = \begin{cases} 1 & \text{when } 0\leqq t\leqq 1 \\ 0 & \text{when } \quad t\geqq 2 . \end{cases}$$

Define $k$ by

$$k(x,y) = \chi(|x-y|)k_1(x,y)+(1-\chi(|x-y|))k_2(x,y) .$$

For fixed $y$ we want to find $K$ of the form $f(k)$, where $f$ is a function of one variable. Since

$$LK = Lf(k) = f''(k) \sum_{i,j=1}^{n} a_{ij} \frac{\partial k}{\partial x_i} \frac{\partial k}{\partial x_j} + f'(k) Lk \,, \tag{13}$$

we need some estimates of $Lk$ and

$$Qk = Qk(x,y) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial k(x,y)}{\partial x_i} \frac{\partial k(x,y)}{\partial x_j} \,.$$

When $|x-y|<1$, $k=k_1$ and we have

$$\begin{aligned} Lk = Lk_1 &= \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} k_1 \\ &= \sum_{i,j=1}^{n} (a_{ij}(x) - a_{ij}(y)) \frac{\partial^2}{\partial x_i \partial x_j} k_1 \,. \end{aligned}$$

Here we used (10). Since

$$\begin{aligned} |a_{ij}(x) - a_{ij}(y)| &= |a_{ij}(x) - a_{ij}(y)|^{\frac{1}{2}} |(a_{ij}(x) - \delta_{ij}) + (\delta_{ij} - a_{ij}(y))|^{\frac{1}{2}} \\ &\leqq c_4 |x-y|^{\varepsilon/2} (c_5 |x|^{-\beta} + c_5 |y|^{-\beta})^{\frac{1}{2}} \end{aligned}$$

and

$$|x| \geqq |y| - |x-y| > |y| - 1 \geqq \tfrac{1}{2}|y| \quad \text{if } |y| \geqq 2 \,,$$

we get, using (12),

$$Lk \leqq c_6 |y|^{-\beta/2} |x-y|^{-n-1+\varepsilon/2} \quad \text{if } |x-y|<1,\ |y| \geqq 2 \,. \tag{14}$$

Similarly, when $|x-y|>2$,

$$Lk = Lk_2 \leqq c_7 |x|^{-\beta} |x-y|^{-n-1} \,.$$

With $s$ and $t$ to be fixed later, $0<s,t<1$, we have, if $|x| \geqq \frac{1}{2}|y|$,

$$|x|^{-\beta} \leqq 2^{t\beta} |y|^{-t\beta} |x|^{-(1-t)\beta} 2^{-s\beta} |x-y|^{s\beta} \,.$$

If $|x| < \frac{1}{2}|y|$, then $|x-y| > \frac{1}{2}|y|$ and we get

$$|x|^{-\beta} \leqq |x|^{-(1-t)\beta} |y|^{-s\beta} 2^{s\beta} |x-y|^{s\beta} \quad \text{if } |x| \geqq 1 \,.$$

Thus

$$Lk \leqq c_8 |x|^{-(1-t)\beta} |y|^{-\min(s,t)\beta} |x-y|^{-n-1+s\beta} \tag{15}$$

when $|x-y|>2$, $|x| \geqq 1$, $|y| \geqq 1$.

When $1 \leqq |x-y| \leqq 2$,

$$
\begin{aligned}
Lk = {} & \chi \sum_{i,j=1}^{n} (a_{ij}(x)-a_{ij}(y))\frac{\partial^2}{\partial x_i \partial x_j}k_1 + \\
& +(1-\chi)\sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij})\frac{\partial^2}{\partial x_i \partial x_j}k_2 + \\
& +\chi'(\varrho_1^{-n}-\varrho_2^{-n})\Bigg[\sum_{i,j=1}^{n} a_{ij}(x)|x-y|^{-3}(|x-y|^2\delta_{ij}-(x_i-y_i)(x_j-y_j))x_n + \\
& +2\sum_{i=1}^{n} a_{in}(x)|x-y|^{-1}(x_i-y_i)\Bigg] - \\
& -2n\chi' x_n \sum_{i,j=1}^{n} a_{ij}(x)|x-y|^{-1}(x_i-y_i)\Bigg(\varrho_1^{-n-2}\sum_{k=1}^{n} A_{jk}(y)(x_k-y_k) - \\
& -\varrho_2^{-n-2}\sum_{k=1}^{n}\delta_{jk}(x_k-y_k)\Bigg) + \\
& +\chi'' x_n(\varrho_1^{-n}-\varrho_2^{-n})\sum_{i,j=1}^{n} a_{ij}(x)|x-y|^{-2}(x_i-y_i)(x_j-y_j)\,.
\end{aligned}
$$

The first two terms are less than $c_9(|x|^{-\beta}+|y|^{-\beta})|x-y|^{-n-1}$, which is dominated by $c_{10}|y|^{-\beta}$ if $|y| \geqq 4$, say, since $1 \leqq |x-y| \leqq 2$.

As

$$
(16) \qquad |\varrho_1^{-n}-\varrho_2^{-n}| \leqq c_{11}|\varrho_1^2-\varrho_2^2|\,|x-y|^{-n-2} = c_{11}|x-y|^{-n-2}\left|\sum_{i,j=1}^{n}(A_{ij}(y)-\delta_{ij})(x_i-y_i)(x_j-y_j)\right|,
$$

the Hölder continuity (8) implies that the terms involving $\chi'$ may be estimated by terms of order $|x-y|^{-n}|y|^{-\beta}$. In the same way we see that the $\chi''$-term is less than $c_{12}|x-y|^{-n+1}|y|^{-\beta}$, so

$$
(17) \qquad Lk \leqq c_{13}|y|^{-\beta} \quad \text{when } 1 \leqq |x-y| \leqq 2,\ |y| \geqq 4\,.
$$

As for $Qk$, we get when $|x-y|<1$,

$$
Qk = Qk_1 = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial k_1}{\partial x_i}\frac{\partial k_1}{\partial x_j} \geqq c_{14}|\mathrm{grad}\, k_1|^2, \qquad c_{14}>0\,.
$$

Now

$$
\begin{aligned}
|\mathrm{grad}\, k_1| \geqq \left|\frac{\partial k_1}{\partial x_n}\right| &= \left|\varrho_1^{-n} - n x_n \varrho_1^{-n-2}\sum_{i=1}^{n} A_{in}(y)(x_i-y_i)\right| \\
&\geqq \varrho_1^{-n} - x_n|x-y|^{-1}c_{15}\varrho_1^{-n} \geqq \tfrac{1}{2}\varrho_1^{-n} \text{ if } x_n|x-y|^{-1} \leqq \frac{c_{15}^{-1}}{2}\,.
\end{aligned}
$$

Further, let $x$ and $y$ be fixed and put $v=(x-y)/|x-y|$. Then for real $t$,

$$k_1(x+tv,y) = k_1(y+(t+|x-y|)v,y) = x_n|x-y|^{-1}(t+|x-y|)^{1-n}c_v^{-n},$$

where $c_v=(\sum_{i,j=1}^n A_{ij}(y)v_iv_j)^{\frac{1}{2}}$, $v=(v_1,v_2,\ldots,v_n)$. Thus the derivative with respect to $x$ in the direction of $v$ at $(x,y)$ is

$$\frac{\partial k_1}{\partial v} = x_n|x-y|^{-1}c_v^{-n}(1-n)|x-y|^{-n} = (1-n)x_n|x-y|^{-1}\varrho_1^{-n},$$

so

$$|\operatorname{grad} k_1| \geqq \frac{n-1}{2}c_{15}^{-1}\varrho_1^{-n} \quad \text{if } x_n|x-y|^{-1} \geqq \frac{c_{15}^{-1}}{2}.$$

Hence, for all $x \in \Omega$, $y \in \partial\Omega$,

$$Qk_1 \geqq c_{16}|x-y|^{-2n},\ c_{16} > 0.$$

With $\delta_{ij}$ instead of $A_{ij}(y)$ above we get the same inequality for $k_2$. Thus, when $|x-y|<1$ or $|x-y|>2$, we have

$$Qk \geqq c_{16}|x-y|^{-2n}, \quad c_{16} > 0. \tag{18}$$

For $1 \leqq |x-y| \leqq 2$, we will need, for large $|y|$,

$$Qk \geqq c_{17} > 0. \tag{19}$$

To see this, we observe that

$$\operatorname{grad} k = \chi'|x-y|^{-1}(k_1-k_2)(x-y)+\chi \operatorname{grad}(k_1-k_2)+\operatorname{grad} k_2.$$

In virtue of (8) the absolute value of the terms containing $\chi$ and $\chi'$ may be estimated by $c_{18}|y|^{-\beta}|x-y|^{1-n} \leqq c_{18}|y|^{-\beta}$. Hence

$$\begin{aligned}|\operatorname{grad} k| &\geqq |\operatorname{grad} k_2|-c_{18}|y|^{-\beta} \geqq c_{19}|x-y|^{-n}-c_{18}|y|^{-\beta}\\ &\geqq c_{19}2^{-n}-c_{18}|y|^{-\beta},\end{aligned}$$

which gives (19), if $|y|$ is large enough.

Now, if $|y| \geqq$ some $r_0$ we conclude when $|x-y|<1$,

$$Ak = Lk(Qk)^{-1} \leqq c_{20}|y|^{-\beta/2}|x-y|^{n-1+\varepsilon/2}, \tag{20}$$

while (20) (for some constant $c_{20}$) follows from (17) and (19) when $1 \leqq |x-y| \leqq 2$. When $|x-y|>2$,

$$k = k_2 = x_n|x-y|^{-n} \leqq |x-y|^{1-n} < 1.$$

In general

$$k \leqq c_{21}x_n|x-y|^{-n} \leqq c_{21}|x-y|^{1-n}$$

so we get

$$(21)\qquad |x-y| \leqq c_{21}^{1/(n-1)}\left(\frac{1}{k}\right)^{1/(n-1)} .$$

Thus (20) gives

$$Ak \leqq c_{22}|y|^{-\beta/2}\left(\frac{1}{k}\right)^{1+\varepsilon'} \quad \text{when } k \geqq 1 ,$$

where $\varepsilon' = \varepsilon/2(n-1)$.

Using (15) and (18) we obtain for $|x-y|>2$, $|x| \geqq 1$,

$$\begin{aligned} Ak &\leqq c_{23}|x|^{-(1-t)\beta}|y|^{-\min(s,t)\beta}|x-y|^{n-1+s\beta} \\ &\leqq c_{23}|y|^{-\min(s,t)\beta}\left(\frac{|x-y|^n}{x_n}\right)^{(1-t)\beta}|x-y|^{n-1+s\beta-(1-t)n\beta} . \end{aligned}$$

If $n-1+\beta(s-n+nt)>0$, (21) gives

$$\begin{aligned} Ak &\leqq c_{24}|y|^{-\min(s,t)\beta}\left(\frac{1}{k}\right)^{(1-t)\beta+1+(s-n+nt)\beta/(n-1)} \\ &= c_{24}|y|^{-\min(s,t)\beta}\left(\frac{1}{k}\right)^{1-\beta'} , \end{aligned}$$

where $\beta' = (1-s-t)\beta/(n-1)>0$ if $s+t<1$.

It is seen from a figure in the $st$-plane that it is possible to choose $s$ and $t$ so that $n-1+\beta(s-n+nt)>0$, $0<s,t<1$ and $s+t<1$. If $\beta \leqq 2$ we may take $\min(s,t)<\frac{1}{2}$ as close to $\frac{1}{2}$ as we want, while if $\beta>2$ we can get $\min(s,t)$ close to $1/\beta$. (Taking $s$ and $t$ so that $n-1+\beta(s-n+nt)<0$, $(1-t)\beta<1$ does not improve the exponent of $|y|$.)

Thus, for any $\gamma<\min(\beta/2, 1)$, there is a constant $c_{24}$ such that

$$(22)\qquad Ak \leqq c_{24}|y|^{-\gamma}\left(\frac{1}{k}\right)^{1-\beta'} \quad \text{if } |x-y|>2,\ |x|, |y| \geqq 1 .$$

When $|x-y| \leqq 2$ and $|y| \geqq r_0$, (22) (for some constant $c_{24}$) follows from (20) and (21). Summing up, if $|x| \geqq 1$, $|y| \geqq r_0$,

$$Ak \leqq |y|^{-\gamma}c_{25}\begin{Bmatrix} k^{-(1+\varepsilon')}, & \text{if } k \geqq 1 \\ k^{-(1-\beta')}, & \text{if } k<1 \end{Bmatrix} = |y|^{-\gamma}\omega(k) ,$$

the last equality defining $\omega$. Hence, if we for fixed $y$ choose $f$ as a solution of the ordinary differential equation

$$(23)\qquad f''(k)+|y|^{-\gamma}\omega(k)f'(k) = 0 \quad \text{with } f' > 0 ,$$

we get

$$LK = Qk(f''(k)+Akf'(k)) \leqq 0 ,$$

so (i) is satisfied. One solution of (23) is

$$f(k) = c' \int_0^k \exp\left(|y|^{-\gamma} \int_t^\infty \omega(\tau)\,d\tau\right) dt , \tag{24}$$

which is defined for $k \geqq 0$ since $\int_0^\infty \omega(\tau)\,d\tau$ converges. Here $c'$ is a positive constant to be chosen later. As we want $K(x,y)=f(k(x,y))$ to be 0 when $x \in \partial\Omega$, $x \neq y$, we have taken $f(0)=0$. It is easy to see that $K$ belongs to $C^2$ when $x \in \bar{\Omega}$, $y \in \partial\Omega \setminus \{0\}$, $x \neq y$.

To prove (ii), we see from (24) that for fixed $y$

$$\lim_{k\to\infty} \frac{f(k)}{k} = c' .$$

This fact and standard procedures of potential theory imply, with $z' = (z_1, \ldots, z_{n-1})$,

$$\lim_{x\to y^0} \int_{\partial\Omega \cap \{|y|\geqq 1\}} K(x,y)\varphi(y)\,dy$$

$$= \varphi(y^0)c' \int_{z'\in \mathbf{R}^{n-1}, z_n=1} \left(\sum_{i,j=1}^{n} A_{ij}(y^0)z_iz_j\right)^{-n/2} dz' .$$

The last integral is equal to

$$(\det A_{ij}(y^0))^{-\frac{1}{2}} \int_{z'\in\mathbf{R}^{n-1}} (1+|z'|^2)^{-n/2}\,dz' = \gamma_n^{-1} .$$

(We omit the details of the calculation.) With $c'=\gamma_n$ we obtain (ii).

The remaining part of the lemma now follows easily. From (24) we get

$$\gamma_n k(x,y) \leqq K(x,y) \leqq \gamma_n\left(\exp|y|^{-\gamma} \int_0^\infty \omega(\tau)\,d\tau\right)k(x,y) .$$

Since $k=k_2+\chi(k_1-k_2)$, we get, using (16) once more,

$$\gamma_n(1-c_{26}|y|^{-\beta})k_2 \leqq K(x,y) \leqq \gamma_n\left(\exp|y|^{-\gamma} \int_0^\infty \omega(\tau)\,d\tau\right)(1+c_{26}|y|^{-\beta})k_2$$

from which (iii) follows.

## 4. Proof of theorem 2.

Put for shortness $m(r)=M^+(u,r)r^{-\lambda}$ and let $\underline{m}=\underline{\lim}_{r\to\infty} m(r)$ and $\overline{m} = \overline{\lim}_{r\to\infty} m(r)$. Supposing $0<\underline{m}<\infty$, we shall first prove that also $\overline{m}<\infty$.

Choose $\gamma$ with $\delta<\gamma<\min(\beta/2,1)$ and take $K$ according to the lemma. Then our gauge

$$v(x) = \int_{|y|\geqq r_0} K(x,y)|y|^\lambda\, dy$$

satisfies

$$M^+(v,r) \leqq C^{-1}r^\lambda + c_{26}\max(r^{\lambda-\gamma},1) \tag{25}$$

and, since $\beta>\lambda$,

$$v(x) \geqq |x|^\lambda - \int_{|y|<r_0} P(x,y)|y|^\lambda\, dy - Br_0^{\lambda-\beta}\,. \tag{26}$$

Let $w=u-M^+(u,r_0)-av$, where $a$ is a positive constant to be chosen close to $\underline{m}$. Then $Lw\geqq 0$ when $x\in\Omega$, $|x|\geqq 1$, and $M^+(w,r_0)=0$. At $\partial\Omega$

$$w(y) = u(y)-M^+(u,r_0)-a|y|^\lambda, \qquad |y| > r_0\,.$$

The boundary condition of $u$ and (25) give, when $y\in\partial\Omega$, $|y|>r_0$,

$$CM^+(w,|y|) \geqq u(y)-CM^+(u,r_0)+CA|y|^{-\delta}M^+(u,|y|)-a|y|^\lambda - -aCc_{26}\max(|y|^{\lambda-\gamma},1)$$

so $w(y)\leqq CM^+(w,|y|)$ if

$$ac_{26}\max(|y|^{\lambda-\gamma},1) \leqq A|y|^{-\delta}M^+(u,|y|)\,. \tag{27}$$

Since $M^+(u,|y|)\geqq(\underline{m}/2)\,|y|^\lambda$ if $|y|\geqq$ some constant and $\delta<\min(\gamma,\lambda)$, (27) is satisfied if $|y|$ is large. Thus, if $r_0$ is chosen big enough, $M^+(w,r)$ is strictly increasing for $r\geqq r_0$. With $a=M^+(u,R)R^{-\lambda}$, $R>r_0$, we have from (26)

$$\begin{aligned} M^+(w,R) &\leqq M^+(u,R)-M^+(u,r_0)- \\ &\quad -M^+(u,R)R^{-\lambda}\left(R^\lambda-\sup_{|x|=R}\int_{|y|<r_0} P(x,y)|y|^\lambda\, dy - Br_0^{\lambda-\beta}\right) \\ &\leqq -M^+(u,r_0)+M^+(u,R)R^{-\lambda}(R(R-r_0)^{-n}c_{27}+c_{28})\,. \end{aligned}$$

Hence, when $r_0\leqq|x|\leqq R$,

$$u(x) \leqq M^+(u,R)R^{-\lambda}(C^{-1}|x|^\lambda+c_{26}\max(|x|^{\lambda-\gamma},1)+R(R-r_0)^{-n}c_{27}+c_{28})$$

and so

$$\begin{aligned} M^+(u,r)r^{-\lambda} &\leqq M^+(u,R)R^{-\lambda}(C^{-1}+c_{26}\max(r^{-\gamma},r^{-\lambda})+ \\ &\quad +R(R-r_0)^{-n}c_{27}r^{-\lambda}+c_{28}r^{-\lambda}) \end{aligned}$$

for $r_0\leqq r\leqq R$. As in the proof of theorem 1 we get $\overline{m}\leqq\underline{m}C^{-1}$. Then also $m^*=$

$\sup_{r \geqq r_0} m(r) < \infty$. We thus have $0 < \underline{m} \leqq \overline{m} \leqq m^* < \infty$ and we shall prove, with a technique similar to Kjellberg's [8], that $\underline{m} = \overline{m}$ $(=m^*)$.

Let us first note that

$$h(x) = u(x) - C \int_{|y| \geqq r_0} K(x,y)(1 - A|y|^{-\delta}) M^+(u,|y|)\, dy$$

is $\leqq 0$ when $x \in \Omega$, $|x| \geqq r_0$, at least if we replace $u$ by $u - M^+(u, r_0)$ which affects neither the boundary condition nor the conclusion $\underline{m} = \overline{m}$. In fact, since $M^+(u,r) = O(r^\lambda)$ the lemma shows that the integral converges. We have $Lh \geqq 0$ when $x \in \Omega$, $|x| \geqq r_0$, and $h$ is $\leqq 0$ on the boundary of this domain. Finally $M^+(h,r)r^{-1}$ tends to zero as $r \to \infty$. Hence, by the Phragmén–Lindelöf principle, $h \leqq 0$ in $\Omega$, $|x| \geqq r_0$. Now, since $\delta < \gamma$, by (iii) of the lemma

$$C^{-1} u(x) \leqq \int_{r_0}^{\infty} P(x,y) M^+(u,|y|)\, dy \qquad \text{if } r_0 \text{ is large} .$$

Here the limits of the integral are those of $|y|$.

Next, let

$$m_R = \max m(r) \qquad \text{for } r_0 \leqq r \leqq R$$

and choose $x = x(R)$, necessarily with $|x| \leqq R$, such that

$$u(x) = m_R |x|^\lambda .$$

Since, by the definition of $C$,

$$C^{-1}|x|^\lambda \geqq \int_0^\infty P(x,y)|y|^\lambda\, dy ,$$

we then have

$$m_R \int_{r_0}^{\infty} P(x,y)|y|^\lambda\, dy \leqq \int_{r_0}^{\infty} P(x,y) m(|y|)|y|^\lambda\, dy .$$

With $\varrho$ to be determined later we estimate the right side as follows

$$\begin{aligned} m(|y|) &\leqq m^* && \text{when } |y| \geqq R \\ m(|y|) &\leqq (R/\varrho)^\lambda m(R) && \text{when } \varrho \leqq |y| \leqq R \\ m(|y|) &\leqq m_R && \text{when } r_0 \leqq |y| \leqq \varrho . \end{aligned}$$

The second of these estimates follows since $M^+(u,r)$ increases. Putting $\omega = P(x,y)|y|^\lambda\, dy$, this gives

$$m_R \int_{r_0}^{\infty} \omega \leqq m_R \int_{r_0}^{\varrho} \omega + (R/\varrho)^\lambda m(R) \int_{\varrho}^{R} \omega + m^* \int_R^\infty \omega ,$$

so that

$$(m_R-(R/\varrho)^{\lambda}m(R))\int_{\varrho}^{R}\omega \leqq (m^*-m_R)\int_{R}^{\infty}\omega .$$

It is easy to see that if

$$|x| \leqq \varrho, \quad \varrho \leqq aR \tag{28}$$

for some $a<1$, then the quotient $\int_R^\infty \omega/\int_\varrho^R \omega$ is bounded. Since $m_R \to m^*$ as $R \to \infty$, (28) implies

$$m^* \leqq \varliminf_{R\to\infty} (R/\varrho)^{\lambda}m(R) .$$

Now, suppose that $\underline{m}<m^*$ and choose $R \to \infty$ so that $m(R) \to \underline{m}$. Then (28) is satisfied with $\varrho=aR$ and $a$ so close to 1 that, for large $R$, $m(R)<m_R a^{\lambda}$. Thus we get $m^* \leqq \underline{m}/a^{\lambda}$ for all $a<1$, a contradiction. Hence $\underline{m}=m^*$ and it follows that $\underline{m}=\overline{m}$.

If $\underline{m}=0$, (27) still holds with $R$ chosen so that $M^+(R)R^{-\lambda}=\min(M^+(r)r^{-\lambda})$, $r_0 \leqq r \leqq R$. Thus we obtain $u \leqq 0$.

Remark. If the assumptions on $L$ and $u$ are satisfied only for $|x| \geqq$ some $R_0$ the conclusion of the theorem reads that either

$$u(x) \leqq M^+(u,R_0) \quad \text{when } |x| \geqq R_0$$

or else

$$\lim_{r\to\infty} M^+(u,r) \text{ exists and is positive .}$$

Finally we give an example where $\lim_{r\to\infty} M^+(u,r)r^{-\lambda}$ is finite and positive. Assume $0<\lambda<1$, $0<\delta<\lambda$ and let

$$u(x) = \int_0^\infty P(x,y)(|y|^{\lambda}-|y|^{\lambda-\delta/2})\,dy ,$$

which is harmonic in $\Omega$. Put $x^0=(0,\ldots,|x|)$ and $C(t)=C(t,n)$. Then

$$\begin{aligned} M^+(u,|x|) \geqq u(x^0) &= C(\lambda)^{-1}|x|^{\lambda}-C(\lambda-\delta/2)^{-1}|x|^{\lambda-\delta/2} \\ &= C(\lambda)^{-1}|x|^{\lambda}(1-a|x|^{-\delta/2}) , \end{aligned}$$

where $a=C(\lambda)C(\lambda-\delta/2)^{-1}<1$, since $C(t)$ is decreasing with respect to $t$. At $\partial\Omega$

$$u(y) = |y|^{\lambda}-|y|^{\lambda-\delta/2} .$$

Thus we have $u(y) \leqq C(\lambda)(1-|y|^{-\delta})M^+(u,|y|)$ if

$$|y|^\lambda - |y|^{\lambda-\delta/2} \leqq |y|^\lambda(1-a|y|^{-\delta/2})(1-|y|^{-\delta})$$

or

$$(1-a|y|^{-\delta/2})|y|^{-\delta} \leqq (1-a)|y|^{-\delta/2} ,$$

which is satisfied if $|y|$ is large. So the assumptions of theorem 2 are fulfilled. That $M^+(u,r)r^{-\lambda} \to C(\lambda)^{-1}$ as $r \to \infty$ is seen directly.

## REFERENCES

1. B. Dahlberg, *Growth properties of subharmonic functions*, thesis, University of Göteborg, 1971.
2. D. Drasin and D. F. Shea, *Convolution inequalities, regular variation and exceptional sets*, J. d'Analyse Math. 23 (1976), 232–293.
3. M. Essén and J. L. Lewis, *The generalized Ahlfors–Heins theorem in certain d-dimensional cones*, Math. Scand. 33 (1973), 113–129.
4. D. Gilbarg, *The Phragmén–Lindelöf theorem for elliptic partial differential equations*, J. Rational Mech. Anal. 1 (1952), 411–417.
5. M. Heins, *On the Phragmén–Lindelöf principle*, Trans. Amer. Math. Soc. 60 (1946), 238–244.
6. U. Hellsten, B. Kjellberg and F. Norstad, *Subharmonic functions in a circle*, Ark. Mat. 8 (1970), 185–193.
7. E. Hopf, *Remarks on the preceding paper of D. Gilbarg*, J. Rational Mech. Anal. 1 (1952), 419–424.
8. B. Kjellberg, *A theorem on the minimum modulus of entire functions*, Math. Scand. 12 (1963), 5–11.
9. J. L. Lewis, *Subharmonic functions in certain regions*, Trans. Amer. Math. Soc. 167 (1972), 191–201.
10. K. Miller, *Extremal barriers on cones with Phragmén–Lindelöf theorems and other applications*, Ann. Mat. Pura Appl. 90 (1971), 297–329.
11. M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Prentice–Hall, Englewood Cliffs, N.J., 1967.
12. J. B. Serrin, *On the Phragmén–Lindelöf principle for elliptic differential equations*, J. Rational Mech. Anal. 3 (1954), 395–413.
13. J. B. Serrin, *On the Harnack inequality for linear elliptic equations*, J. d'Analyse Math. 4 (1955/56), 292–308.

MATEMATISKA INSTITUTIONEN
BOX 725
S-22007 LUND
SWEDEN