LUND UNIVERSITY

# A rate R=5/20 hypergraph-based woven convolutional code with free distance 120

Hug, Florian; Bocharova, Irina; Johannesson, Rolf; Kudryashov, Boris

*Published in:*
IEEE Transactions on Information Theory

*DOI:*
10.1109/TIT.2010.2040966

2010

Link to publication

*Citation for published version (APA):*
Hug, F., Bocharova, I., Johannesson, R., & Kudryashov, B. (2010). A rate R=5/20 hypergraph-based woven convolutional code with free distance 120. *IEEE Transactions on Information Theory*, *56*(4), 1618-1623. https://doi.org/10.1109/TIT.2010.2040966

*Total number of authors:*
4

**General rights**
Unless other specific re-use rights are stated the following general rights apply:
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.
- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Read more about Creative commons licenses: https://creativecommons.org/licenses/

**Take down policy**
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

Download date: 16. Jul. 2024

IEEE COPYRIGHT NOTICE

©2010 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works.

This material is presented to ensure timely dissemination of scholarly and technical work. Copyright and all rights therein are retained by authors or by other copyright holders. All persons copying this information are expected to adhere to the terms and constraints invoked by each authors copyright. In most cases, these works may not be reposted without the explicit permission of the copyright holder.

*Last Update: April 25, 2010*

# A Rate $R = 5/20$ Hypergraph-Based Woven Convolutional Code with Free Distance 120

Florian Hug, *Student Member, IEEE*, Irina E. Bocharova,
Rolf Johannesson, *Fellow, IEEE*, and Boris D. Kudryashov

***Abstract*—A rate $R = 5/20$ hypergraph-based woven convolutional code with overall constraint length $67$ and constituent convolutional codes is presented. It is based on a $3$-partite, $3$-uniform, $4$-regular hypergraph and contains rate $R^{c} = 3/4$ constituent convolutional codes with overall constraint length $5$. Although the code construction is based on low-complexity codes, the free distance of this construction, computed with the BEAST algorithm, is $d_{\rm free} = 120$, which is remarkably large.**

***Index Terms*—BEAST, convolutional codes, graphs, graph codes, hypergraphs, tailbiting codes, woven codes.**

## I. Introduction

THE idea of constructing good long codes from short ones is well developed in the theory of block codes (see [1] and references therein). Product codes [2], concatenated codes [3], and generalized concatenated codes [4] are constructions often referred to. It is also well-known that, for example, among the generalized concatenated codes there exist near-optimum codes since this class contains codes achieving the Varshamov-Gilbert bound [5, Ch. 1].

The same idea applied to convolutional codes leads to so-called turbo-codes (parallel concatenated convolutional codes), serial concatenated convolutional codes [6], [7], and woven convolutional codes (with inner or outer wraps [8]). Turbo codes appear in practical applications due to their good bit error rate (BER)/complexity tradeoff. However, not much is known about the distances of these codes. Moreover, it is clear that the free distance/delay tradeoff for these codes is far from the asymptomatic lower bounds [9, Ch. 3].

Recently, several graph- and hypergraph-based asymptotically optimum code constructions were found [10], [11]. It is shown in [12], [13] that woven hypergraph-based codes represent classes of asymptotically good block and convolutional codes satisfying the Varshamov-Gilbert and Costello bounds on the minimum and free distances, respectively.

Woven block codes based on $s$-partite, $s$-uniform, $c$-regular hypergraphs containing constituent block codes whose block length is equal to $lc$, where $l$ is an integer, were introduced in [12]. In particular, when $l$ tends to infinity, we obtain hypergraph-based woven convolutional codes containing convolutional constituent codes.

In Section II, we describe a two-dimensional construction of a hypergraph-based woven convolutional code that is a tailbiting (TB) block code in one dimension and a convolutional code in the other. The problem of finding its free distance is considered in Section III. Section IV concludes the paper with some final remarks.

## II. Code construction

Hypergraph-based woven convolutional codes [12], [13] are based on $s$-partite, $s$-uniform, $c$-regular hypergraphs and contain constituent convolutional codes of rate $R^{c} = b/c$, while their rate is given by $R = 1 - s(1 - R^{c})$. Moreover, they can be described as two-dimensional (2-D) convolutional codes which are tailbitten in one dimension.

The concrete code example which we study here is based on the 3-partite, 3-uniform, 4-regular hypergraph as illustrated in Fig. 1, with the incidence matrix given by (1).

Each edge in Fig. 1 corresponds to a column in (1), whereas the vertices are represented by the rows. As each edge in this hypergraph connects three different vertices (3-uniform), there are exactly three ones in each column. These three vertices, connected by the same edge, belong to three different, nonintersecting sets, as the hypergraph is 3-partite. The hypergraph is chosen such that, for each set of five rows of its incidence matrix (1), the rows are quasi-cyclic shifts of each other. These different sets are separated by dashed horizontal lines in (1). Moreover, since each vertex is connected to four edges, the graph is $4$-regular.

Hereinafter we will consider (1) as a parity-check matrix of the corresponding hypergraph-based code. The 20 edges of the graph represent 20 code symbols and each vertex represents a parity check. In the woven code construction, for each vertex the $4$-tuples corresponding to the hypergraph edges represent the "branches" of the constituent convolutional codes.

Manuscript received June 27, 2008. Current version published March 17, 2010. This work was supported in part by the Swedish Research Council by Grant 621-2007-61281.

F. Hug and R. Johannesson are with the Department of Electrical and Information Technology, Lund University, SE-22100 Lund, Sweden (e-mail: florian@eit.lth.se; rolf@eit.lth.se).

I. E. Bocharova and B. D. Kudryashov are with the Department of Information Systems, St. Petersburg University of Information Technologies, Mechanics and Optics, St. Petersburg 197101, Russia (e-mail: irina@eit.lth.se; boris@eit.lth.se).

Communicated by L. M. G. M. Tolhuizen, Associate Editor for Coding Theory.

Digital Object Identifier 10.1109/TIT.2010.2040966

0018-9448/$26.00 ©2010 IEEE

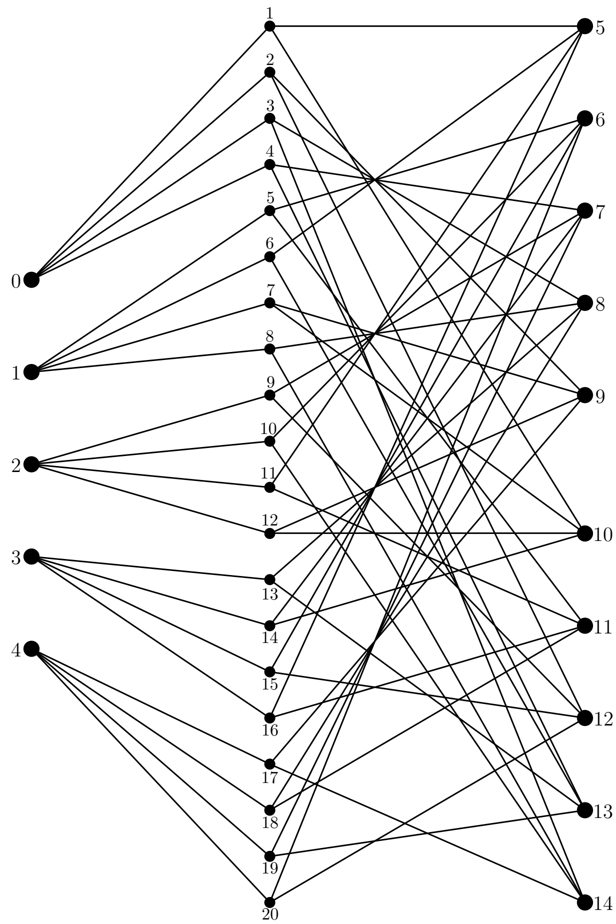


Fig. 1. A 3-partite, 3-uniform, 4-regular hypergraph.

By row permutations we obtain the parity-check matrix (2), which is equal to the binary representation of the parity-check matrix of the rate $R = 1/4$ convolutional code (3) terminated by tailbiting (TB) to obtain a $(20, 7)$ linear block code. [See (2) at the bottom of the next page.] [Notice that the rows of (2) are linearly dependent which yields the block code rate $R = 7/20 > 1/4$.]

$$
\begin{aligned}
H(Z) &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} Z \\
&\quad + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} Z^2 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} Z^3 \\
&= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & Z & Z^2 & Z^3 \\ 1 & Z^3 & Z & Z^2 \end{pmatrix}. \qquad (3)
\end{aligned}
$$

In order to obtain a 2-D rate $R = 5/20$ convolutional code, up to three rate $R^{\mathrm{c}} = 3/4$ constituent convolutional codes are specified by their parity-check matrices (since this construction is based on a 3-partite hypergraph; we use three convolutional codes per hyperedge):

$$H(D) = \left( h_1(D) \;\; h_2(D) \;\; h_3(D) \;\; h_4(D) \right) \qquad (4)$$
$$T(D) = \left( t_1(D) \;\; t_2(D) \;\; t_3(D) \;\; t_4(D) \right) \qquad (5)$$
$$Q(D) = \left( q_1(D) \;\; q_2(D) \;\; q_3(D) \;\; q_4(D) \right). \qquad (6)$$

Such a 2-D convolutional code can be specified by its parity-check matrix

$$H(D, Z) = \{h_{ij}(D, Z)\}, i = 1, 2, \ldots, c - b, j = 1, 2, \ldots, c$$

where $h_{ij}(D, Z)$ are polynomials over the formal variables $D$ and $Z$. Combining the parent hypergraph-based code with parity-check matrix (3) and the constituent convolutional codes with parity-check matrices (4)-(6), we obtain a rate

$$
H_{\mathrm{hg}} = \begin{array}{cc}
 & \begin{array}{cccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} &
\left(\begin{array}{cccc|cccc|cccc|cccc|cccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right)
\end{array} \qquad (1)
$$

$R = 1/4$ 2-D hypergraph-based woven convolutional code with parity-check matrix

$$\begin{aligned} H(D,Z) &= \begin{pmatrix} h_1(D) & h_2(D) & h_3(D) & h_4(D) \\ t_1(D) & t_2(D)Z & t_3(D)Z^2 & t_4(D)Z^3 \\ q_1(D) & q_2(D)Z^3 & q_3(D)Z & q_4(D)Z^2 \end{pmatrix} \\ &= H_3(D) + H_2(D)Z + H_1(D)Z^2 + H_0(D)Z^3 \end{aligned} \quad (7)$$

where

$$\begin{aligned} H_3(D) &= \begin{pmatrix} h_1(D) & h_2(D) & h_3(D) & h_4(D) \\ t_1(D) & 0 & 0 & 0 \\ q_1(D) & 0 & 0 & 0 \end{pmatrix} \\ H_2(D) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t_2(D) & 0 & 0 \\ 0 & 0 & q_3(D) & 0 \end{pmatrix} \\ H_1(D) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & t_3(D) & 0 \\ 0 & 0 & 0 & q_4(D) \end{pmatrix} \\ H_0(D) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_4(D) \\ 0 & q_2(D) & 0 & 0 \end{pmatrix}. \end{aligned}$$

In our example, $T(D)$ and $Q(D)$ are column permutations of $H(D)$, namely,

$$T(D) = \begin{pmatrix} h_1(D) & h_3(D) & h_4(D) & h_2(D) \end{pmatrix} \quad (8)$$

$$Q(D) = \begin{pmatrix} h_3(D) & h_4(D) & h_1(D) & h_2(D) \end{pmatrix} \quad (9)$$

where

$$\begin{aligned} h_1(D) &= 1 + D + D^2 + D^4 \\ h_2(D) &= 1 + D + D^2 + D^3 + D^4 \\ h_3(D) &= 1 + D + D^3 + D^5 \\ h_4(D) &= 1 + D^2 + D^5. \end{aligned}$$

*Remark*. The parity-check matrix (7), written in reciprocal form, is equal to the generator matrix $G^{\perp}(D,Z)$ of the dual code [9, Ch. 2].

The free distances of the two codes with parity-check matrices (3) and (4) are equal to 8 and 5, respectively. In the sequel we will show that, although both the parent and the constituent convolutional codes have rather small free distances, our construction yields a convolutional code with remarkably large free distance.

A straightforward use of a 2-D convolutional code determined by (7) requires that the information sequence is organized in the form of a 2-D array, semi-infinite in both dimensions. We choose, however, to tailbite the 2-D convolutional code in the $Z$-dimension and obtain the following parity-check matrix $H_{\mathrm{wg}}(D)$ for our rate $R = 5/20$ hypergraph-based woven convolutional code,

$$H_{\mathrm{wg}}(D) = \begin{pmatrix} H_3(D) & H_2(D) & H_1(D) & H_0(D) & \mathbf{0} \\ \mathbf{0} & H_3(D) & H_2(D) & H_1(D) & H_0(D) \\ H_0(D) & \mathbf{0} & H_3(D) & H_2(D) & H_1(D) \\ H_1(D) & H_0(D) & \mathbf{0} & H_3(D) & H_2(D) \\ H_2(D) & H_1(D) & H_0(D) & \mathbf{0} & H_3(D) \end{pmatrix}$$

where $\mathbf{0}$ denotes the all-zero matrix of size $3 \times 4$.

By Gaussian elimination we obtain the corresponding generator matrix of this convolutional code. The overall constraint length of this generator matrix is equal to 75 but it is catastrophic and, thus, nonminimal. We apply the Smith form decomposition in order to obtain an equivalent basic encoding matrix, which can be reduced to its minimal-basic form (Algorithm MB [9, Ch. 2]) with an overall constraint length of 67 and it is given by

$$G_{\mathrm{wg}}(D) = \begin{pmatrix} G_0(D) & G_1(D) & G_2(D) & G_3(D) & G_4(D) \\ G_4(D) & G_0(D) & G_1(D) & G_2(D) & G_3(D) \\ G_3(D) & G_4(D) & G_0(D) & G_1(D) & G_2(D) \\ G_2(D) & G_3(D) & G_4(D) & G_0(D) & G_1(D) \\ G_5(D) & G_5(D) & G_5(D) & G_5(D) & G_5(D) \end{pmatrix}$$

where (in our octal notation 64 corresponds to $1+D+D^3$ [9])

$$\begin{aligned} G_0 &= \begin{pmatrix} 1473 & 40453 & 16256 & 62224 \end{pmatrix} \\ G_1 &= \begin{pmatrix} 44364 & 50324 & 36077 & 30173 \end{pmatrix} \\ G_2 &= \begin{pmatrix} 53717 & 4266 & 30434 & 32352 \end{pmatrix} \\ G_3 &= \begin{pmatrix} 37464 & 14262 & 6517 & 71254 \end{pmatrix} \end{aligned}$$

$$H_{\mathrm{hg}} = \begin{array}{cc} & \begin{array}{cccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \end{array} \\ \begin{array}{c} 0 \\ 5 \\ 10 \\ 1 \\ 6 \\ 11 \\ 2 \\ 7 \\ 12 \\ 3 \\ 8 \\ 13 \\ 4 \\ 9 \\ 14 \end{array} & \left(\begin{array}{cccc|cccc|cccc|cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right) \end{array} \quad (2)$$

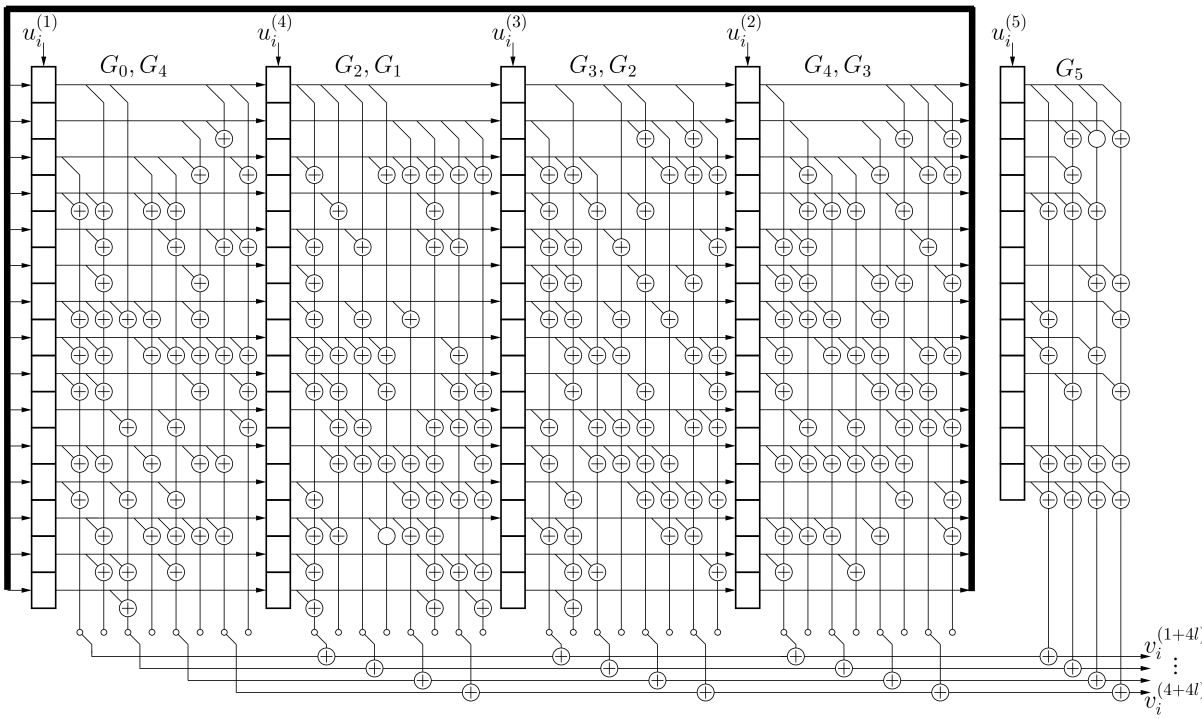


Fig. 2. Implementation of an R=5/20 hypergraph-based woven convolutional encoder.

$$G_4 = \begin{pmatrix} 47726 & 14624 & 31724 & 5234 \end{pmatrix}$$
$$G_5 = \begin{pmatrix} 4463 & 7413 & 6523 & 6153 \end{pmatrix}.$$

An implementation of this encoder is illustrated in Fig. 2. The input 5-tuple $u_i^{(1)}, u_i^{(2)}, \ldots, u_i^{(5)}$, $i = 0, 1, 2, \ldots$, enters the encoder every fifth clock pulse. The output connections with modulo-2 adders of each of the registers 1-4 (counted from left to right) are time-varying and determined by the matrices $G_0, G_1, \ldots, G_4$. The connections of register 5 are time-invariant and are determined by the matrix $G_5$. During each round of five clock pulses we begin with four circular shifts and obtain the four output 4-tuples $v_i^{(1+4l)}, v_i^{(2+4l)}, \ldots, v_i^{(4+4l)}$, $l = 0, 1, 2, 3$, by adding the 4-tuples of the outputs from the registers 1-4 to the 4-tuples of outputs from register 5. After these four circular shifts, $u_i^{(1)}$ is back at register 1 and the 4-tuple $v_i^{(1+4l)}, v_i^{(2+4l)}, \ldots, v_i^{(4+4l)}$ is generated for $l = 4$. The corresponding time-varying connections are described by Table I.

All registers can be considered as enlarged delay elements of the encoder of a TB code. After one clock cycle of five clock pulses a 20-tuple, $v_i^{(1)}, v_i^{(2)}, \ldots, v_i^{(20)}$, of the rate $R = 5/20$ convolutional code is generated. Then we shift a new input 5-tuple into the five registers without a circular shift and the next 20-tuple of output symbols is generated similarly.

## III. Free Distance

To evaluate the free distance we used the BEAST algorithm [14], which is, to the best of our knowledge, the most powerful existing tool for computing distance spectra of convolutional codes. However, even for this algorithm the overall constraint

| | Register | | | |
|---|---|---|---|---|
| clock pulse | 1 | 2 | 3 | 4 |
| 1 | $G_0$ | $G_2$ | $G_3$ | $G_4$ |
| 2 | $G_0$ | $G_1$ | $G_3$ | $G_4$ |
| 3 | $G_0$ | $G_1$ | $G_2$ | $G_4$ |
| 4 | $G_0$ | $G_1$ | $G_2$ | $G_3$ |
| 5 | $G_4$ | $G_1$ | $G_2$ | $G_3$ |

TABLE I
Time-Varying Connections.

length 67 seemed to be too large. Therefore the task was a challenge.

We started by finding different upper bounds on the free distance. According to the Griesmer bound for convolutional codes, the free distance for *any* binary, rate $R = b/c$ convolutional code encoded by a minimal-basic encoding matrix of memory $m$ satisfies [9, Ch. 3]

$$\sum_{j=0}^{bi-1} \left\lceil \frac{d_{\text{free}}}{2^j} \right\rceil \leq (m+i)c \tag{10}$$

for $i = 1, 2, \ldots$. Applying (10) to any rate $R = 5/20$, memory $m = 14$ convolutional code, we obtain $d_{\text{free}} \leq 154$. For *particular* convolutional codes we can strengthen the bound (10) as follows. Consider our convolutional code with encoding matrix

$$G_{\text{wg}}(D) = G_{\text{wg},0} + G_{\text{wg},1}D + \cdots + G_{\text{wg},m}D^m$$

or, written as a semi-infinite binary matrix,

$$\boldsymbol{G}_{\mathrm{wg}} = \begin{pmatrix} G_{\mathrm{wg},0} & G_{\mathrm{wg},1} & \dots & G_{\mathrm{wg},m} & & \\ & G_{\mathrm{wg},0} & G_{\mathrm{wg},1} & \dots & G_{\mathrm{wg},m} & \\ & & \ddots & \ddots & & \ddots \end{pmatrix}$$

where $G_{\mathrm{wg},i}$, $i = 0, 1, \dots, m$, are $5 \times 20$ binary matrices. Then represent our woven hypergraph-based convolutional code in minimal span form [15], obtained by applying linear row operations to the encoding matrix given in minimal-basic form. Each row of this encoding matrix is reduced in such a way that it starts and ends at different positions than all other rows. This allows us to obtain a bound tighter than (10) since this matrix has the shortest possible span, that is, the shortest possible length of the nontrivial part of a row. Let $N$ denote the minimum span (support) of dimension $K$ subcodes, where $K = 1, 2, \dots$. Then we have the following upper bound on the free distance

$$d_{\mathrm{free}} \le \min_{K \ge 1} \{d(N, K)\}$$

where $d(N, K)$ denotes the Griesmer bound [5, Ch. 17] on the minimum distance of an $(N, K)$ linear block code, and $N$ is the number of columns of the $K \times N$ submatrix of the minimal-span form of $\boldsymbol{G}_{\mathrm{wg}}$. For our woven hypergraph-based convolutional code, we obtain the minimum value 150 for $K = 6$ and $\mathrm{span} = N = 298$; for these values of $K$ and $N$ the Griesmer bound is achievable [5, Ch.17. Th. 25]. Thus, by using this approach we conclude that $d_{\mathrm{free}} \le 150$.

Another approach is based on the row distances of convolutional encoders. It is well-known that the row distances obtained for convolutional encoders are upper bounds on the free distances and that they can be used very efficiently to reduce the set of promising candidates when we are searching for convolutional encoders [9, Ch. 8]. For our encoder we obtain $d_0^r = d_1^r = d_2^r = 130$ and $d_3^r = \dots = d_6^r = 120$, thus, we have $d_{\mathrm{free}} \le 120$. The row-distance approach yields a much stronger upper-bound at the cost of rather heavy computer computations.

Finally, the BEAST algorithm is used for the code analysis. Finding the free distance for such a code would take prohibitively long time without using parallel computations on many processors. Using about 100 processors in parallel yields the free distance, $d_{\mathrm{free}} = 120$, of this woven hypergraph-based convolutional code, where the individual forward and backward sets of the BEAST algorithms are sorted and merged by individual processors. We thereby obtain that there exists only one codeword of weight 120 with corresponding length $(1+3+11)20 = 300$. Based on our experience obtained from studying less complex woven hypergraph-based convolutional codes, we conjecture that the next nonzero spectral component occurs at weight 130 (*cf.* the sequence of row distances shown above).

Costello's asymptotic lower bound on the free distance establishes the existence of a rate $R = b/c$, memory $m$ convolutional code in the ensemble of binary, periodically time-varying (period $T >> m$) convolutional codes having a free distance satisfying

$$\frac{d_{\mathrm{free}}}{mc} \ge \frac{R}{-\log_2(2^{1-R} - 1)} + O\left(\frac{\log_2 m}{m}\right).$$

For rate $R = 5/20$, the main term in Costello's asymptotic bound is 0.452 but for our convolutional code we have $d_{\mathrm{free}}/mc = 0.429$. However, from the derivation of Costello's lower bound in [9, Ch. 3] we borrow the following:

$$d_{\mathrm{free}} > \frac{-mb}{\log_2(2^{1-R} - 1)} - \frac{\log_2(2^R - 1) + \log_2\left(m^{-2} - 2^{\left(h(\frac{1}{m}) + R - 1\right)m^2 c}\right)}{\log_2(2^{1-R} - 1)}$$

which for the parameters $R = 5/20$ and memory $m = 14$ of our woven hypergraph-based convolutional code yields $d_{\mathrm{free}} \ge 109$.

## IV. Conclusions

In this paper we have described a construction of a rate $R = 5/20$ hypergraph-based woven convolutional code. It has memory $m = 14$ and its free distance is $d_{\mathrm{free}} = 120$. We specified a minimal-basic encoding matrix with an overall constraint length $\nu = 67$.

Constructing such a powerful convolutional code is interesting per se. Notice, that the best known convolutional code with rate $R = 1/4$ has overall constraint length $\nu = 15$ and free distance $d_{\mathrm{free}} = 40$ [14]. Clearly, convolutional codes with large distances can be constructed as the product of short codes, but the corresponding rate of such a construction is the product of the rates of the constituent codes. In contrary to this, the rate of our woven construction is determined by the rate of the underlying graph (hypergraph). However, whether we can decode such a convolutional code (suboptimally) with reasonable complexity is still an open question.

For verifying such a huge free distance, it was crucial to have an extremely powerful algorithm. We used the BEAST algorithm where the sorting and merging of the individual forward and backward sets were distributed among several processors, but we would also like to mention that by running the BEAST algorithm on a single laptop we could, in a few minutes, obtain that the free distance was at least 80. Tho verify $d_{\mathrm{free}} = 120$ was a task worthy (a) BEAST.

## Acknowledgment

This work was supported in part by the Swedish Research Council under Grant 621-2007-61281. The authors would like to thank Ludo Tolhuizen for his detailed comments on the manuscript and Lunarc, the Center for Scientific and Technical Computing at Lund University, for their generosity in making their computational cluster available.

## References

[1] I. Dumer, "Concatenated codes and their multilevel generalizations," in *Handbook of Coding Theory*, V. Pless and W. C. Huffman, Eds. Amsterdam: Elsevier, Nov. 1998, vol. 2, pp. 1911–1988.

[2] P. Elias, "Error-free coding," *IRE Trans. Inf. Theory*, pp. 29–37, 1954.

[3] G. D. Forney Jr., *Concatenated Codes*. Cambridge, MA: MIT Press, 1966.

[4] E. L. Blokh and V. V. Zyablov, *Concatenated Codes (in Russian)*. Moscow, U.S.S.R.: Nauka, 1982.

[5] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. Amsterdam: North-Holland, 1977.

[6] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near shannon limit error-correcting coding and decoding: Turbo codes," in *Proc. IEEE Int. Conf. Commun. (ICC '93)*, Geneva, Switzerland, May 1993, pp. 1063–1070.

[7] S. Benedetto, D. Divsalar, G. Montorsi, and F. Pollara, “Serial concatenation of interleaved codes: Performance analysis, design, and iterative decoding,” *IEEE Trans. Inf. Theory*, vol. 44, no. 3, pp. 909–926, May 1998.

[8] S. Höst, R. Johannesson, and V. V. Zyablov, “Woven convolutional codes I: Encoder properties,” *IEEE Trans. Inf. Theory*, vol. 48, no. 1, pp. 149–161, Jan. 2002.

[9] R. Johannesson and K. S. Zigangirov, *Fundamentals of Convolutional Coding*. Piscataway, NJ: IEEE Press, 1999.

[10] A. Barg and G. Zemor, “Distance properties of expander codes,” *IEEE Trans. Inf. Theory*, vol. 52, no. 1, pp. 78–90, Jan. 2006.

[11] I. E. Bocharova, B. D. Kudryashov, R. Johannesson, and V. V. Zyablov, “Asymptotically good woven codes with fixed constituent convolutional codes,” in *Proc. IEEE Int. Symp. Information Theory (ISIT ’07)*, Nice, France, 1997.

[12] I. E. Bocharova, R. Johannesson, B. D. Kudryashov, , and V. V. Zyablov, “Woven graph codes: Asymptotic performances and examples,” *IEEE Trans. Inf. Theory*, vol. 56, no. 1, pp. 121–129, Jan. 2010..

[13] I. E. Bocharova, R. Johannesson, B. D. Kudryashov, and V. V. Zyablov, “Woven graph codes over hypergraphs,” in *Proc. 7th Int. ITG Conf. Source and Channel Coding (SCC ’08)*, Ulm, Germany, Jan. 2008.

[14] I. Bocharova, M. Handlery, R. Johannesson, and B. Kudryashov, “A BEAST for prowling in trees,” *IEEE Trans. Inf. Theory*, vol. 50, no. 6, pp. 1295–1302, Jun. 2004.

[15] F. R. Kschischang and V. Sorokine, “On the trellis structure of block codes,” *IEEE Trans. Inf. Theory*, vol. 41, no. 6, pp. 1924–1937, Nov. 1995.

**Florian Hug** (S’08) was born in Memmingen, Germany, on May 21, 1983. He received his Dipl.-Ing. degree from the Faculty of Electrical Engineering, University of Ulm, Ulm, Germany, in 2008. Since then, he has been with the Department of Electrical and Information Technology, Lund University, Lund, Sweden, where he is working towards the Ph.D. degree in information theory. His research interests covers the field of information and coding theory. Currently, he is focusing on codes over graphs.

**Irina E. Bocharova** was born in Leningrad, U.S.S.R., on July 31, 1955. She received the Diploma in Electrical Engineering in 1978 from the Leningrad Electro-technical Institute and the Ph.D. degree in technical sciences in 1986 from Leningrad Institute of Aircraft Instrumentation.

Since 1986 until 2007, she has been a Senior Researcher, an Assistant Professor, and then Associate Professor at the Leningrad Institute of Aircraft Instrumentation (now State University of Aerospace Instrumentation, St.-Petersburg, Russia). Since 2007 she has been an Associate Professor at the State University of Information Technologies, Mechanics and Optics. Her research interests include convolutional codes, communication systems, source coding and its applications to speech, audio and image coding. She has more than 50 papers published in journals and proceedings of international conferences, and seven U.S. patents in speech, audio and video coding.

Professor Bocharova was awarded the Lise Meitner Visiting Chair in engineering at Lund University, Lund, Sweden (January–June 2005).

**Rolf Johannesson** (F’98) was born in Hassleholm, Sweden, on July 3, 1946. He received the M.S. and Ph.D. degrees in 1970 and 1975, respectively, both from Lund University, Lund, Sweden, and the degree of Professor, *honoris causa*, from the Institute for Information Transmission Problems, Russian Academy of Sciences, in 2000.

Since 1976, he has been a faculty member with Lund University where he has the Chair of Information Theory. From 1976 to 2003, he was Head of the Department of Electrical Engineering and Computer Sciences and during 1988-1993, was Dean of the School of Electrical Engineering and Computer Sciences. During 1990-1995, he served as a member of the Swedish Research Council for Engineering Sciences. His scientific interests include information theory, error-correcting codes, and cryptography. In addition to papers in the area of convolutional codes and cryptography, he has authored two textbooks on switching theory and digital design (both in Swedish) and one on information theory (in both Swedish and German) and coauthored *Fundamentals of Convolutional Coding* (New York: IEEE Press, 1999), and *Understanding Information Transmission* (Hoboken, NJ: IEEE Press/Wiley-Interscience, 2005).

Professor Johannesson has been an Associate Editor for the *International Journal of Electronics and Communications*. During 1983-1995 he co-chaired seven Russian-Swedish Workshops, which were the chief interactions between Russian and Western scientists in information theory and coding during the final years of the Cold War. He became an elected member of the Royal Swedish Academy of Engineering Sciences in 2006.

**Boris D. Kudryashov** was born in Leningrad, U.S.S.R., on July 9, 1952. He received the Diploma in Electrical Engineering in 1974 and the Ph.D in technical sciences degree in 1978 both from the Leningrad Institute of Aerospace Instrumentation, and the Doctor of Science degree in 2005 from Institute of Problems of Information Transmission, Moscow.

In 1978, he was an assistant professor and then Associate professor and Professor in the Leningrad Institute of Aerospace Instrumentation (now the State University on Aerospace Instrumentation, St.-Petersburg, Russia). Since November 2007, he has been a Professor with the State University on Information Technologies, Mechanics and Optics, St.-Petersburg, Russia. His research interests include coding theory, information theory and applications to speech, audio and image coding. He has more than 70 papers published in journals and proceedings of international conferences, 15 U.S. patents and published patent applications in image, speech and audio coding.

Professor Kudryashov served as a member of Organizing Committee of ACCT International Workshops.