An Analytic Solution to Fixed-Time Point-to-Point Trajectory Planning

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An Analytic Solution to Fixed-Time Point-to-Point Trajectory Planning

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***Abstract*— We derive an analytic solution to the problem of fixed-time trajectory generation with a quadratic cost function under velocity and acceleration constraints. This problem has a wide range of applications within motion planning. The advantage of the analytic solution compared to numerical optimization of the discretized problem is the unlimited resolution of the solution and the efficiency of the calculation, allowing sensor-based replanning and on-line trajectory generation.**

I. INTRODUCTION

A desired characteristic of motion planning in uncertain environments is the ability to react to sensor inputs [1]. Motion replanning based on sensor information requires algorithms that can quickly generate a motion trajectory. The concept of on-line trajectory generation, *i.e.*, updating the trajectory in each control cycle, is discussed in [2]. There are many methods for on-line trajectory generation with the objective of time-optimality based on analytic expressions [3], [4], [5], [1]. The difference between these methods lies in their generality with regard to the constraints and the initial and final states of the desired motion.

Using the existing analytic solutions, time-optimal or nearly time-optimal trajectories can be computed extremely fast. While the minimum-time trajectories are of interest for defining an upper bound for the productivity of a robotic system, they tend to maximize the wear of the system. In practice, other factors such as coordination between different units often determine the required time. Hence, the solution to fixed-time problems with a cost function motivated by the application can prove valuable by reducing the wear of the robotic system. A common approach to fixed-time problems is fitting a piece-wise polynomial between the starting point and a final point [6], [7], [8] without considering an explicit cost function. Optimizing the energy or power consumption [9] or the effort [10] was suggested in other approaches. The solutions were obtained by numerical methods either by discretization or optimizing parameters over a set of basis functions. A sub-optimal solution to the fixed-time trajectory planning considering a more generic cost function was proposed in [11].

In this article, we consider the fixed-time trajectory-generation problem with a quadratic cost under velocity and acceleration constraints. The resulting trajectory can for example be used in pick-and-place tasks to transfer the current state (position and velocity) of a manipulator to a new one in a given time. The purpose is to find a computationally efficient solution, such that a new trajectory can be computed quickly if it is called upon by new sensor measurments. We cast this problem into the framework of optimal control with state variable inequality constraints (SVIC). To find the solution, we use the direct adjoining approach described in [12], [13], which is based on the maximum principle [14]. This approach leads to a system of equations that determines a set of parameters for an analytic solution. The system of equations is solved numerically and the resulting trajectories are compared with the numerical solution obtained by discretizing the model and using an interior-point method [15].

A. Problem Formulation

Since we are concerned with kinematic variables, specifically constraining the velocity and acceleration, a double integrator is a sufficient model for each degree of freedom (DoF). This implies that we assume that the acceleration of each DoF can be independently controlled. The robotic system has an initial error with respect to a desired final state (a certain position at rest), which is supposed to become zero at a given fixed time. The error of the position denoted by x_1 is unconstrained but there are constraints on both the acceleration u and on the velocity x_2. We assume a quadratic cost function specified by $R > 0$ and a diagonal matrix $Q := \operatorname{diag}(q_1, q_2) > 0$. The problem can be compactly written as:

$$\underset{u}{\text{minimize}} \quad \int_0^{t_f} x^T Q x + u^T R u \, dt \tag{1}$$

$$\text{subject to} \quad \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \tag{2}$$

$$|u(t)| \leq 1 \tag{3}$$

$$|x_2(t)| \leq c \tag{4}$$

$$x(0) = \begin{pmatrix} x_0, & v_0 \end{pmatrix}^T \quad x(t_f) = 0_{2\times 1} \tag{5}$$

II. PRELIMINARIES

Following the presentation of [12], [13], the control problem with SVIC is specified by an objective functional J to be maximized subject to constraints on the states and the control signal:

$$\begin{aligned} J(u) &= \int_0^{t_f} F(x(t), u(t), t) dt + S(x(t_f), t_f) \\ \dot{x}(t) &= f(x(t), t), \quad x(0) = x_i \end{aligned} \tag{6}$$

$$\begin{aligned} g(x(t), u(t), t) &\geq 0 \\ h(x(t), t) &\geq 0 \\ a(x(t_f), t_f) &\geq 0 \\ b(x(t_f), t_f) &= 0, \end{aligned} \tag{7}$$

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where

$$h:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^q,\qquad g:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^s \tag{8}$$

$$a:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^\ell,\qquad b:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^{\ell'}. \tag{9}$$

A. General Definitions and Conditions

Following [13], the order of pure state constraints as well as junction times are defined here. Moreover, a constraint qualification condition is presented [13]. Note that when two symbols appear after each other, depending on the dimensions, dot product or matrix multiplication is intended.

1) Order of Pure State Constraints:

$$h^0(x,u,t)=h=h(x,t)$$

$$h^1(x,u,t)=\dot{h}=h_x(x,t)f(x,u,t)+h_t(x,t)$$

$$\vdots$$

$$h^p(x,u,t)=h^{(p)}=h_x^{p-1}(x,t)f(x,u,t)+h_t^{p-1}(x,t) \tag{10}$$

The state constraint is of order p iff

$$h_u^i(x,u,t)=0,\ \text{for}\ 0\le i\le p-1,\quad h_u^p(x,u,t)\neq 0, \tag{11}$$

2) Junction Times: With respect to the ith constraint, an interval $[\tau_1,\tau_2]$ is called a *boundary interval* if $h_i(x(t),t)=0$ for all $t\in[\tau_1,\tau_2]$. A subinterval $(\tau_1,\tau_2)\subset[0,t_f]$ is called an *interior interval* of a trajectory $x(\cdot)$ if $h_i(x(t),t)>0$ for all $t\in(\tau_1,\tau_2)$. If an interior interval ends at τ_1 and a boundary interval starts at τ_1, the instant τ_1 is called an *entry time*. Correspondingly, τ_2 is called an *exit time* if there is a boundary interval ending at $t=\tau_2$ and an interior interval starting at τ_2. A *contact time* is the instant that the trajectory just touches the boundary, *i.e.*, $h(x(\tau),\tau)=0$ and the trajectory is in the interior just before and after τ. Entry, exit, and contact times are called *junction times*.

3) Constraint Qualification: The constraint qualification for terminal constraints requires

$$\operatorname{rank}\begin{bmatrix}\partial a/\partial x & \operatorname{diag}(a)\\ \partial b/\partial x & 0\end{bmatrix}=\ell+\ell'. \tag{12}$$

Additionally, for mixed constraints, *i.e.*, the constraints involving both states and input signals, we require

$$\operatorname{rank}[\partial g/\partial u\quad \operatorname{diag}(g)]=s. \tag{13}$$

B. Direct Adjoining approach

In this approach the mixed constraints as well as the pure constraints are directly adjoined to the Hamiltonian H to form the Lagrangian L. The Hamiltonian and the so called D-form Lagrangian are defined as [13]

$$H(x,u,\lambda_0,\lambda,t)=\lambda_0 F(x,u,t)+\lambda f(x,u,t) \tag{14}$$

$$L(x,u,\lambda_0,\lambda,\mu,\nu,t)=H+\mu g(x,u,t)+\nu h(x,u,t). \tag{15}$$

The costate is a mapping $\lambda(\cdot):[0,t_f]\to\mathbb{R}^n$ and multiplier functions $\mu(\cdot)$ and $\nu(\cdot)$ are mappings from $[0,t_f]$ into $\mathbb{R}^s$ and $\mathbb{R}^q$, respectively. The control region is defined as:

$$\Omega(x,t)=\{u\in\mathbb{R}^m\,|\,g(x,u,t)\ge 0\} \tag{16}$$

Theorem 1 (Informal Theorem 4.1 from [13]): Let $\{x^*(\cdot),u^*(\cdot)\}$ be an optimal pair for problem (6)–(9) over $[0,t_f]$ such that

- $u^*(\cdot)$ is right-continuous with left-hand limits,
- the constraint qualification holds for every tripple $\{t,x^*(t),u\},t\in[0,t_f],u\in\Omega(x^*(t),t)$;
- Assume $x^*(t)$ has only finitely many junction times.

Then there exists

- a constant $\lambda_0\ge 0$,
- a piecewise absolutely continuous costate trajectory $\lambda(\cdot)$ mapping $[0,t_f]$ into $\mathbb{R}^n$,
- a piecewise continuous multiplier functions $\mu(\cdot)$ and $\nu(\cdot)$ mapping $[0,t_f]$ into $\mathbb{R}^s$ and $\mathbb{R}^q$, respectively,
- a vector $\eta(\tau_i)\in\mathbb{R}^q$ for each point τ_i of discontinuity of $\lambda(\cdot)$,
- $\alpha\in\mathbb{R}^\ell$ and $\beta\in\mathbb{R}^{\ell'}$, $\gamma\in\mathbb{R}^q$, not all zero,

such that the following conditions hold almost everywhere: Hamiltonian maximization

$$u^*(t)=\underset{u\in\Omega(x^*(t),t)}{\arg\max}\ H(x^*(t),u,\lambda_0,\lambda(t),t) \tag{17}$$

and conditions on the optimal Hamiltonian and Lagrangian, costates and multipliers

$$L_u^*[t]=H_u^*[t]+\mu g_u^*[t]=0 \tag{18}$$

$$\dot{\lambda}(t)=-L_x^*[t] \tag{19}$$

$$\mu(t)\ge 0,\quad \mu(t)g^*[t]=0 \tag{20}$$

$$\nu(t)\ge 0,\quad \nu(t)h^*[t]=0 \tag{21}$$

$$\text{and}\quad dH^*[t]/dt=dL^*[t]/dt=L_t^*[t]:=\partial L^*[t]/\partial t; \tag{22}$$

At the terminal time t_f, the transversality conditions hold:

$$\lambda(t_f^-)=\lambda_0 S_x^*[t_f]+\alpha a_x^*[t_f]+\beta b_x^*[t_f]+\gamma h_x^*[t_f] \tag{23}$$

$$\alpha\ge 0,\ \gamma\ge 0,\quad \alpha a^*[t_f]=\gamma h^*[t_f]=0; \tag{24}$$

For any time τ in the boundary interval and for any contact time τ, the costate trajectory may have a discontinuity given by the following conditions

$$\lambda(\tau^-)=\lambda(\tau^+)+\eta(\tau)h_x^*[\tau] \tag{25}$$

$$H^*[\tau^-]=H^*[\tau^+]-\eta(\tau)h_t^*[\tau] \tag{26}$$

$$\eta(\tau)\ge 0,\quad \eta(\tau)h^*[\tau]=0. \tag{27}$$

Remark 2.1: If the control signal appears linearly, formal proofs for the necessity of the conditions are available [16].

III. Analytic Solution

In this section, we apply the direct adjoining approach to our problem. First, the variables and parameters are identified with the general problem (6)–(9). Then, we evaluate one by one the different conditions imposed on the solution by Theorem 1. From the Hamiltonian maximization, we find out that the solution is divided into several regions. The regions are determined by active constraints. In each region, the conditions are evaluated and expressions for the control signal, states, costates, and multipliers are derived. Furthermore, the conditions at the boundaries and the continuity of the control signal and states are considered.

Depending on the problem, various scenarios may arise in which no constraint, one of the constraints or all become active. For the sake of brevity of this article, we just present the scenario where both the constraints on the control signal and the state become active. The solution to other scenarios can be derived similarly.

A. Parameters, Hamiltonian and Lagrangian

By comparing (1)–(5) with (6)–(7), we identify

$$F\left(x(t),u(t),t\right) = -(x^TQx + u^TRu) \tag{28}$$

$$S(t_f) = 0 \tag{29}$$

$$f_1\left(x(t),u(t),t\right) = x_2(t) \tag{30}$$

$$f_2\left(x(t),u(t),t\right) = u(t). \tag{31}$$

In addition, from (14) and (15)

$$\begin{aligned} H = &-\lambda_0\left(q_1x_1^2(t) + q_2x_2^2(t) + ru^2(t)\right) \\ &+ \lambda_1(t)x_2(t) + \lambda_2(t)u(t) \end{aligned} \tag{32}$$

$$\begin{aligned} L = &\, H + \mu_1(t)(1-u(t)) + \mu_2(t)(1+u(t)) \\ &+ \nu_1(t)(c - x_2(t)) + \nu_2(t)(c + x_2(t)). \end{aligned} \tag{33}$$

B. Constraints

Identifying the constraints with (7) results in

$$h(x(t)) = \begin{pmatrix} c - x_2(t) \\ c + x_2(t) \end{pmatrix} \tag{34}$$

$$g(u(t)) = \begin{pmatrix} 1 - u(t) \\ 1 + u(t) \end{pmatrix} \tag{35}$$

$$b(x(t)) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}. \tag{36}$$

The order of the constraints is found to be $p = 1$ by the following calculation

$$h^0 = \begin{pmatrix} c - x_2 \\ c + x_2 \end{pmatrix} \Rightarrow h_u^0 = 0 \tag{37}$$

$$h^1 = \begin{pmatrix} -\dot{x}_2(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -u(t) \\ u(t) \end{pmatrix} \Rightarrow h_u^1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{38}$$

The constraint qualification conditions (12)–(13) hold, since

$$\operatorname{rank}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2 = \ell' \tag{39}$$

$$\operatorname{rank}\begin{pmatrix} -1 & 1-u & 0 \\ 1 & 0 & 1+u \end{pmatrix} = 2 = s \quad \forall u \tag{40}$$

C. Hamiltonian Maximization

Without constraints, the Hamiltonian is maximized for

$$\hat{u}(t) = \frac{1}{2}\frac{1}{r}\frac{\lambda_2(t)}{\lambda_0}. \tag{41}$$

Considering the constraints, the *optimal solution* is

$$u^*(t) = \begin{cases} -1 & \text{if } \lambda_2(t)/\lambda_0 < -2r \\ \frac{1}{2r}\frac{\lambda_2(t)}{\lambda_0} & \text{if } -2r \le \lambda_2(t)/\lambda_0 \le 2r \\ 1 & \text{if } \lambda_2(t)/\lambda_0 > 2r \end{cases} \tag{42}$$

Given the scenario that both the constraints on the control signal and the state become active, we conclude the following time-dependent optimal solution u^* (see Fig. 1 for an illustration)

$$u^*(t) = \begin{cases} -1 & \text{for } t < \tau_1 \\ \frac{1}{2r}\frac{\lambda_2(t)}{\lambda_0} & \text{for } \tau_1 \le t \le \tau_2 \\ 0 & \text{for } \tau_2 < t < \tau_3 \\ \frac{1}{2r}\frac{\lambda_2(t)}{\lambda_0} & \text{for } \tau_3 \le t \le \tau_4 \\ 1 & \text{for } \tau_4 < t \end{cases} \tag{43}$$

where $\tau_i, i \in \{1..4\}$ are junction times.

D. The Conditions

We evaluate conditions (18)–(21) here:

$$\begin{aligned} &L_u^*[t] = H_u^*[t] + \mu g_u^*[t] = 0 \\ &\quad \Rightarrow -2ru(t) + \lambda_2(t) - \mu_1(t) + \mu_2(t) = 0 \end{aligned} \tag{44}$$

$$\begin{aligned} &\dot{\lambda}(t) = -L_x^*[t] \\ &\quad \Rightarrow \begin{cases} \dot{\lambda}_1(t) = 2\lambda_0 q_1 x_1(t) \\ \dot{\lambda}_2(t) = 2\lambda_0 q_2 x_2(t) - \lambda_1(t) + \nu_1(t) - \nu_2(t) \end{cases} \end{aligned} \tag{45}$$

$$\begin{aligned} &\mu(t) = \begin{pmatrix} \mu_1(t) & \mu_2(t) \end{pmatrix}^T \ge 0,\ \mu(t)g^*[t] = 0 \\ &\quad \Rightarrow (\mu_1(t) + \mu_2(t)) - (\mu_1(t) - \mu_2(t))u^*(t) = 0 \end{aligned} \tag{46}$$

$$\begin{aligned} &\nu(t) = \begin{pmatrix} \nu_1(t) & \nu_2(t) \end{pmatrix}^T \ge 0,\ \dot{\nu}(t) \le 0,\ \nu(t)h^*(t) = 0 \\ &\quad \Rightarrow (\nu_1(t) + \nu_2(t))c - (\nu_1(t) - \nu_2(t))x_2(t) = 0 \end{aligned} \tag{47}$$

Additionally, (22) results in

$$\begin{aligned} &\frac{dH^*[t]}{dt} = \frac{dL^*}{dt} = L_t^*[t] \Rightarrow \\ &-2q_1x_1(t)x_2(t) - 2q_2x_2(t)u(t) - 2ru(t)\dot{u}(t) \\ &+ \lambda_1(t)u(t) + \dot{\lambda}_1(t)x_2(t) + \lambda_2(t)\dot{u}(t) + \dot{\lambda}_2(t)u(t) = 0 \end{aligned} \tag{48}$$

$$\begin{aligned} &\dot{u}(t)(\mu_1(t) - \mu_2(t)) + (\dot{\mu}_1(t) - \dot{\mu}_2(t))u(t) \\ &\quad - (\dot{\mu}_1(t) + \dot{\mu}_2(t)) + (\nu_1(t) - \nu_2(t))u(t) \\ &\quad + (\dot{\nu}_1(t) - \dot{\nu}_2(t))x_2(t) - c(\dot{\nu}_1(t) + \dot{\nu}_2(t)) = 0 \end{aligned} \tag{49}$$

E. Transversality conditions

Equations (23)–(24) are evaluated here:

$$\lambda(t_f^-) = \beta b_x^*[t_f] + \gamma h_x^*[t_f]$$

$$\begin{pmatrix} \lambda_1[1^-] \\ \lambda_2[1^-] \end{pmatrix} = I_{2\times 2}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \tag{50}$$

$$\gamma \ge 0, \gamma h[t_f] = 0 \Rightarrow (\gamma_1 + \gamma_2)c - (\gamma_1 - \gamma_2)x_2(t_f) = 0 \tag{51}$$

Since $x_2(t_f) = 0$, we conclude $\gamma_1 = \gamma_2 = 0$.

F. Regions

In view of (43), the solution is divided into different regions. We consider each case separately and using the knowledge of $u^*(t)$ in each specific region and (44)–(49), expressions for the states, the costates and the multiplier functions are calculated. Hereafter, we set $\lambda_0 = 1$.

1) Case 1, $u^*(t) = -1$, $t < \tau_1$: Assuming that $x_0 > 0$

$$\begin{aligned} x_1(t) &= -\frac{1}{2}t^2 + v_0 t + x_0 \\ x_2(t) &= -t + v_0 \end{aligned} \tag{52}$$

From (44) follows

$$2r + \lambda_2(t) - \mu_1(t) + \mu_2(t) = 0. \tag{53}$$

From (46) follows

$$\mu_1(t) = 0,\ \mu_2(t) \ge 0 \text{ free.} \tag{54}$$

From (47) follows

$$\nu_1(t) = \frac{t-c}{t+c}\nu_2(t). \tag{55}$$

From (49) follows

$$2\dot{\mu}_1 + (t+c)\dot{\nu}_1 - (t-c)\dot{\nu}_2 + (\nu_1 - \nu_2) = 0. \tag{56}$$

From (45) follows

$$\begin{cases} \dot{\lambda}_1(t) = -q_1t^2 + 2q_1x_0 \\ \dot{\lambda}_2(t) = -2q_2t - \lambda_1(t) + \nu_1(t) - \nu_2(t), \end{cases} \tag{57}$$

and combined with (48), we conclude

$$\nu_1(t) = \nu_2(t). \tag{58}$$

From (55) and (58) it follows that

$$\nu_1(t) = \nu_2(t) = 0.$$

From (57) follows

$$\begin{cases} \lambda_1(t) = -\frac{1}{3}q_1 t^3 + q_1 x_0 t + K_1 \\ \lambda_2(t) = \frac{1}{12}q_1 t^4 - (\frac{1}{2}q_1 x_0 + q_2)t^2 - tK_1 + K_2, \end{cases} \tag{59}$$

where K_1 and K_2 are appropriate constants of integration.

2) Case 2, $u^*(t) = 0$, $\tau_2 < t < \tau_3$:

$$\begin{aligned} x_1(t) &= -ct + K_3 \\ x_2(t) &= -c \end{aligned} \tag{60}$$

From (44) follows

$$\lambda_2(t) - \mu_1(t) + \mu_2(t) = 0. \tag{61}$$

From (46) follows

$$\mu_1(t) = \mu_2(t) = 0. \tag{62}$$

From (47) follows

$$\nu_1(t) = 0,\ \nu_2(t) \geq 0 \text{ free}. \tag{63}$$

From (45) follows

$$\begin{aligned} \dot{\lambda}_1(t) &= -2q_1 ct + 2q_1 K_3 \\ \dot{\lambda}_2(t) &= -\lambda_1(t) - 2q_2 c - \nu_2(t). \end{aligned} \tag{64}$$

From (61) and (62) we conclude $\lambda_2(t) = 0$. Considering this, (64) simplifies to

$$\nu_2(t) = -\lambda_1(t) - 2q_2 c \tag{65}$$

$$\lambda_1(t) = -q_1 ct^2 + 2q_1 K_3 t + K_4. \tag{66}$$

3) Case 3, $u^*(t) = 1$, $t > \tau_4$: By integrating the control signal, we obtain

$$\begin{aligned} x_1(t) &= \frac{1}{2}t^2 + K_5 t + K_6 \\ x_2(t) &= t + K_5. \end{aligned} \tag{67}$$

From (44) follows

$$-2r + \lambda_2(t) - \mu_1(t) + \mu_2(t) = 0. \tag{68}$$

From (46) follows

$$\mu_1(t) \geq 0 \text{ free},\ \mu_2(t) = 0. \tag{69}$$

From (47) follows

$$(c-t)\nu_1(t) + (c+t)\nu_2(t) - (\nu_1(t) - \nu_2(t))K_5 = 0. \tag{70}$$

From (49) follows

$$\nu_1(t) - \nu_2(t) + (\dot{\nu}_1(t) - \dot{\nu}_2(t))(t - c + K_5) = 0. \tag{71}$$

From (45) follows

$$\begin{cases} \dot{\lambda}_1(t) = 2q_1(\frac{1}{2}t^2 + K_5 t + K_6) \\ \dot{\lambda}_2(t) = 2q_2(t + K_5) - \lambda_1(t) + \nu_1(t) - \nu_2(t), \end{cases} \tag{72}$$

and combined with (48), we conclude

$$\nu_1(t) = \nu_2(t). \tag{73}$$

From (70) and (73) it follows that $\nu_1(t) = \nu_2(t) = 0$. From (72) follows

$$\begin{cases} \lambda_1(t) = \frac{1}{3}q_1 t^3 + q_1 K_5 t^2 + 2q_1 K_6 t + K_7 \\ \lambda_2(t) = -\frac{1}{12}q_1 t^4 - \frac{1}{3}q_1 K_5 t^3 + (q_2 - q_1 K_6)t^2 \\ \qquad + (2q_2 K_5 - K_7)t + K_8 \end{cases} \tag{74}$$

4) Case 4, $u^*(t) = \frac{1}{2r}\lambda_2(t)$, $\tau_1 < t < \tau_2$ or $\tau_3 < t < \tau_4$: From (44) follows

$$u(t) = \frac{\lambda_2(t) - \mu_1(t) + \mu_2(t)}{2r} \Rightarrow \mu_1(t) = \mu_2(t). \tag{75}$$

From (46) follows

$$\mu_1(t) + \mu_2(t) = 0. \tag{76}$$

Therefore, $\mu_1(t) = \mu_2(t) = 0$.
From (48) follows

$$\left(-2q_2 x_2(t) + \lambda_1(t) + \dot{\lambda}_2(t)\right)\lambda_2(t) = 0. \tag{77}$$

In this region, $\lambda_2(t)$ cannot be identically equal to zero. Thus,

$$-2q_2 x_2(t) + \lambda_1(t) + \dot{\lambda}_2(t) = 0. \tag{78}$$

Comparing with (45) and using (47), we conclude that $\nu_1(t) = \nu_2(t) = 0$. Using (78), the system equations (2), and $u^*(t) = \frac{1}{2r}\lambda_2(t)$, we conclude

$$\lambda_2^{(4)}(t) = \frac{1}{r}(q_2\ddot{\lambda}_2(t) - q_1\lambda_2(t)). \tag{79}$$

From this follows

$$\lambda_2(t) = C_1 e^{-\sigma_1 t} + C_2 e^{\sigma_1 t} + C_3 e^{-\sigma_2 t} + C_4 e^{\sigma_2 t}, \tag{80}$$

where

$$\sigma_1 = \sqrt{\frac{q_2 + \sqrt{q_2^2 - 4rq_1}}{2r}},\ \sigma_2 = \sqrt{\frac{q_2 - \sqrt{q_2^2 - 4rq_1}}{2r}},$$

and C_i are appropriate constants of integration.

Let us define

$$\begin{aligned} A(C,t) &:= \frac{C_1}{\sigma_1^2}e^{-\sigma_1 t} + \frac{C_2}{\sigma_1^2}e^{\sigma_1 t} + \frac{C_3}{\sigma_2^2}e^{-\sigma_2 t} + \frac{C_4}{\sigma_2^2}e^{\sigma_2 t} \\ A'(C,t) &= -\frac{C_1}{\sigma_1}e^{-\sigma_1 t} + \frac{C_2}{\sigma_1}e^{\sigma_1 t} - \frac{C_3}{\sigma_2}e^{-\sigma_2 t} + \frac{C_4}{\sigma_2}e^{\sigma_2 t} \\ A''(C,t) &= C_1 e^{-\sigma_1 t} + C_2 e^{\sigma_1 t} + C_3 e^{-\sigma_2 t} + C_4 e^{\sigma_2 t} \\ B(C,t) &:= \frac{\sigma_2^2}{\sigma_1}\left(C_1 e^{-\sigma_1 t} - C_2 e^{\sigma_1 t}\right) + \frac{\sigma_1^2}{\sigma_2}\left(C_3 e^{-\sigma_2 t} - C_4 e^{\sigma_2 t}\right). \end{aligned}$$

Thus, we can write

$$u(t) = \frac{1}{2r}A''(C,t). \tag{81}$$

From the system equations (2)

$$\dot{x}_2(t) = u(t) \qquad \Rightarrow x_2(t) = \frac{1}{2r}A'(C,t) + \kappa_1 \tag{82}$$

$$\dot{x}_1(t) = x_2(t) \qquad \Rightarrow x_1(t) = \frac{1}{2r}A(C,t) + \kappa_1 t + \kappa_2. \tag{83}$$

Considering (78), it follows

$$\lambda_1(t) = -B(C,t) + 2q_2\kappa_1. \tag{84}$$

Moreover, substituting (83) and (84) into (45), and using the assumption $q_1, r \neq 0$ result in

$$\kappa_1 t + \kappa_2 = 0. \tag{85}$$

The equations derived in this part apply to two regions. For the first interval, $\tau_1 < t < \tau_2$, we will use constants C_i, $i \in \{1..4\}$ and for the second interval, $\tau_3 < t < \tau_4$, constants D_i. Note that since it is assumed that the junction times τ_i are distinct, (85) must hold in more than one point. Accordingly,

$$\kappa_1 = \kappa_2 = 0. \tag{86}$$

G. Initial and Final Conditions:

The initial condition on x is already used in (52). Without loss of generality, we normalize the final time to 1. Thus, the final conditions for x are

$$x_1(1) = 0,\ x_2(1) = 0. \tag{87}$$

Therefore, from (67), when $u^* = 1$ follows

$$K_5 = -1,\ K_6 = \frac{1}{2}. \tag{88}$$

H. Continuity of u^*:

The Hamiltonian (32) is regular, *i.e.* the maximization of H w.r.t. u is unique. Therefore, u^* is continuous everywhere including the points on the boundary according to Proposition 4.3 in [13]. We evaluate the control signal at junction times from below and above and equate the expressions:

$$u^*(\tau_1^+) = -1,\ u^*(\tau_2^-) = 0,\ u^*(\tau_3^+) = 0,\ u^*(\tau_4^-) = 1 \Rightarrow$$

$$A''(C,\tau_1) + 2r = 0 \tag{89}$$

$$A''(C,\tau_2) = 0 \tag{90}$$

$$A''(D,\tau_3) = 0 \tag{91}$$

$$A''(D,\tau_4) - 2r = 0 \tag{92}$$

I. Continuity of x^*:

By integrating u^*, we arrive at the states. Hence, x^* is continuous too. The values of the states at junction times from below and above are equated:

$$x_1^*(\tau_i^-) = x_1^*(\tau_i^+),\ x_2^*(\tau_i^-) = x_2^*(\tau_i^+),\ i \in \{1..4\} \Rightarrow$$

$$A(C,\tau_1) + r\left(\tau_1^2 - v_0\tau_1 - 2x_0\right) = 0 \tag{93}$$

$$A'(C,\tau_1) + 2r(\tau_1 - v_0) = 0 \tag{94}$$

$$A(C,\tau_2) + 2r(c\tau_2 - K_3) = 0 \tag{95}$$

$$A'(C,\tau_2) + 2rc = 0 \tag{96}$$

$$A(D,\tau_3) + 2r(c\tau_3 - K_3) = 0 \tag{97}$$

$$A'(D,\tau_3) + 2rc = 0 \tag{98}$$

$$A(D,\tau_4) - r(\tau_4 - 1)^2 = 0 \tag{99}$$

$$A'(D,\tau_4) - 2r(\tau_4 - 1) = 0 \tag{100}$$

J. Continuity of λ:

The continuity of the adjoint function λ at junction times is guaranteed since our problem fulfills the conditions of Proposition 4.2 in [13]. To show this, from the dimensions of the constraint functions $h(\cdot)$ and $g(\cdot)$, we have $s = q = 2$. The control signal u^* is continuous and

$$\begin{aligned}
&\operatorname{rank}\begin{pmatrix} \partial g^*[\tau]/\partial u & \operatorname{diag}(g^*[\tau]) & 0 \\ \partial h^{1*}[\tau]/\partial u & 0 & \operatorname{diag}(h^*[\tau]) \end{pmatrix} \\
&= \operatorname{rank}\begin{pmatrix} -1 & 1-u(\tau) & 0 & 0 & 0 \\ 1 & 0 & 1+u(\tau) & 0 & 0 \\ -1 & 0 & 0 & c - x_2(\tau) & 0 \\ 1 & 0 & 0 & 0 & c + x_2(\tau) \end{pmatrix} \\
&= 4 = s + q.
\end{aligned} \tag{101}$$

Now, we evaluate the costates at junction times from below and above and equate the expressions:

$$\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+),\ \lambda_2(\tau_i^-) = \lambda_2(\tau_i^+),\ i \in \{1..4\} \Rightarrow$$

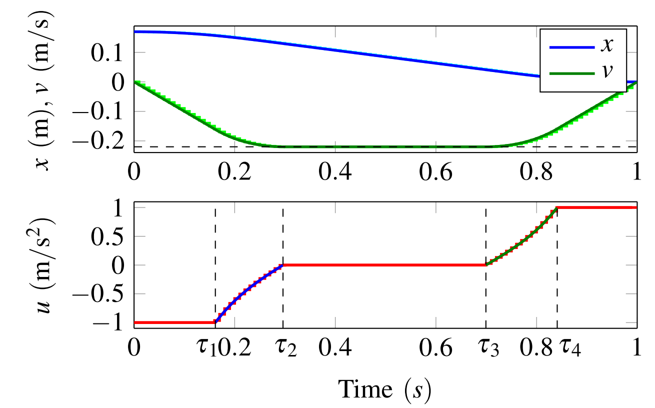

Fig. 1. Comparison of the result of the interior-point method with 100 discretization points and the analytic solution to (1)–(5) with the parameters given in (108)–(109). The smooth curves in the upper plot as well as the blue and green pieces in the lower plot belong to the analytic solution.

$$B(C,\tau_1) - \frac{1}{3}q_1\tau_1^3 + q_1x_0\tau_1 + K_1 = 0 \tag{102}$$

$$\frac{1}{12}q_1\tau_1^4 + (q_1x_0 + q_2)\tau_1^2 - \tau_1K_1 - K_2 - 2r = 0 \tag{103}$$

$$B(C,\tau_2) - q_1c\tau_2^2 + 2q_1K_3\tau_2 + K_4 = 0 \tag{104}$$

$$B(D,\tau_3) - q_1c\tau_3^2 + 2q_1K_3\tau_3 + K_4 = 0 \tag{105}$$

$$B(D,\tau_4) + \frac{1}{3}q_1\tau_4^3 - q_1\tau_4^2 + q_1\tau_4 + K_7 = 0 \tag{106}$$

$$\begin{aligned}&\frac{1}{12}q_1\tau_4^4 + \frac{1}{3}q_1\tau_4^3 + (\frac{1}{2}q_1 - q_2)\tau_4^2 + (2q_2 + K_7)\tau_4 \\ &\quad - K_8 + 2r = 0\end{aligned} \tag{107}$$

IV. Results

The conditions of the theorem lead to 24 unknowns ($K_i, i \in \{1..8\}$, $\kappa_i, i \in \{1..2\}$ for two regions, $C_i, D_i, i \in \{1..4\}$, and $\tau_i, i \in \{1..4\}$) and 24 equations, (86) and (88)–(107). By solving these equations, the junction times and the integration constants are determined and hence the solution to the optimal control problem. There are six trivial equations corresponding to (86) and (88). The rest are nonlinear in τ_i but linear in the other parameters. Note that we have only presented the result of the scenario where $0 \le \tau_1 < \tau_2 < \tau_3 < \tau_4 \le 1$. If there is no solution to the equations, other scenarios where no or only some of the state/input constraints become active, must be considered. The problem is infeasible if there is no solution to any of these scenarios.

A. Example

We report the results for the following example:

$$R = r = 0.1, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix} \tag{108}$$

$$c = 0.22, \quad x_0 = 0.17, \quad v_0 = 0, \quad t_f = 1. \tag{109}$$

The numerical solution obtained by the interior method and the analytic solution are compared in Fig. 1. We ran 10 experiments starting from a random initial guess for the solution of the nonlinear equations. The `fsolve` function in Matlab was able to find a solution on average in 0.0825 s on an Intel(R) Core(TM) i7-3770K CPU @ 3.50GHz with 32GB RAM running Fedora 20. The same problem was solved

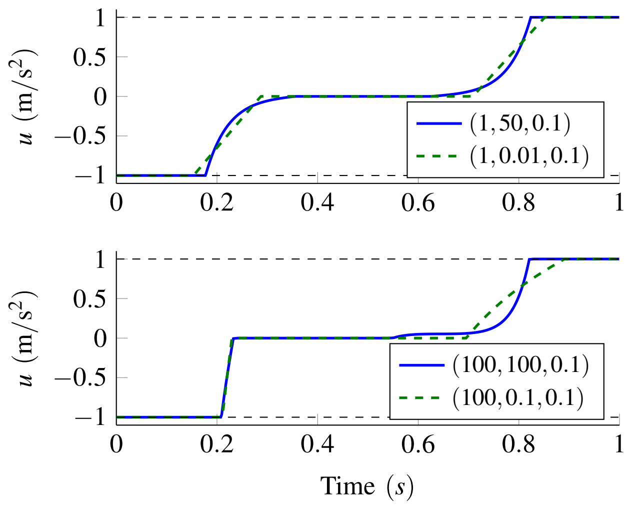

Fig. 2. An illustrative example for the effect of various cost functions (q_1, q_2, r). The other parameters for the problem are according to (109).

using CVX package [17] in Matlab, with the sampling time of $h = 1$ ms. The interior point method approach took on average 14.232 s (including the overheads 53.753 s). The cost obtained by the analytical approach was $385.352h$ which was confirmed by the numerical result. The effect on the control signal by varying Q and R is shown in Fig. 2.

V. Discussions

Since the Hamiltonian H in (32) is concave, we conclude that the solution to our example found by the direct approach is optimal according to the Mangasarian-type sufficient condition [18], [13]. Note that, although we use a numerical approach to find the constants, this approach is independent of the number of discretization points. In other words, there is an analytic expression for the solution that is parametrized by the unknowns.

The minimum-time solutions proposed in [4], [5], [1] result in "bang-bang" solutions. By including constraints on jerk or higher order derivatives of the position, it is possible to make the transitions smoother. However, no explicit cost on the states and control signal is taken into account. Moreover, the variation in the final time for the minimum-time problems demands a special solution for the synchronization between different degrees of freedom [1]. Compared to the piece-wise polynomial approaches [6], [7], [8], our solution takes into account a quadratic cost function as well as the constraints on velocity and acceleration.

In many industrial manipulators, individual joint velocities and positions are used as reference inputs for each DOF. The required torque values can also be calculated if the inverse dynamics is available. Thus, our approach is applicable for a large class of robots. However, the assumption of decoupled DOF does not hold in general when there are obstacles. In such cases, via-points can be included or a multiple DOF problem with the obstacles represented as state constraints needs to be solved. Finding an analytic solution in the latter approach is challenging.

VI. Conclusions

An analytic solution to the fixed-time optimal point-to-point trajectory planning problem with velocity and acceleration constraints is derived. The benefit of the analytic solution is that its computation time is independent of its resolution, while a numerical approach based on discretization might fail for a high sampling rate. Since the solution can be computed efficiently, there is an opportunity for an on-line trajectory generation. Compared to time-optimal solutions, the fixed-time problem allows for an application-specific cost function. Additionally, the synchronization between several degrees of freedom is for free since all movements must follow the same given fixed-time.

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