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The Effect of Symbol Rate on Constrained Capacity for Linear Modulation

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***Abstract*— We consider the effect of symbol rate on the constrained capacity of linear modulation with a fixed spectral density. We show that constrained capacity grows with the symbol rate for some modulation pulses but shrinks with others. Sufficient conditions on the pulse are derived for the constrained capacity to be monotonically increasing with faster symbol rate. Most standard pulses fulfill these.**

I. Introduction

Consider signals of the linear modulation form

$$s(t) = \sum_{k=-\infty}^{\infty} a_k h(t - k\tau T), \qquad (1)$$

where $h(t)$ is any real-valued pulse that satisfies the following assumptions:

1) $h(t)$ has unit energy, i.e. $\int |h(t)|^2 dt = 1$.
2) $h(t)$ is T-orthogonal, i.e. $\int h(t)h(t-kT)dt = \delta_k$ and there exists no number $T' < T$ such that $h(t)$ is T'-orthogonal.
3) $h(t)$ is bandlimited to $1/T$ Hz (not $1/2T$), i.e. $H(f)= 0,\ |f|>1/T$.

2) implies that the symbol sequence $\{a_k\}$ can be detected independently if $\tau = 1$. Throughout, the case $\tau = 1$ will be referred to as Nyquist signaling. Optimal detection of $\{a_k\}$ requires a matched filter and τT-sampling at the receiver.

A common way to implement (1) is with binary data $\{a_k\}$, i.e. $a_k \in \{\pm\sqrt{E}\}$ where E is the symbol energy; the data can be coded or not. This choice of signaling has a certain constrained capacity, a rate less than 1 bit/channel use. In order to increase the rate further, without changing the average symbol energy or the power spectral density, the standard technique today is to employ a larger modulation alphabet; the optimal alphabet is the Gaussian one. In order to increase the rate even further correlated symbols can be used; the optimal correlation is found via the standard waterfilling technique. In this paper we do not perform waterfilling; the symbols $\{a_k\}$ are assumed independent and identically distributed (i.i.d.), as in Shannon's random coding technique. The correlated case is, however, important and will be looked into in the near future. The bandwidth $W = 1/T$ Hz in 3) makes available $W \log_2(1 + P/N_0W)$ bits/s according to Shannon's classic formula. This is the underlying channel capacity. Our signals work within this bandwidth and have a constrained, smaller capacity. For brevity we will refer to our constrained capacity simply as 'capacity'.

Instead of increasing the signaling constellation, a different approach is to *increase the symbol rate*, $1/\tau T$. This makes the signaling faster-than-Nyquist, and the receiver must combat intersymbol interference (ISI). Unfortunately, it is not known today how to compute the resulting capacity except in the case of Gaussian signaling values. The capacities for discrete level constellations are certainly of interest but are not addressed in this paper. In the case of i.i.d. Gaussian $\{a_k\}$ with variance $P\tau T$ the capacity of signals specified by (1) in additive white Gaussian noise with density $N_0/2$ equals [1]

$$C(P,\tau) = \int_0^{\frac{1}{2T\tau}} \log_2\left(1 + \frac{2P}{N_0}\left|H_{\rm fo}(f,\tau)\right|^2\right) df \qquad (2)$$

where $|H_{\rm fo}(f,\tau)|^2$ denotes the folded pulse spectrum

$$|H_{\rm fo}(f,\tau)|^2 \triangleq \begin{cases} |H(f)|^2 + \left|H\left(\frac{1}{\tau T} - f\right)\right|^2, & |f| \le \frac{1}{2\tau T} \\ 0, & |f| > \frac{1}{2\tau T} \end{cases}$$

It is important to note that [1] only considers finite duration ISI while the ISI stemming from $\tau < 1$ is of infinite duration. The extension to the infinite duration case is straightforward and can be found in [2]. The assumption that $h(t)$ is T-orthogonal is equivalent to $\left|H_{\rm fo}(f,1)\right|^2 = T$ all f [3]. The more general conditions from [4] are not needed due to 3). This is in turn equivalent to $|H(f)|^2 = T - |H(1/T - f)|^2$, $0 \le f \le 1/2T$, that is, $|H(f)|^2$ is antisymmetric about the point $f = 1/2T$.

For Nyquist signaling ($\tau = 1$) with some h the capacity is the familiar

$$C_N(P) \triangleq C(P,1) = \frac{1}{2T}\log_2(1 + 2PT/N_0). \qquad (3)$$

We will refer to $C_N(P)$ as the Nyquist capacity. It does not depend on the pulse h, so long as h is T-orthogonal and unit-energy. But it *is* commonly possible to achieve a higher capacity in the same power and spectral density. One way is to signal faster than rate $1/T$ in Eq. (1). The higher capacity can be significant, and this is the motivation for our paper.

The paper investigates when the capacity integral $C(P,\tau)$ is higher; especially, the impact of τ on $C(P,\tau)$ will be studied in detail. A number of results for T-orthogonal pulses are known from previous literature and we list some facts.

Fact 1. The shape $|H_{\rm fo}(f,\tau)|^2$ that maximizes (2) (under a power constraint) is $|H_{\rm fo}(f,\tau)|^2_{\rm opt} = \tau T$. But any τT-orthogonal pulse has a folded pulse shape of this form; consequently, for any τ the optimal pulse shape is a τT-orthogonal shape.

Fact 2. For any T-orthogonal $H(f)$, bandlimited or not, except the sinc pulse we have $C(P,1/q) > C_N(P)$, for any $P > 0$ and q an integer [2].

Fact 3. For any T-orthogonal $H(f)$ except the sinc pulse, bandlimited to Ω Hz, we have $C(P,1/2\Omega T) > C_N(P)$, $P > 0$. Moreover, $C(P,\tau) = C(P,1/2\Omega T)$, $\tau < 1/2\Omega T$.

By using Fact 1 we can immediately extract information about the case $\tau > 1$. For any $\tau > 1$, $C(P,\tau)$ can be written as

$$\begin{aligned} C(P,\tau) &= \int_0^{1/2\tau T} \log_2\left(1+\frac{2P}{N_0}\left|H_{\rm fo}(f,\tau)\right|^2\right) df \\ &= \int_0^{1/2T} \log_2\left(1+\frac{2P}{N_0}\left|H_{\rm fo}(f,\tau)\right|^2\right) df \quad (4)\end{aligned}$$

However, Fact 1 states that the optimal shape in the last expression of (4) is flat over the entire range $[0, 1/2\tau T]$. This is true for the shape that leads to $C_N(P)$ but not for the one leading to $C(P,\tau)$. We have therefore shown

Proposition 1: For a T-orthogonal pulse and any $\tau > 1$, we have $C(P,\tau) < C_N(P)$ for any $P > 0$.

If we combine Assumption 2 and Facts 2 and 3 with Proposition 1, the following knowledge about $C(P,\tau)$ for an arbitrary T-orthogonal pulse is at hand: For $\tau > 1$, $C(P,\tau)$ is smaller than the Nyquist capacity $C_N(P)$, and at $\tau = 1/2$ the capacity saturates at a capacity that is larger than $C_N(P)$. Consequently we know that the capacity must increase when τ decreases from 1 to 1/2 but it is not clear in what fashion it increases.

The scope of this paper is to resolve the ambiguity in the range $1/2 < \tau < 1$. In principle three different behaviors can be imagined. The capacity can be monotonically increasing, i.e. $\partial C(P,\tau)/\partial\tau < 0$, $1/2 < \tau < 1$; this is behavior 1 in Figure 1. Another behavior that could be imagined is that $C(P,\tau)$ is not monotonic in τ but is always larger than $C_N(P)$; this is behavior 2. The third behavior is a capacity curve that goes below $C_N(P)$, but due to Fact 2 it must eventually go above it; this is behavior 3.

In the next section, we present our main results on the analysis of $C(P,\tau)$. Numerical examples are given in Section III. The proofs of the theorems in Section II are deferred to an appendix.

II. Analysis of $C(P,\tau)$

The first question we address is whether there are any pulses for which $C(P,\tau)$ increases monotonically with *decreasing* τ (behavior 1), that is, whether there exist pulses that have

$$\frac{\partial C(P,\tau)}{\partial \tau} < 0, \quad \text{for } 1/2 < \tau < 1$$

In what follows we assume that $|H(f)|^2$ is continuous in $f \in \left[0, \frac{1}{T}\right]$, while $d\,|H(f)|^2/df$ is discontinuous in at most a finite number of points in the interval. We summarize the results of our analysis as follows.

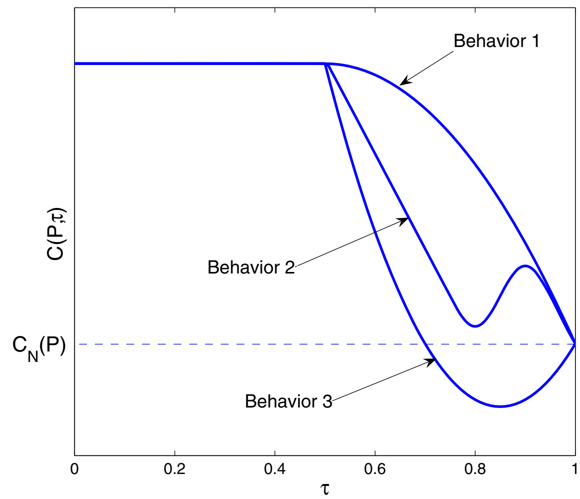

Fig. 1. Three different cases for $C(P,\tau)$ with respect to τ.

Theorem 1: Assume that $|H(f)|^2$ is decreasing in $[0, \frac{1}{T}]$, that the smallest value of $|H_{\rm fo}(f,\tau)|^2$ in the interval $\left[\frac{1}{\tau T} - \frac{1}{T}, 1/2\tau T\right]$ occurs when $f = 1/2\tau T$ and that $|H_{\rm fo}(f,\tau)|^2$ is not identically zero in that interval. Then $C(P,\tau)$ is monotonically increasing for decreasing τ and larger than $C_N(P)$ for $\tau < 1$. If $|H_{\rm fo}(f,\tau)|^2 = 0$ in $\left[\frac{1}{\tau T} - \frac{1}{T}, 1/2\tau T\right]$, then $C(P,\tau)$ is non-decreasing for decreasing τ.

A generalization of this theorem for spectra that are discontinuous in $\left[0, \frac{1}{T}\right]$ is discussed in the appendix. From the theorem, we deduce a corollary.

Corollary 1: Assume that $|H(f)|^2$ is decreasing in $[0, \frac{1}{T}]$. If $d^2\,|H(f)|^2/df^2 \le 0$ for $f \in [0, 1/2T]$, then $C(P,\tau)$ is non-decreasing for decreasing τ.

Proof:

It is enough to prove that the smallest value is at $f = 1/2\tau T$. We have $|H(f)|^2 + |H(1/T - f)|^2 = T$ which gives $(|H(f)|^2)'' + (|H(1/T - f)|^2)'' = 0$. This gives that $(|H(f)|^2)'' \ge 0$ for $f \in [1/2T, 1/T]$ since $(|H(f)|^2)'' \le 0$. From this it follows that $(|H_{\rm fo}(f,\tau)|^2)' = (|H(f)|^2)' - (|H(1/\tau T - f)|^2)' \le 0$ for $f \in [0, 1/2\tau T]$, with equality for $f = 1/2\tau T = 1/2\tau T$, which proves the corollary. ■

In Section III we will apply these results to two triangular spectra and the root raised cosine spectrum.

We next consider behavior 2. Assume that $|H(f)|^2$ is at most discontinuous at a finite number of points in the interval $f \in [0, 1/T]$. We state our results as follows.

Theorem 2: If $|H_{\rm fo}(f,\tau)|^2 \le T$ in $[0, 1/2\tau T]$ then $C(P,\tau) > C_N(P)$ for $P > 0$. Moreover, $C(P,\tau) - C_N(P)$ is monotonically increasing with P.

From Theorem 2 we deduce the following corollary.

Corollary 2: Assume that $|H(f)|^2$ is T-orthogonal. If $|H(f)|^2$ is decreasing in $[0, 1/T]$ then $C(P,\tau) - C_N(P)$ is monotonically increasing with P.

Proof:
Since $|H(f)|^2$ is T-orthogonal and decreasing in $[0, 1/T]$ we have $T = |H(f)|^2+|H(1/T-f)|^2 \geq |H(f)|^2+|H((1/\tau T)-f)|^2$ for $\tau < 1$, because $|H(f)|^2$ is bandlimited to $1/T$ Hz. Hence the conditions in Theorem 2 are satisfied and the corollary is proved. ∎

Finally, we state a sufficient condition for behavior 3, that is, $C(P, \tau) < C_N(P)$ for some P.

Theorem 3: Assume that

$$\int_0^{1/2\tau T} |H_{\text{fo}}(f,\tau)|^4 df > \frac{T}{2}$$

Then there exists a $P > 0$ such that $C(P, \tau) < C_N(P)$.

Hence, to have $C(P, \tau) \geq C_N(P)$ for all P, a necessary condition is $\int_0^{1/2\tau T} |H_{\text{fo}}(f,\tau)|^4 df \leq T/2$. Observe that pulses satisfying Theorem 2 cannot fulfill Theorem 3, since $\int_0^{1/2\tau T} |H_{\text{fo}}(f,\tau)|^4 df \leq \int_0^{1/2\tau T} |H_{\text{fo}}(f,\tau)|^2 T df = T/2$.

III. Numerical Calculations

In this section we apply the results from Section II to actual pulse spectra. We will use two triangular spectra as well as the frequently used root raised cosine pulse (rtRC). The two triangular ones are

$$|H^{\text{tri1}}(f)|^2 = \begin{cases} T - T^2 f, & 0 \leq f \leq \frac{1}{T} \\ 0, & f > \frac{1}{T} \end{cases} \tag{5}$$

and

$$|H^{\text{tri2}}(f)|^2 = \begin{cases} T^2 f, & 0 \leq f \leq \frac{1}{T} \\ 0, & f > \frac{1}{T} \end{cases} \tag{6}$$

Because these spectra are antisymmetric about $f = 1/2T$ it follows that both pulses are T-orthogonal. What can be said about these two? Since both are unit energy and T-orthogonal it follows that they yield equal capacity at $\tau = 1$. Moreover, at $\tau = 1/2$, we have

$$|H_{\text{fo}}^{\text{tri1}}(f, 1/2)|^2 = |H^{\text{tri1}}(f)|^2 = |H_{\text{fo}}^{\text{tri2}}(1/T - f, 1/2)|^2.$$

If we insert these folded shapes in (2) we obtain the same integral and it follows that both spectra yield the same capacity for $\tau = 1/2$.

Although the two spectra are very similar, we will show that for $1/2 < \tau < 1$ they have very different capacity properties.

The folded spectrum of (5) is:

$$|H_{\text{fo}}^{\text{tri1}}(f,\tau)|^2 = \begin{cases} T - T^2 f, & 0 \leq f \leq \frac{1}{\tau T} - \frac{1}{T} \\ 2T - \frac{T}{\tau}, & \frac{1}{\tau T} - \frac{1}{T} < f \leq \frac{1}{2\tau T} \end{cases} \tag{7}$$

From (5), we see that the pulse spectrum is continuous in $[0, \frac{1}{T}]$. The spectrum is also decreasing in $[0, 1/T]$ and the smallest value of (7) in the interval $[\frac{1}{\tau T} - \frac{1}{T}, 1/2\tau T]$ occurs when $f = 1/2\tau T$. Hence the conditions of Theorem 1 are fulfilled and we conclude that the triangular pulse spectrum in (5) has monotonically increasing capacity with decreasing τ. Computing the capacity numerically, we get the curve in Figure 2. The capacity indeed increases with decreasing τ and is larger than $C_N(P)$ for $\tau < 1$. We also see that the capacity

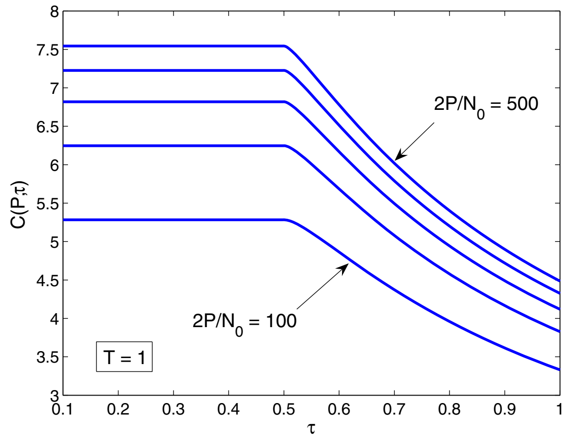

Fig. 2. Constrained capacity $C(P, \tau)$ versus τ for $|H^{\text{tri1}}(f)|^2$.

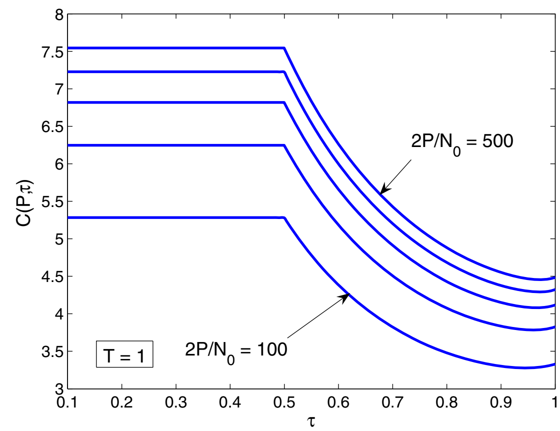

Fig. 3. Constrained capacity $C(P, \tau)$ versus τ for $|H^{\text{tri2}}(f)|^2$.

saturates when $\tau < 1/2$; this is because $|H_{\text{fo}}(f,\tau)|^2 = |H(f)|^2$ for all pulses that are bandlimited to $1/T$ Hz.

Inspecting (7) we can easily conclude that the conditions in Theorem 2 are satisfied and it follows that $C(P, \tau) - C_N(P)$ increases monotonically with P.

The folded spectrum of $|H_{\text{fo}}^{\text{tri2}}(f,\tau)|^2$ is

$$|H_{\text{fo}}^{\text{tri2}}(f,\tau)|^2 = \begin{cases} T^2 f, & 0 \leq f < \frac{1}{\tau T} - \frac{1}{T} \\ \frac{T}{\tau}, & \frac{1}{\tau T} - \frac{1}{T} \leq f \leq \frac{1}{2\tau T} \end{cases} \tag{8}$$

We see that $|H_{\text{fo}}^{\text{tri2}}(f,\tau)|^2$ does not satisfy the conditions in Theorem 1 as it is not decreasing. Neither does it satisfy Theorem 2. However, it satisfies Theorem 3 for some τ values, which implies that this spectrum yields a capacity which is smaller than $C_N(P)$ for some τ and P. Thus, the two spectra have significantly different behaviors: $|H^{\text{tri1}}(f,\tau)|^2$ has behavior 1 but $|H^{\text{tri2}}(f,\tau)|^2$ has behavior 3. In Figure 3 we have plotted $C(\tau, P)$ versus τ for $|H^{\text{tri2}}(f)|^2$ for some values of P; it can be clearly seen that the capacity is indeed not always increasing for this pulse spectrum.

Next we analyse the well known rtRC pulse. Its spectrum is given by

$$|H(f)|^2 = \begin{cases} T, & |f| \le \frac{1-\beta}{2T} \\ T\cos^2\left(\frac{\pi T}{2\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right), & \frac{1-\beta}{2T} < |f| \le \frac{1+\beta}{2T} \\ 0, & |f| > \frac{1+\beta}{2T} \end{cases} \quad (9)$$

where $0 \le \beta \le 1$. The first conditions in Theorem 1 are trivially satisfied, since the rtRC spectrum is continuously differentiable and also has a decreasing spectrum. We prove that the spectrum satisfies Corollary 1. Since we are studying positive frequency values, we can drop the $|\cdot|$ sign in (9). The second derivative of (9) is

$$\frac{d^2\,|H(f)|^2}{df^2} = \begin{cases} 0, & f \le \frac{1-\beta}{2T} \\ -\frac{\pi^2 T^3}{2\beta^2}\cos\left(\frac{\pi T}{\beta}\left(f - \frac{1-\beta}{2T}\right)\right), & \frac{1-\beta}{2T} < f \le \frac{1+\beta}{2T} \\ 0, & f > \frac{1+\beta}{2T} \end{cases} \quad (10)$$

From (10) we observe that $d^2\,|H(f)|^2/df^2 \le 0$ when $\frac{\pi T}{\beta}\left(f - \frac{1-\beta}{2T}\right) \le \frac{\pi}{2}$. This reduces to $f \le 1/2T$, which shows that Corollary 1 is satisfied; thus a rtRC has behavior 1 for any value of β. By inspection it can be seen that the rtRC satisfies Corollary 2 and hence $C(P,\tau) - C_N(P)$ is monotonically increasing with P for any τ. In a similar way, we can analyze pulses with discontinuous spectra (which includes the sinc).

We conclude with the remark that we have a spectrum at hand which has behavior 2; due to lack of space we omit it in this paper.

IV. Conclusions

In this paper we have investigated whether a faster signaling rate always gives a larger capacity than a slower one in linear modulation, for the same power spectral density.

The answer is no; there exist pulses for which a slower rate is beneficial. We have stated sufficient conditions that guarantee that a faster rate always increases the capacity. It is fair to say that all practical pulses (continuous, with decreasing spectrum) fulfill these conditions; for example the rtRC family fulfills them. We have also stated a sufficient condition to have worse capacity when lowering the symbol rate; an actual spectrum that meets this condition was given.

Appendix: Proofs

We investigate for which pulses it is true that $C(P,\tau)$ increases with decreasing τ. We study the sign of the derivative of $C(P,\tau)$ in order to answer this question.

Consider a pulse with a continuously differentiable spectrum in the frequency interval $(0, 1/T)$. If the spectrum is discontinuous at $f = 1/T$, we assume that the pulse values approaching the point from left and right are finite and positive. In the folded pulse shape, when $f = \frac{1}{\tau T} - \frac{1}{T}$, we might then have a discontinuity in the value of the integrand in $C(P,\tau)$ and its derivative with respect to τ. Therefore, we split $C(P,\tau)$ into two integrals:

$$C(P,\tau) = \lim_{\epsilon\to 0}\int_0^{\frac{1}{\tau T}-\frac{1}{T}+\epsilon} \ln\left(1 + \frac{2P}{N_0}|H_{\text{fo}}(f,\tau)|^2\right) df + \lim_{\epsilon\to 0}\int_{\frac{1}{\tau T}-\frac{1}{T}-\epsilon}^{1/2\tau T} \ln\left(1 + \frac{2P}{N_0}|H_{\text{fo}}(f,\tau)|^2\right) df \quad (11)$$

In the first integral, we approach the discontinuity point $f = \frac{1}{\tau T} - \frac{1}{T}$ from left and note that the integrand is bounded and well-defined for that limit, since the pulse spectrum also is. The same holds when approaching the discontinuity from the right, in the second integral. Both integrands are continuously differentiable in respective (open) integration intervals with respect to f and τ, and the first does not depend on τ. From this we get that $C(P,\tau)$ is a differentiable function with respect to τ. This allows application of the Leibniz integral rule on $C(P,\tau)$, which gives the following for the derivative of $C(P,\tau)$ with respect to τ:

$$\begin{aligned}\frac{\partial C(P,\tau)}{\partial \tau} = {} & \ln\left(1+\frac{2P}{N_0}\left|H_{\text{fo}}^{+}\left(\frac{1}{\tau T}-\frac{1}{T},\tau\right)\right|^2\right)\left(-\frac{1}{\tau^2 T}\right) + \\ & + \ln\left(1+\frac{2P}{N_0}\,|H_{\text{fo}}(1/2\tau T,\tau)|^2\right)\left(-\frac{1}{2\tau^2 T}\right) + \\ & + \ln\left(1+\frac{2P}{N_0}\left|H_{\text{fo}}^{-}\left(\frac{1}{\tau T}-\frac{1}{T},\tau\right)\right|^2\right)\left(\frac{1}{\tau^2 T}\right) + \\ & + \lim_{\epsilon\to 0}\int_{\frac{1}{\tau T}-\frac{1}{T}-\epsilon}^{1/2\tau T} \frac{\frac{2P}{N_0}(|H\left(\frac{1}{\tau T}-f\right)|^2)'}{1+\frac{2P}{N_0}|H_{\text{fo}}(f,\tau)|^2}\left(-\frac{1}{\tau^2 T}\right) df\end{aligned} \quad (12)$$

where H^+ and H^- denote limits from the right and left. In the case when we have several discontinuity points in the interval $(0, 1/T]$, we split the integral as above at the discontinuity points and do similar calculations. This leads to more complicated expressions, which are left for future investigation. Consider now pulse spectra that are continuous at $f = 1/T$, i.e., $|H(1/T)|^2 = 0$. In that case, the derivative (12) reduces to

$$\begin{aligned}\frac{\partial C(P,\tau)}{\partial \tau} = {} & \ln\left(1+\frac{2P}{N_0}|H_{\text{fo}}(1/2\tau T,\tau)|^2\right)\left(-\frac{1}{2\tau^2 T}\right) \\ & + \lim_{\epsilon\to 0}\int_{\frac{1}{\tau T}-\frac{1}{T}-\epsilon}^{1/2\tau T} \frac{\frac{2P}{N_0}(|H\left(\frac{1}{\tau T}-f\right)|^2)'}{1+\frac{2P}{N_0}|H_{\text{fo}}(f,\tau)|^2}\left(-\frac{1}{\tau^2 T}\right) df\end{aligned} \quad (13)$$

From (13) we can prove Theorem 1.

Proof of Theorem 1:

We prove that the derivative in (13) is smaller than or equal to 0. We have that

$$\lim_{\epsilon\to 0}\int_{\frac{1}{\tau T}-\frac{1}{T}-\epsilon}^{1/2\tau T}\frac{\frac{2P}{N_0}(|H\left(\frac{1}{\tau T}-f\right)|^2)'}{1+\frac{2P}{N_0}|H_{\rm fo}(f,\tau)|^2}\left(-\frac{1}{\tau^2 T}\right)df$$

$$\leq \lim_{\epsilon\to 0}\int_{\frac{1}{\tau T}-\frac{1}{T}-\epsilon}^{1/2\tau T}\frac{\frac{2P}{N_0}(|H\left(\frac{1}{\tau T}-f\right)|^2)'}{1+\frac{2P}{N_0}|H_{\rm fo}(1/2\tau T,\tau)|^2}\left(-\frac{1}{\tau^2 T}\right)df,$$

because $|H(f)|^2$ is decreasing in $\left[1/2\tau T,\frac{1}{T}\right)$ and the smallest value of $|H_{\rm fo}(f,\tau)|^2$ in the interval $\left(\frac{1}{\tau T}-\frac{1}{T},1/2\tau T\right]$ is $|H_{\rm fo}(1/2\tau T,\tau)|^2=2|H(1/2\tau T)|^2$. Now

$$\lim_{\epsilon\to 0}\int_{\frac{1}{\tau T}-\frac{1}{T}-\epsilon}^{1/2\tau T}\frac{\frac{2P}{N_0}(|H\left(\frac{1}{\tau T}-f\right)|^2)'}{1+\frac{2P}{N_0}|H_{\rm fo}(1/2\tau T,\tau)|^2}\left(-\frac{1}{\tau^2 T}\right)df =$$

$$\frac{\frac{2P}{N_0}}{\tau^2T(1+\frac{2P}{N_0}|H_{\rm fo}(1/2\tau T,\tau)|^2)}\;|H(1/2\tau T)|^2 \qquad (14)$$

We also have that

$$\frac{\frac{2P}{N_0}|H(1/2\tau T)|^2}{\tau^2T(1+\frac{2P}{N_0}|H_{\rm fo}(1/2\tau T,\tau)|^2)}$$
$$\leq \ln(1+\frac{2P}{N_0}|H_{\rm fo}(1/2\tau T,\tau)|^2)\left(\frac{1}{2\tau^2T}\right) \qquad (15)$$

which reduces to

$$\frac{\frac{4P}{N_0}|H(1/2\tau T)|^2}{1+\frac{4P}{N_0}|H(1/2\tau T)|^2}\leq \ln(1+\frac{4P}{N_0}|H(1/2\tau T)|^2) \qquad (16)$$

and this is true for $\frac{4P}{N_0}|H(1/2\tau T)|^2\geq 0$. Hence, we conclude that

$$\begin{aligned}\frac{\partial C(P,\tau)}{\partial\tau}&\leq \frac{\frac{2P}{N_0}|H(1/2\tau T)|^2}{\tau^2T(1+\frac{2P}{N_0}|H_{\rm fo}(1/2\tau T,\tau)|^2)}\\&+\ln\left(1+\frac{2P}{N_0}|H_{\rm fo}(1/2\tau T,\tau)|^2\right)\left(-\frac{1}{2\tau^2T}\right)\\&\leq 0 \qquad (17)\end{aligned}$$

In the case when $|H_{\rm fo}(f,\tau)|^2$ is not identically 0 in $\left[\frac{1}{\tau T}-\frac{1}{T},1/2\tau T\right]$, we have strict inequality in the first step in the proof above, which gives us that the derivative is strictly smaller than 0. This proves the theorem. ■

Next we fix the value of τ and investigate $C(P,\tau)-C_N(P)$, i.e., we prove Theorem 2.

Proof of Theorem 2:

To simplify notation, we put $\xi=2P/N_0$. Define

$$g(\xi)=\int_0^{1/2\tau T}\ln(1+\xi|H_{\rm fo}(f,\tau)|^2)df-\frac{1}{2T}\ln(1+\xi T) \quad (18)$$

Without loss of generality, we can assume that the integrand is a continuously differentiable function with respect to ξ and f. If it is discontinuous in at most finitely many f, we split the integral into intervals where the integrand is continuous. Hence, differentiation under the integral sign is allowed. We have that $g(0)=0$ and

$$g'(\xi)=\int_0^{1/2\tau T}\frac{|H_{\rm fo}(f,\tau)|^2}{1+\xi|H_{\rm fo}(f,\tau)|^2}df-\frac{1}{2(1+T\xi)} \qquad (19)$$

Hence $g'(0)=0$, since $\int_0^{1/2\tau T}|H_{\rm fo}(f,\tau)|^2df=1/2$ because $h(t)$ has unit energy. From (19) we infer that

$$g'(\xi)>\int_0^{1/2\tau T}\frac{|H_{\rm fo}(f,\tau)|^2}{1+\xi T}df-\frac{1}{2(1+\xi T)}=0, \qquad (20)$$

since $|H_{\rm fo}(f,\tau)|^2\leq T$ with strict inequality in some interval. Since $g(\xi)$ and $g'(\xi)$ are continuous, we infer from above that $g(\xi)>0$ when $\xi>0$, which proves the theorem. ■

We observe that it cannot be the case that $|H_{\rm fo}(f,\tau)|^2=T$, since then $\int_0^{1/2\tau T}|H_{\rm fo}(f,\tau)|^2df=\int_0^{1/2\tau T}Tdf=1/2\tau>1/2$, because $\tau<1$. This contradicts that the pulse has unit energy.

Finally, we prove Theorem 3.

Proof of Theorem 3:

We Taylor-expand $g(\xi)$ introduced in the proof of Theorem 2. Taylor expansion is allowed for sufficiently small ξ values, since the integrand in $g(\xi)$ is analytic and converging with respect to ξ, and this also holds for the second term in $g(\xi)$:

$$\begin{aligned}g(\xi)&=\int_0^{1/2\tau T}(\xi|H_{\rm fo}(f,\tau)|^2-\frac{1}{2}(\xi|H_{\rm fo}(f,\tau)|^2)^2+O(\xi^3))df\\&-\frac{1}{2T}(\xi T-\frac{1}{2}(\xi T)^2+O(\xi^3))\\&=\frac{\xi^2}{2}\left(\frac{T}{2}-\int_0^{1/2\tau T}|H_{\rm fo}(f,\tau)|^4df\right)+O(\xi^3) \qquad (21)\end{aligned}$$

because $\int_0^{1/2\tau T}\xi|H_{\rm fo}(f,\tau)|^2df=\xi/2$. $O(\xi)$ is such that $O(\xi^3)/\xi^3$ is bounded when $\xi\to 0$. From above, we conclude that if $\int_0^{1/2\tau T}|H_{\rm fo}(f,\tau)|^4df>T/2$, then by choosing ξ sufficiently small we can make the last expression in (21) negative. This proves the theorem. ■

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