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1. Introduction

Many control systems have a global dynamical behavior that in addition to a desired stable equilibrium has one or more unstable equilibria or other exceptional trajectories. Typical examples of such systems are pendulums or so called phase locked loops.

The objective of this paper is to compare two different methods for analysis of the global behavior in such systems. The first method is LaSalle's invariant set theorem [3]. The second method is the criterion for almost global stability introduced by the author in [4].

The phase locked loop (PLL), Figure 1, is a widespread technique which has contributed significantly to communications and servo control for many years [2]. The purpose is to synchronize two signals by adjusting the phase of the second signal $s_{\text{out}}(t)$ by comparing it to the first signal $s_{\text{in}}(t)$. Most often, such loops are designed using linearized models, but the nonlinearities are of fundamental importance and put constraints on achievable performance. The application of Lyapunov redesign to PLL's was studied in [1].

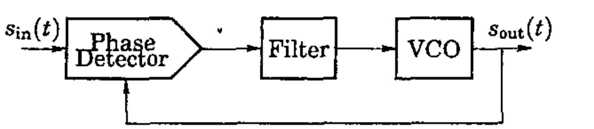

Figure 1: Schematic picture of a phase locked loop for synchronization of the signals $s_{\text{in}}(t)$ and $s_{\text{out}}(t)$

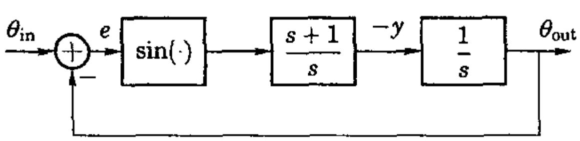

Figure 2: Approximate model for a phase locked loop with the filter $(s+1)/s$

2. PLL using LaSalle's theorem

Our objective is to study the global behaviour of a system with several equilibria. The following result by LaSalle is classical.

Proposition 1 ([3]) *Suppose that* $V \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R})$ *with*

$$\nabla V \cdot f(x) \le 0 \qquad \textit{for all } x, t$$

Let $E = \{x : \nabla V(x) \cdot f(x) = 0\}$ *and let* M *be the largest set in* E *that is invariant to the dynamics* $\dot{x}(t) = f(x)$. *Then* $|x(t)| \to M \cup \{\infty\}$ *as* $t \to \infty$.

This theorem will now be used to analyze a phase-locked loop. With idealized models for the phase detector and the oscillator (VCO) and a phase reference equal to zero, the phase locked loop with filter $(s+1)/s$ can be modelled as (Figure 2)

$$\begin{bmatrix} \dot{e} \\ \dot{y} \end{bmatrix} = f(e,y) = \begin{bmatrix} y \\ -\sin e - y\cos e \end{bmatrix}$$

(The same equations can be used to describe a pendulum on a cart with proportional feedback.) Introduce the Lyapunov function

$$V(e,y) = 1 - \cos(e) + (y + \sin e)^2/2$$

Then

$$\begin{aligned} \dot{V} &= \begin{bmatrix} \frac{\partial V}{\partial e} & \frac{\partial V}{\partial y} \end{bmatrix} f(e,y) \\ &= \begin{bmatrix} \sin e + (y+\sin e)\cos e & y + \sin e \end{bmatrix} f(e,y) \\ &= -\sin^2(e) \end{aligned}$$

The possibility that $(y,e) \to \infty$ can be excluded, so LaSalle's theorem shows that all trajectories eventually will approach the set

$$M = \{(k\pi, 0) : k = 1, 2, \dots\}$$

Note, however, that only the even multiples of π correspond to the desired stable equilibria of synchronization. Indeed some initial states will (in theory) give rise to trajectories that end up in a phase error of 180°. To prove that this event is exceptional, one could use the theory of stable manifolds to verify that the set of such initial states is a manifold of lower dimension.

3. PLL analysis using density functions

The following is a modification of a theorem in [4]:

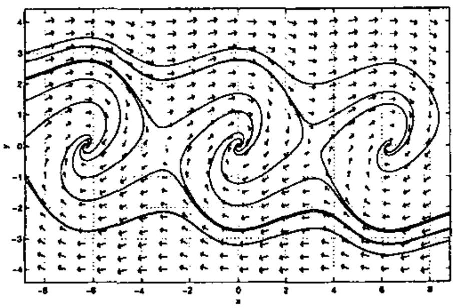

Figure 3: Phase diagram for the system in Figure 2

Theorem 1 *Consider the equation* $\dot{x}(t) = f(x(t))$, *where* $f \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R}^n)$, $f(x)/(1+|x|)$ *is bounded and* $f(0) = 0$. *Suppose* $\rho \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R}^+)$ *with*

$$\int_{|x|\geq 1} \rho(x)dx < \infty \qquad (1)$$

$$[\nabla \cdot (f\rho)](x) > 0 \qquad \textit{for almost all } x \qquad (2)$$

Then, for almost all initial states $x(0)$ *the trajectory* $x(t)$ *tends to zero as* $t \to \infty$.

Moreover, if $f(x_1, \ldots, x_n) = f(x_1 + \tau, x_2, \ldots, x_n)$ *for all* x, *let* $X = ([0,\tau) \times \mathbf{R}^{n-1}) \setminus \{0\}$. *If there exists* $\rho \in \mathbf{C}^1(X, \mathbf{R}^+)$ *such that (1)-(2) hold in domain* X *and* $\rho(0, x_2, \ldots, x_n) = \rho(\tau, x_2, \ldots, x_n)$ *for all* x, *then for almost all initial states* $x(0)$ *the trajectory* $x(t)$ *tends to an integer multiple of* $(\tau, 0, \ldots, 0)$.

Proof. The first statement is a special case of Theorem 1 in [4]. The second statement is proved analogously: Theorem 2 in [4], restated below, is applied with $P = \{x \in X, : |x| > r\}$, $\mu(Y) = \int_Y \rho(x)dx$ and $T : X \to X$ defined by $T(x) = \phi_1(x) \bmod \tau$.

Theorem 2 ([4]) *Consider a measure space* $(X, \mathcal{A}, \mu)$, *a set* $P \subset X$ *of finite measure and a measurable map* $T : X \to X$. *Suppose that*

$$\mu(T^{-1}Y) \leq \mu(Y) \quad \textit{for all measurable } Y \subset X \quad (3)$$

Define Z *as the set of elements* $x \in P$ *such that* $T^n(x) \in P$ *for infinitely many integers* $n \geq 0$. *Then* $\mu(T^{-1}Z) = \mu(Z)$.

The same phase-locked loop as before can be studied using the density function

$$\rho(e,y) = \frac{1}{\psi} = \frac{1}{y^2 + 2 - 2\cos e + y \sin e}$$

Calculations give

$$\psi = \left(y + \frac{1}{2}\sin e\right)^2 + \sin^2\frac{e}{2}\left(4 - \cos^2\frac{e}{2}\right) \geq 0$$

and

$$\begin{aligned}
&\psi^2(\nabla \cdot f\rho) \\
&= (\nabla \cdot f)\psi - \nabla\psi \cdot f \\
&= (-\cos e)(y^2 + 2 - 2\cos e + y\sin e) \\
&\quad - \begin{bmatrix} 2\sin e + y\cos e & 2y + \sin e \end{bmatrix} f(e,y) \\
&= -2\cos e + 2\cos^2 e + \sin e \\
&= (1 - \cos e)^2 \geq 0
\end{aligned}$$

with strict inequality except for $e = 2k\pi$, $k = 1, 2, \ldots$. Hence Theorem 1 shows that almost every trajectory tends toward a stable equilibrium.

4. Comparison and conclusions

At first sight, it looks like the two approaches give the same result. In both cases, the conclusion is that for almost all initial states the trajectory tends towards the stable equilibrium. However, the robustness properties of the two criteria are different. Consider the modified system

$$\begin{bmatrix} \dot{e} \\ \dot{y} \end{bmatrix} = f(e,y) = \begin{bmatrix} y \\ -\sin e - y\cos e + \delta(e) \end{bmatrix}$$

where $\delta(\cdot)$ is a perturbation. Then $\delta(e)$ must vanish when e is integer multiple of π in order for LaSalle's theorem to work with V unchanged. The condition with density functions may still work as long as $\delta(e)$ vanishes when e is an integer multiple of 2π. Hence the location of the unstable equilibrium may change.

In conclusion, the phase locked loop provides a very illustrative example for global analysis of systems with several equilibria. Much more remains to be said and investigated in this field.

Acknowledgment

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