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# Semi-plenary at CDC08: Distributed Control using Decompositions and Games

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#### Building theoretical foundations for distributed control

A centralized paradigm dominates theory and curriculum today



We need methodology for

- Decentralized specifications
- Decentralized design
- Validation of global behavior

# Approximating the Centralized Controller

Bellman's equation  $|x|_{P}^{2} = \min_{u} (|Ax + Bu|_{P}^{2} + |x|^{2} + |u|^{2})$ gives u = -Lx where

	0.3420	0.0737	0.0046	0.0002	
τ_	0.1839	0.3448	0.0736	0.0047	
L =	0.0103	0.1840	0.3447	0.0726	
	0.0008	0.0104	0.1808	0.3296	

Diagonal dominance of L suggests natural approximations:

	0.34	0	0	0	ΓC	).34	0.07	0	0 ]
Ŧ	0	0.34	0	0	$\bar{\tau}$ 0	0.18	0.34	0.07	0
$L_0 =$	0	0	0.34	0	$L_1 =$	0	0.18	0.34	0.07
	0	0	0	0.33	L	0	0	0.18	0.33

Today's challenges: Distributed controller validation Distributed control synthesis

#### Outline

- Introduction o
- Game theory and dual decomposition •
- Dynamic dual decomposition 0
- Distributed validation for wind farm example 0
- 0 Distributed synthesis

Three major challenges:

- Rapidly increasing complexity
- Dynamic interaction
- Information is decentralized





# A "Wind Farm" Case Study



Minimize 
$$V = \mathbf{E} \sum_{i=1}^{n} \left( |x_i|^2 + |u_i|^2 \right)$$

# Inspiration from other fields

- Congestion control in networks
- Collective motion in biology
- Oscillator synchronization in physics
- Parallelization in optimization theory
- Saddle points and equilibria in economics
- Cooperative and non-cooperative game theory

Much focus on convergence to equilibria, less on dynamic performance.

#### 50 years old idea: Dual decomposition

#### $\min_{x,y,z,w} [V_1(x,y) + V_2(x,z) + V_3(x,w)]$

 $= \max_{n,q} \min_{x_1,x_2,x_3,y,z,w} [V_1(x_1,y) + V_2(x_2,z) + V_3(x_3,w) + p(x_1 - x_2) + q(x_2 - x_3)]$ 

The optimum is a Nash equilibrium of the following game:

The three computers try to minimize their respective costs

Computer 1:	$\min_{x_{1},y} \left[ V_{1}(x_{1},y) + px_{1} \right]$
Computer 2:	$\min_{x_{2},z}\left[V_{2}(x_{2},z)-px_{2}+qx_{2} ight]$
Computer 3:	$\min_{x_3,w}\left[V_3(x_3,w)-qx_3\right]$

while the "market makers" try to maximize their payoffs

Between computer 1 and 2:	$\max_p \left[ p(x_1 - x_2) \right]$
Between computer 2 and 3:	$\max_{q} \left[ q(x_2 - x_3) \right]$

Update in gradient direction:

The three computers try to minimize the potential function

$$V_1(x_1, y) + V_2(x_2, z) + V_3(x_3, w) + p(x_1 - x_2) + q(x_2 - x_3)$$

while the market makers try to maximize it.

Finding a *Nash equilibrium* (where no player has a incentive to change strategy) is greatly simplified by existence of a potential function.

Globally convergent if  $V_i$  convex! [Arrow, Hurwicz, Usawa 1958] Lyapunov function:  $\mathbf{V} = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{y}^2 + \dot{z}^2 + \dot{w}^2 + \dot{p}^2 + \dot{q}^2$ 

#### What do we achieve?

Given any  $p, q, \bar{x}, \bar{y}, \bar{z}, \bar{w}$ , the distributed test

$$\begin{split} V_1(\bar{x}, \bar{y}) + p\bar{x} &\leq \alpha \min_{x_1, y} \left[ V_1(x_1, y) + px_1 \right] \\ V_2(\bar{x}, \bar{z}) - p\bar{x} + q\bar{x} &\leq \alpha \min_{x_2, z} \left[ V_2(x_2, z) - px_2 + qx_2 \right] \\ V_3(\bar{x}, \bar{w}) - q\bar{x} &\leq \alpha \min_{x_2, w} \left[ V_3(x_3, w) - qx_3 \right] \end{split}$$

**Decentralized Bounds on Suboptimality** 

implies that the globally optimal cost  $J^*$  is bounded as

$$V_1(\bar{x}, \bar{y}) + V_2(\bar{x}, \bar{z}) + V_3(\bar{x}, \bar{w}) \le \alpha \min_{x, y, z, w} \left[ V_1(x, y) + V_2(x, z) + V_3(x, w) \right]$$

Proof: Add both sides up!

### Outline

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# Performance criteria for individual nodes

- Suboptimality bounds indicate where things went wrong
- Prices show the relative importance of different terms
- Sparsity structure useful for efficient computations

# A General Optimal Control Problem

Minimize 
$$V(u) = \mathbf{E} \sum_{i} \ell_i (x_i(t), u_i(t))$$

subject to

$$\begin{cases} x_1(t+1) = f_1(x_1, v_{1j}, u_1, w_1) \\ \vdots \\ x_J(t+1) = f_J(x_J, v_{Jj}, u_J, w_J) \end{cases}$$

where

$$v_{ij} = x_j$$

holds for all i, j.

#### **Decomposing the Cost Function**

$$\begin{split} & \max_{p} \min_{u,v} \sum_{i} \mathbf{E} \Big[ \ell_i \big( x_i(t), u_i(t) \big) + 2 \sum_{j} (p_{ij})^T (x_j - v_{ij}) \Big] \\ &= \max_{p} \sum_{i} \min_{u_i, v_{ij}} \mathbf{E} \Big[ \ell_i \big( x_i(t), u_i(t) \big) - 2 \sum_{j} (p_{ij})^T v_{ij} + 2 \big( \sum_{j} p_{ji} \big)^T x_i \Big] \end{split}$$

so agent *i* should minimize the stationary value of

$$\mathbf{E}\left(\underbrace{\ell_i(x_i(t), u_i(t))}_{\text{his own cost}} \xrightarrow{-2\sum_j [p_{ij}(t)]^T v_{ij}(t)} \underbrace{+2[\sum_j p_{ji}(t)]^T x_i(t)}_{\text{what he pays others}}\right)$$

# **Distributed Verification**

$$\max_{p} \sum_{i} \min_{u_{i}, v_{ij}} \mathbf{E} \underbrace{\left[ \ell_{i} (x_{i}(t), u_{i}(t)) - 2 \sum_{j} (p_{ij})^{T} v_{ij} + 2 (\sum_{j} p_{ji})^{T} x_{i} \right]}_{J_{i}(x_{i}, u_{i}, v_{(i)}, p)}$$

Each agent *i* makes the comparison

$$\underbrace{\mathbf{E}J_{i}(\bar{x}_{i}, \bar{u}_{i}, \bar{x}_{j}, \bar{p})}_{\text{Actual cost in node } i} \leq \alpha \underbrace{\min_{x_{i}, u_{i}, v_{ij}} \mathbf{E}J_{i}(x_{i}, u_{i}, v_{ij}, \bar{p})}_{\text{Optimal cost in node } i}$$

where minimization is subject to the local dynamics

$$x_i(t+1) = f_i(x_i, v_{ii}, u_i, w_i)$$

If no actual cost exceeds the expected cost by more than 10%, then the global cost is within 10% from optimal.

### **Theorem on Verification**

Consider control laws 
$$\bar{u}_i = \mu_i(\bar{x})$$
 and stationary solutions to

$$\bar{x}_i(t+1) = f_i(\bar{x}_i, \bar{x}_j, \mu_i(\bar{x}), w_i)$$

where  $w_i(t)$  is stationary white noise. If  $\alpha \ge 0$ , then (I) implies (II) :

(*I*) There exists  $\bar{p} = \lambda(\bar{x})$  satisfying

 $\mathbf{E}J_i(ar{x}_i,ar{u}_i,ar{x}_j,ar{p}) \leq lpha \min_{x_i,u_i,v_{ij}} \mathbf{E}J_i(x_i,u_i,v_{ij},ar{p})$ 

when minimizing over stationary solutions to

 $x_i(t+1) = f_i(x_i, v_{ij}, u_i, w_i)$ 

(11) 
$$\sum_i \mathbf{E} \ell_i(\bar{x}_i, \bar{u}_i) \leq \alpha \min_u \sum_i \mathbf{E} \ell_i(x_i, u_i)$$
 when minimizing over stationary solutions to

$$\begin{cases} x_1(t+1) = f_1(x_1, x_j, u_1, w_1) \\ \vdots \\ x_J(t+1) = f_J(x_J, x_j, u_J, w_J) \end{cases}$$

If dynamics is linear,  $\ell_i \ge 0$  convex and  $\alpha = 1$ , then (II) implies (I).

#### A "Wind Farm" Case Study

Minimize 
$$V = \mathbf{E} \sum_{i=1}^{4} (|x_i|^2 + |u_i|^2)$$

$ \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} $	=	$\begin{bmatrix} 0.6 \\ 0.3 \\ 0 \end{bmatrix}$	$0.1 \\ 0.6 \\ 0.3$	0 0.1 0.6	0 0 0.1	$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} +$	$\begin{bmatrix} u_1(t) + w_1(t) \\ u_2(t) + w_2(t) \\ u_3(t) + w_3(t) \end{bmatrix}$
$x_4(t+1)$		0	0	0.3	0.6	$x_4(t)$	$ u_4(t) + w_4(t) $

Today's challenges: Distributed controller validation Distributed control synthesis

	0.34	0	0	0	[0.34	0.07	0	0 ]
Ī _	0	0.34	0	0	$\bar{r}$ _ 0.18	0.34	0.07	0
$L_0 =$	0	0	0.34	0	$L_1 = \begin{bmatrix} 0 \end{bmatrix}$	0.18	0.34	0.07
	0	0	0	0.33	0	0	0.18	0.34

# Decomposing the turbine dynamics

Minimize **E**  $\sum_{i=1}^{4} (|x_i|^2 + |u_i|^2)$ 

subject to



Problem solved by the first turbine

$$\stackrel{x_1}{\bigcirc} \xrightarrow{p_{12}} p_{21}$$

Minimize 
$$\mathbf{E}(|x_1|^2 + |u_1|^2 + 2p_{12}v_{12} - 2p_{21}x_1)$$
  
when  $x_1^+ = 0.6x_1 + 0.1v_{12} + u_1 + w_1$ 

Test for suboptimality:

$$\begin{split} \mathbf{E}(|x_1|^2 + |u_1|^2 + 2p_{12}x_2 - 2p_{21}x_1) \\ &\leq \alpha \min_{u_1, v_{12}} \mathbf{E}(|x_1|^2 + |u_1|^2 + 2p_{12}v_{12} - 2p_{21}x_1) \end{split}$$

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#### Validation Using Centralized Model

The variance  $\mathbf{E}\sum_{i=1}^{4}\left(|x_i|^2+|u_i|^2\right)$  for the optimal centralized controller becomes

 $V_* = 4.9904$ 

while the values for the decentralized approximations become

$$V_0 = 5.2999$$
  $V_1 = 4.9917$ 

These numbers were calculated using a global model.

We will next use dual decomposition to see that the control laws can be both validated and synthesized in a distributed way.

### Problem solved by the first turbine

$$\bigcirc^{x_1} \qquad p_{12} \\ \bigcirc \checkmark \qquad p_{21} \\ \hline$$

Minimize  $\mathbf{E}(|x_1|^2 + |u_1|^2 + 2p_{12}v_{12} - 2p_{21}x_1)$ when  $x_1^+ = 0.6x_1 + 0.1v_{12} + u_1 + w_1$ 

using measurements of x and knowledge of the joint spectral density of x, w,  $p_{13}$  and  $p_{21}$ .

Notice: Once the price sequences  $p_{12}(t)$ ,  $p_{21}(t)$  are given, no other knowledge of the outside world is relevant. However, since future prices are usually not available, knowledge of other states can be useful for price prediction.

#### Performance degradation due to decentralization

Compare expected and actual costs for the two control laws:

$u = -\bar{L}_0 x$ and $\bar{p} = \bar{M} x$ :	$u = -\bar{L}_1 x$ and $\bar{p} = \bar{M} x$ :
$1.5647 \leq 1.5350\alpha$	$1.5741 \leq 1.5740\alpha$
$1.0853 \leq 0.8558\alpha$	$0.9132 \leq 0.9217\alpha$
$1.0853 \leq 0.8558\alpha$	$0.9132 \leq 0.9217\alpha$
$1.5647 \leq 1.5350\alpha$	$1.5741 \leq 1.5740\alpha$

$$1.062 = \frac{V}{V_*} \le \alpha = 1.27 \qquad 1.0003 = \frac{V}{V_*} \le \alpha = 1.0094$$

Minimize  $V = \mathbf{E} \sum_{i=1}^{n} \left( |x_i|^2 + |u_i|^2 \right)$ 

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$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_n(t+1) \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 & 0 \\ 0.3 & \ddots & \ddots \\ & \ddots & \ddots & 0.1 \\ 0 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_n(t+1) \end{bmatrix} + \begin{bmatrix} u_1(t) + w_1(t) \\ u_2(t) + w_2(t) \\ \vdots \\ u_n(t) + w_n(t) \end{bmatrix}$$

We will optimize a tri-diagonal control structure

$$ar{L} = egin{bmatrix} * & * & 0 \ * & \ddots & \ & \ddots & \ & \ddots & * \ 0 & * & * \ \end{pmatrix}$$

# **Optimal Prices by Dynamic Programming**

Optimal control problem:

Minimize 
$$\mathbf{E}(|x|^2 + |u|^2)$$

when 
$$x^+ = \overline{A}x + Av + Bu + w$$
 and  $v = Sx$ 

Dynamic programming gives control law as well as prices:

$$|x|_{P}^{2} = \max_{n} \min_{u,v} \left[ |\bar{A}x + \bar{A}v + Bu|_{P}^{2} + |x|^{2} + |u|^{2} - 2p^{T}(v - Sx) \right]$$

$$p(t) = \begin{bmatrix} p_{12}(t) \\ p_{21}(t) \\ p_{23}(t) \\ p_{32}(t) \\ p_{43}(t) \end{bmatrix} = \begin{bmatrix} 0.0342 & 0.2574 & 0.0010 & 0.0002 \\ 0.5545 & 0.1013 & 0.0382 & 0.0038 \\ 0.0364 & 0.0676 & 0.2755 & 0.0025 \\ 0.0025 & 0.2755 & 0.0676 & 0.0364 \\ 0.0038 & 0.0382 & 0.1013 & 0.5545 \\ 0.0002 & 0.0010 & 0.2574 & 0.0342 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The same P and u(t) = -Lx(t) as in classical solution.

# Distributed gradient iteration for control law

By the maximum principle, optimal solutions to

Minimize 
$$\mathbf{E}(|x_1|^2 + |u_1|^2 + 2p_{12}v_{12} - 2p_{21}x_1)$$
  
when  $x_1^+ = 0.6x_1 + 0.1v_{12} + u_1 + w_1$ 

must minimize the Hamiltonian

$$\mathbf{E} \left[ |x_1|^2 + |u_1|^2 + 2p_{12}v_{12} - 2p_{21}x_1 - \lambda_1(0.6x_1 + 0.1v_{12} + u_1 + w_1) \right]$$

This allows us to modify the control law

$$u_1 = \begin{bmatrix} l_{11} & l_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

in the gradient direction using correlation estimates from the time interval t = 1, ..., T.

# Gradient iteration for the wind park

cost =					cost
5.3183					
L =					L =
0.0366	0.0411	0	0	0	
0.0386	0.0623	0.0546	0	0	
0	0.0555	0.0686	0.0544	0	
0	0	0.0554	0.0620	0.0405	
0	0	0	0.0385	0.0363	

# Prices by distributed gradient iteration

Diagonal dominance suggests a tri-diagonal structure for  ${\it M}$ 

$\bar{p}_{12}(t)$		$m_{11}$	$m_{12}$	0	0		
$\bar{p}_{21}(t)$		$m_{21}$	$m_{22}$	0	0	$\begin{bmatrix} x_1(t) \end{bmatrix}$	
$\bar{p}_{23}(t)$	_	0	$m_{32}$	$m_{33}$	0	$x_2(t)$	
$\bar{p}_{32}(t)$	-	0	$m_{42}$	$m_{43}$	0	$ x_3(t) $	
$\bar{p}_{34}(t)$		0	0	$m_{53}$	$m_{54}$	$x_4(t)$	
$\bar{p}_{43}(t)$		0	0	$m_{63}$	$m_{64}$		

After running the system with fixed prices and control laws during a time interval t = 1, ..., T, the correlation between state measurements and constraint violations can be estimated as

$$\mathbf{E}\begin{bmatrix} x_1\\ x_2 \end{bmatrix} (v_{12}-x_2) \quad \approx \quad \frac{1}{T} \sum_{t=1}^T \begin{bmatrix} x_1(t)\\ x_2(t) \end{bmatrix} [v_{12}(t)-x_2(t)]$$

If the correlation is non-zero, the prices  $\begin{bmatrix} m_{11} & m_{12} \end{bmatrix}$  are adjusted.

# Gradient iteration for the wind park

cost =

7.4944

T. =

0.0138	0.0195	0	0	0
0.0162	0.0283	0.0294	0	0
0	0.0264	0.0333	0.0294	0
0	0	0.0264	0.0283	0.0195
0	0	0	0.0162	0.0138

#### Gradient iteration for the wind park

cost =				
4.4277				
L =				
0.0709	0.0629	0	0	0
0.0666	0.1025	0.0749	0	0
0	0.0853	0.1070	0.0744	0
0	0	0.0851	0.1016	0.0611
0	0	0	0.0662	0.0697

Gradient iteration	n for the wind park
--------------------	---------------------

Gradient iteration for the wind park

cost =					
3.9476					
L =					
0.1187	0.0812	0	0	0	
0.0987	0.1494	0.0885	0	0	
0	0.1146	0.1509	0.0879	0	
0	0	0.1144	0.1479	0.0777	
0	0	0	0.0976	0.1155	

L

cost =				
3.6674				
L =				
0.1820	0.0903	0	0	0
0.1324	0.2041	0.0920	0	0
0	0.1419	0.2032	0.0917	0
0	0	0.1416	0.2023	0.0853
0	0	0	0.1296	0.1743

Gradient iteration for the wind park				Grad	Gradient iteration for the wind park				
cost =					cost =				
030 -					0050 -				
3.5166					3.4732				
L =					L =				
0.2654	0.0777	0	0	0	0.2347	0.0393	0	0	0
0.1611	0.2684	0.0755	0	0	0.1152	0.2363	0.0449	0	0
0	0.1607	0.2674	0.0759	0	0	0.1187	0.2393	0.0444	0
0	0	0.1604	0.2672	0.0731	0	0	0.1189	0.2369	0.0410
0	0	0	0.1549	0.2479	0	0	0	0.1103	0.2131

aaat -				
cost -				
3.4949				
L =				
0.0570	0.0070	0	0	0
0.2579	0.0679	0	0	0
0.1464	0.2673	0.0704	0	0
0	0.1507	0.2676	0.0702	0
0	0	0.1504	0.2664	0.0664
0	0	0	0.1414	0.2389

Gradient iteration for the wind park

# Conclusions

We have seen dynamic dual decomposition used for

- Distributed validation
- Distributed synthesis

Benefits to be obtained

- Reduced complexity
- Control structure reflects plant structure
- Flexibility and robustness

We have the tools to deal with dynamics!

# Convergence rate versus state dimension



For a fixed number of iterations and fixed sparsity structure of L, M, the computational cost grows linearly with n!

# Welcome to join the efforts!

Much (most) remains to be done and much is happening already at this conference!

See [Rantzer CDC07]

[Rantzer ACC09] covers much of this lecture. Working paper on www.control.lth.se/user/anders.rantzer

Lund University funds postdocs and will also hire new faculty members to complement the competence of our current staff.

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