A Perturbation Analysis of Non-Linear Diffusion from a Permeable Solid into a Finite Volume Containing a Liquid

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The equation

$$\dot{c}(x,t) = c''(x,t)\ , \tag{1}$$

with the initial conditions

$$c(x,0) = 1 \qquad \text{in} \quad 0 < x < 1\ , \tag{2}$$

and the nested boundary conditions

$$c(0,t) = c(1,t) = \chi(1 - \int_0^1 c(x,t)dx)\ . \tag{3}$$

has a solution which is split into

$$c(x,t) = u(x,t) + v(x,t)\ . \tag{4}$$

The part $u(x,t)$ fulfils the equation

$$\dot{u}(x,t) = u''(x,t)\ , \tag{5}$$

with the initial conditions

$$u(x,0) = 1 \qquad \text{in} \quad 0 < x < 1\ , \tag{6}$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0\ . \tag{7}$$

The part $v(x,t)$ fulfils the equation

$$\dot{v}(x,t) = v''(x,t)\ , \tag{8}$$

with the initial conditions

$$v(x,0) = 0 \qquad \text{in} \quad 0 < x < 1\ , \tag{9}$$

and the boundary conditions

$$v(0,t) = v(1,t) = \chi(1 - \int_0^1 c(x,t)dx)\ , \tag{10}$$

where χ is the ratio of the volume of the bone, V_B versus the volume of the container V_C, i.e., $\chi = V_B/V_C$. Obviously $v(x,t)$ may be expanded in a Taylor's series as follows:

$$v(x,t) = \chi\varphi_1(x,t) + \chi^2\varphi_2(x,t) + \chi^3\varphi_3(x,t) + ... \tag{11}$$

Since χ is a free parameter the functions $\varphi_i(x,t)$ have to fulfil the equation

$$\dot{\varphi}_i(x,t) = \varphi_i''(x,t)\ , \tag{12}$$

with the initial conditions

$$\dot{\varphi}_i(x,0) = 0 \qquad \text{in} \quad 0 < x < 1\ . \tag{13}$$

The boundary conditions become recursive according to the following:

$$\varphi_1(0,t) = \varphi_1(1,t) = \phi_1(t) = (1 - \int_0^1 u(x,t)dx)\ , \tag{14}$$

and

$$\varphi_i(0,t) = \varphi_i(1,t) = \phi_i(t) = -\int_0^1 \varphi_{i-1}(x,t)dx, \qquad \text{for} \quad i = 2,3,4,... \tag{15}$$

Duhamel's theorem allow us to solve the φ_i:s as an application of single step boundary conditions $\mathrm{d}\phi_i(\tau) = \frac{\mathrm{d}\phi_i(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau$ representing a concentration $\mathrm{d}\phi_i(t')$ applied at the time t',

$$\mathrm{d}\varphi_i(x,t) = u(x,t-\tau)\frac{\mathrm{d}\phi_i(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau \, . \tag{16}$$

Integration from $\tau' = \tau$ to $\tau' = t$ gives

$$\varphi_i(x,t) = \int_0^t u(x,t-\tau)\frac{\mathrm{d}\phi_i(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau = [u(x,t-\tau)\phi_i(\tau)]_0^t - \int_0^t \frac{\mathrm{d}u(x,t-\tau)}{\mathrm{d}\tau}\phi_i(\tau)\mathrm{d}\tau. \tag{17}$$

Considering that $u(x,0) = 0$ and $\varphi_i(0,\tau) = 0$. The general solution is given as

$$\varphi_i(x,t) = -\int_0^t \phi_i(\tau)\frac{\mathrm{d}u(x,t-\tau)}{\mathrm{d}\tau}\mathrm{d}\tau. \tag{18}$$

Insertion of the solution

$$u = \sum_{n=1,3,5,..} \frac{4e^{-n^2\pi^2 t}\sin(n\pi x)}{n\pi} \, , \tag{19}$$

gives

$$\varphi_i(x,t) = -\int_0^t \phi_i(\tau) \sum_{n=1,3,5,..} 4\pi n e^{-n^2\pi^2(t-\tau)}\sin(n\pi x)\mathrm{d}\tau. \tag{20}$$

For large containers and, hence, small values of χ the solution for the concentration $u(x,t)$ according to (19), clearly gives the approximative solution as $\chi \to 0$.
The boundary conditions for φ_1 are obtained after integration of (19),

$$\phi_1(t) = 1 - \sum_{m=1,3,5,..} \frac{8}{m^2\pi^2}e^{-m^2\pi^2 t} \, . \tag{21}$$

The solution of $\varphi_1(x,t)$ with the boundary conditions $\phi_1(t)$ $v(0,t)$ is the concentration in the container considering the escape of matter from the bone sample and is

$$\varphi_1(x,t) = -\int_0^t (1 - \sum_{m=1,3,5,..} \frac{8}{m^2\pi^2}e^{-m^2\pi^2\tau}) \times \sum_{n=1,3,5,..} 4\pi n e^{-n^2\pi^2(t-\tau)}\sin(n\pi x)\mathrm{d}\tau. \tag{22}$$

$$\varphi_1(x,t) = -[\sum_{n=1,3,5,..} \frac{4}{n\pi}(1 - e^{-n^2\pi^2 t}) - \sum_{\substack{m=1,3,5,.. \\ m \neq n}} \frac{32n}{m^2(n^2-m^2)\pi^3}(e^{-m^2\pi^2 t} - e^{-n^2\pi^2 t}) + f(n,t)]\sin(n\pi x) \, , \tag{23}$$

where

$$f(n,t) = \frac{64}{m^2(n^2-m^2)\pi^4}(e^{-m^2\pi^2 t} - e^{-n^2\pi^2 t})] \quad \text{as} \quad m^2 \to n^2 \, , \tag{24}$$

which is written

$$f(n,t) = \lim_{n^2-m^2 \to \epsilon} \frac{64}{n^2\epsilon\pi^4}e^{-n^2\pi^2 t}(e^{-\epsilon\pi^2 t} - 1) = -\frac{64t}{n^2\pi^2}e^{-n^2\pi^2 t} \, . \tag{25}$$

The contribution to the concentration in the container from φ_1 becomes

$$\phi_2 = \int_0^1 \varphi_1 \mathrm{d}x = -\sum_{n=1,3,5,..} \tfrac{8}{n^2\pi^2}(1-e^{-n^2\pi^2 t})$$
$$+[\sum_{\substack{m=1,3,5,..\\ m\neq n}} \tfrac{64}{m^2(n^2-m^2)\pi^4}(e^{-m^2\pi^2 t}-e^{-n^2\pi^2 t})] + f(n,t), \tag{26}$$

This now reads

$$\phi_2 = -1 - \tfrac{8}{\pi^2} + \sum_{n=1,3,5,..} \tfrac{8}{n^2\pi^2}(1-8t)e^{-n^2\pi^2 t} - $$
$$=$$
$$-1 - \tfrac{8}{\pi^2} + \sum_{n=1,3,5,..} \{\tfrac{8}{n^2\pi^2}(1-8t)e^{-n^2\pi^2 t} +$$
$$\sum_{\substack{m=1,3,5,..\\ m\neq n}} \tfrac{128e^{-n^2\pi^2 t}}{n^2(n^2-m^2)\pi^4}\} = -1 - \tfrac{8}{\pi^2} + \tag{27}$$
$$\sum_{n=1,3,5,..} \tfrac{8}{n^2\pi^2}e^{-n^2\pi^2 t}\{1-8t+\sum_{\substack{m=1,3,5,..\\ m\neq n}} \tfrac{16}{(n^2-m^2)\pi^2}\},$$

that may serve as boundary conditions for the function φ_2. Considering a fairly small ratio $\chi \approx 0.01$ we anticipate the third order term $\chi^2\varphi_2$ to be of the order of 10^{-4}. The work to do this does not seem meaningful, with the accuracy of the present measurements in mind. Figure 1 shows the difference between the solution without considering the increasing concentration in the container and same with a first order correction.

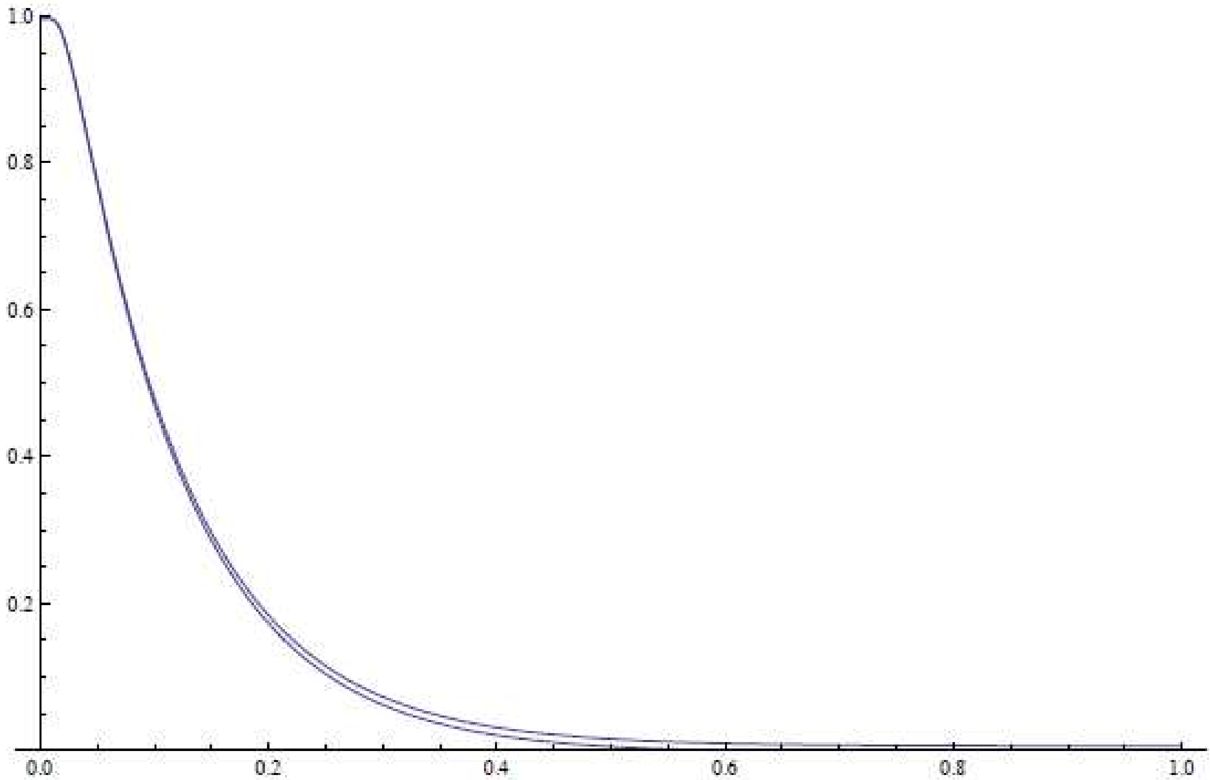

Figure 1: The solution for escaped ions for a sample in an infinite container $1 - \int_0^1 u \mathrm{d}x$ and the same with first order correction $1 - \int_0^1 (u + \chi\varphi_1)\mathrm{d}x$.