



LUND UNIVERSITY

A Perturbation Analysis of Non-Linear Diffusion from a Permeable Solid into a Finite Volume Containing a Liquid

Lindberg, Gustav; Ståhle, Per; Svensson, Ingrid

2013

Document Version:

Publisher's PDF, also known as Version of record

[Link to publication](#)

Citation for published version (APA):

Lindberg, G., Ståhle, P., & Svensson, I. (2013). *A Perturbation Analysis of Non-Linear Diffusion from a Permeable Solid into a Finite Volume Containing a Liquid*. (LUTFD2/(TFHF-3088)/1-8/(2013); Vol. TFHF-3088). Solid Mechanics, Faculty of Engineering, Lund University.

Total number of authors:

3

General rights

Unless other specific re-use rights are stated the following general rights apply:

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Read more about Creative commons licenses: <https://creativecommons.org/licenses/>

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

A Perturbation Analysis of Non-Linear Diffusion from
a Permeable Solid into a Finite Volume Containing a Liquid

Gustav Lindberg, Per Ståhle, Ingrid Svensson

The equation

$$\dot{c}(x, t) = c''(x, t), \quad (1)$$

with the initial conditions

$$c(x, 0) = 1 \quad \text{in } 0 < x < 1, \quad (2)$$

and the nested boundary conditions

$$c(0, t) = c(1, t) = \chi(1 - \int_0^1 c(x, t) dx). \quad (3)$$

has a solution which is split into

$$c(x, t) = u(x, t) + v(x, t). \quad (4)$$

The part $u(x, t)$ fulfils the equation

$$\dot{u}(x, t) = u''(x, t), \quad (5)$$

with the initial conditions

$$u(x, 0) = 1 \quad \text{in } 0 < x < 1, \quad (6)$$

and the boundary conditions

$$u(0, t) = u(1, t) = 0. \quad (7)$$

The part $v(x, t)$ fulfils the equation

$$\dot{v}(x, t) = v''(x, t), \quad (8)$$

with the initial conditions

$$v(x, 0) = 0 \quad \text{in } 0 < x < 1, \quad (9)$$

and the boundary conditions

$$v(0, t) = v(1, t) = \chi(1 - \int_0^1 c(x, t) dx), \quad (10)$$

where χ is the ratio of the volume of the bone, V_B versus the volume of the container V_C , i.e., $\chi = V_B/V_C$. Obviously $v(x, t)$ may be expanded in a Taylor's series as follows:

$$v(x, t) = \chi\varphi_1(x, t) + \chi^2\varphi_2(x, t) + \chi^3\varphi_3(x, t) + \dots \quad (11)$$

Since χ is a free parameter the functions $\varphi_i(x, t)$ have to fulfil the equation

$$\dot{\varphi}_i(x, t) = \varphi_i''(x, t), \quad (12)$$

with the initial conditions

$$\dot{\varphi}_i(x, 0) = 0 \quad \text{in } 0 < x < 1. \quad (13)$$

The boundary conditions become recursive according to the following:

$$\varphi_1(0, t) = \varphi_1(1, t) = \phi_1(t) = (1 - \int_0^1 u(x, t) dx), \quad (14)$$

and

$$\varphi_i(0, t) = \varphi_i(1, t) = \phi_i(t) = - \int_0^1 \varphi_{i-1}(x, t) dx, \quad \text{for } i = 2, 3, 4, \dots \quad (15)$$

Duhamel's theorem allow us to solve the φ_i 's as an application of single step boundary conditions $d\phi_i(\tau) = \frac{d\phi_i(\tau)}{d\tau}d\tau$ representing a concentration $d\phi_i(t')$ applied at the time t' ,

$$d\varphi_i(x, t) = u(x, t - \tau) \frac{d\phi_i(\tau)}{d\tau} d\tau. \quad (16)$$

Integration from $\tau' = \tau$ to $\tau' = t$ gives

$$\varphi_i(x, t) = \int_0^t u(x, t - \tau) \frac{d\phi_i(\tau)}{d\tau} d\tau = [u(x, t - \tau)\phi_i(\tau)]_0^t - \int_0^t \frac{du(x, t - \tau)}{d\tau} \phi_i(\tau) d\tau. \quad (17)$$

Considering that $u(x, 0) = 0$ and $\varphi_i(0, \tau) = 0$. The general solution is given as

$$\varphi_i(x, t) = - \int_0^t \phi_i(\tau) \frac{du(x, t - \tau)}{d\tau} d\tau. \quad (18)$$

Insertion of the solution

$$u = \sum_{n=1,3,5,..} \frac{4e^{-n^2\pi^2t}\sin(n\pi x)}{n\pi}, \quad (19)$$

gives

$$\varphi_i(x, t) = - \int_0^t \phi_i(\tau) \sum_{n=1,3,5,..} 4\pi n e^{-n^2\pi^2(t-\tau)} \sin(n\pi x) d\tau. \quad (20)$$

For large containers and, hence, small values of χ the solution for the concentration $u(x, t)$ according to (19), clearly gives the approximative solution as $\chi \rightarrow 0$.

The boundary conditions for φ_1 are obtained after integration of (19),

$$\phi_1(t) = 1 - \sum_{m=1,3,5,..} \frac{8}{m^2\pi^2} e^{-m^2\pi^2t}. \quad (21)$$

The solution of $\varphi_1(x, t)$ with the boundary conditions $\phi_1(t)$ $v(0, t)$ is the concentration in the container considering the escape of matter from the bone sample and is

$$\varphi_1(x, t) = - \int_0^t (1 - \sum_{m=1,3,5,..} \frac{8}{m^2\pi^2} e^{-m^2\pi^2\tau}) \times \sum_{n=1,3,5,..} 4\pi n e^{-n^2\pi^2(t-\tau)} \sin(n\pi x) d\tau. \quad (22)$$

$$\begin{aligned} \varphi_1(x, t) = - [& \sum_{n=1,3,5,..} \frac{4}{n\pi} (1 - e^{-n^2\pi^2t}) - \\ & - \sum_{\substack{m=1,3,5,.. \\ m \neq n}} \frac{32n}{m^2(n^2-m^2)\pi^3} (e^{-m^2\pi^2t} - e^{-n^2\pi^2t}) + f(n, t)] \sin(n\pi x), \end{aligned} \quad (23)$$

where

$$f(n, t) = \frac{64}{m^2(n^2 - m^2)\pi^4} (e^{-m^2\pi^2t} - e^{-n^2\pi^2t}) \quad \text{as } m^2 \rightarrow n^2, \quad (24)$$

which is written

$$f(n, t) = \lim_{n^2 - m^2 \rightarrow \epsilon} \frac{64}{n^2 \epsilon \pi^4} e^{-n^2\pi^2t} (e^{-\epsilon\pi^2t} - 1) = - \frac{64t}{n^2\pi^2} e^{-n^2\pi^2t}. \quad (25)$$

The contribution to the concentration in the container from φ_1 becomes

$$\begin{aligned}
\phi_2 = \int_0^1 \varphi_1 dx = & - \sum_{n=1,3,5,\dots} \frac{8}{n^2 \pi^2} (1 - e^{-n^2 \pi^2 t}) \\
& + \left[\sum_{\substack{m=1,3,5,\dots \\ m \neq n}} \frac{64}{m^2(n^2-m^2)\pi^4} (e^{-m^2 \pi^2 t} - e^{-n^2 \pi^2 t}) \right] + f(n, t),
\end{aligned} \tag{26}$$

This now reads

$$\begin{aligned}
\phi_2 = & -1 - \frac{8}{\pi^2} + \sum_{n=1,3,5,\dots} \frac{8}{n^2 \pi^2} (1 - 8t) e^{-n^2 \pi^2 t} - \\
& = \\
& -1 - \frac{8}{\pi^2} + \sum_{n=1,3,5,\dots} \left\{ \frac{8}{n^2 \pi^2} (1 - 8t) e^{-n^2 \pi^2 t} + \right. \\
& \left. \sum_{\substack{m=1,3,5,\dots \\ m \neq n}} \frac{128 e^{-n^2 \pi^2 t}}{n^2(n^2-m^2)\pi^4} \right\} = -1 - \frac{8}{\pi^2} + \\
& \sum_{n=1,3,5,\dots} \frac{8}{n^2 \pi^2} e^{-n^2 \pi^2 t} \left\{ 1 - 8t + \sum_{\substack{m=1,3,5,\dots \\ m \neq n}} \frac{16}{(n^2-m^2)\pi^2} \right\},
\end{aligned} \tag{27}$$

that may serve as boundary conditions for the function φ_2 . Considering a fairly small ratio $\chi \approx 0.01$ we anticipate the third order term $\chi^2 \varphi_2$ to be of the order of 10^{-4} . The work to do this does not seem meaningful, with the accuracy of the present measurements in mind. Figure 1 shows the difference between the solution without considering the increasing concentration in the container and same with a first order correction.

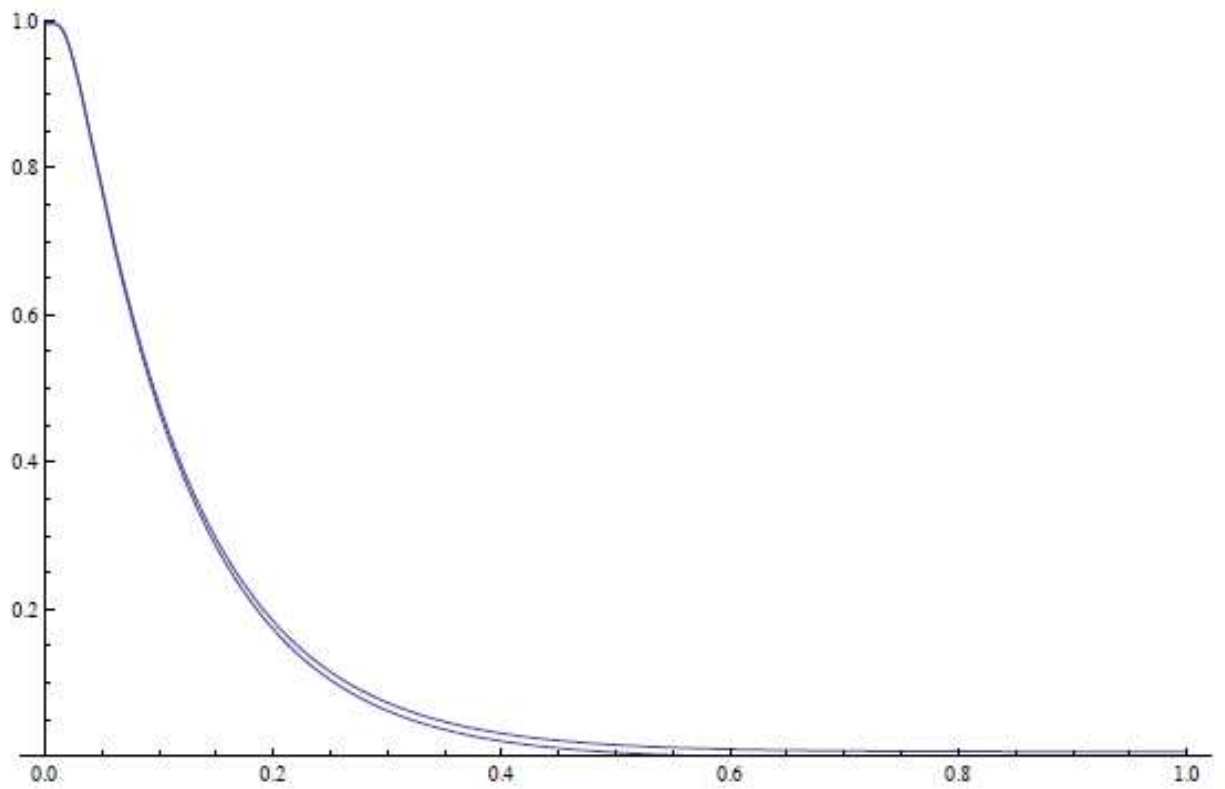


Figure 1: The solution for escaped ions for a sample in an infinite container $1 - \int_0^1 u dx$ and the same with first order correction $1 - \int_0^1 (u + \chi\varphi_1) dx$.