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ANALYSIS AND SYNTHESIS OF INERTIAL PLATFORMS WITH SINGLE AXIS GYROSCOPES

BY

KARL JOHAN ÅSTRÖM

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ELANDERS BOKTRYCKERI AKTIEBOLAG

Synopsis

An inertial stabilized platform system with three single axis gyroscopes is analysed. Special attention is given to the influence of the reaction torques of the gyroscopes on the dynamics of the system. On the basis of the analysis a synthesis method is developed using standard techniques for servomechanism synthesis.

Preface

The work presented in this report was performed as part of a research project in the TTN-Group (Inertial Navigation Group) of the division of Applied Hydromechanics of the Royal Institute of Technology in Stockholm. The TTN-Group is managed by Professor Bengt Joel Andersson and supported by the Swedish Defense Authorities (the Research Institute of National Defense, the Army, the Navy and the Air Force).

The objective of the work was to develop a method for the analysis and synthesis of platform systems. It was considered highly desirable to make the analysis as general as possible in order to cover a large variety of the systems occurring in the practical applications, both for the purpose of judging existing systems and to guide the work on new ones. In particular the analysis should cover systems where the reaction torques of the gyros are appreciable in order to investigate the possibilities of utilizing them in heavy gyro systems. The proper matching of the electrical and the mechanical properties should also be analyzed to form a foundation for a study of systems with extremely small stable elements.

The main drawbacks of the existing analyses are the restrictive assumptions made in the treatment of the mechanics of the system motivated by the special structure of the specific designs. In addition the notations used are usually very cumbersome. It was therefore suggested to start the work by developing a compact notation for the analysis of the mechanics of the system.

The author was greatly helped by the tireless discussions and questioning of Professor Bengt Joel Andersson who gave the premises and supervised the work. I also would like to thank my friend Svante Jahnberg for the stimulation obtained from discussions with him. Many thanks are also due to Mrs. Ulla Nyberg who typed the manuscript and to Mr. Karl Svahn who prepared the drawings.

Stockholm, December, 1961

Karl Johan Åström

1. Introduction

The function of a platform system is to establish a reference coordinate system for guidance, navigation, fire control or other purpose. The desired reference system may be fixed to inertial space, fixed to the earth, etc. The part of the platform system which mechanizes the reference is called the *controlled member* or the *stable element*. This is isolated from the motions of the base by a system of gimbals. The deviation of the controlled member from the desired reference system is sensed and counteracted by torque-producing devices. Much research has been devoted to the development of suitable sensors; according to the unclassified literature the gyroscopes have been the most successful. In the platform systems which use gyros, the reference system is basically fixed to inertial space. If other references are required, additional information must be provided. In earth fixed systems e.g. this can be obtained from accelerometers or pendulums. Both free gyros and single-axis gyros have been used as sensing devices. For systems with free gyros the control problem is to align the controlled member to the spin axes of the gyros. The solution to this type of problem is fairly straight forward. Systems with single axis gyros are more complex as the gyros give reaction torques when they produce signals. In many systems the gyros are very small compared to the controlled member which means that the reaction torques are negligible and that even the single axis gyros can be treated as pure sensing devices. This greatly simplifies the analysis. See e.g. DRAPER and WOODBURY (1956) and MITSUTOMI (1958).

In this report we will study an inertial stabilized platform system where the deviations of the controlled member from the inertial space are sensed by three single-axis gyros. It is assumed that the controlled member is isolated from the motions of the base by a system of gimbals arranged in such a way that it is possible to apply torques to the stable element. It is further assumed that the stable element and the gyros can be treated as rigid bodies. However, we do not make any assumptions concerning the relative magnitudes of the mechanical parameters of the system. Also we do not assume any special orientation of the gyros. The fact that the orientation of the gyros might affect to dynamics of the system was pointed out by ZACHRISSON (1957).

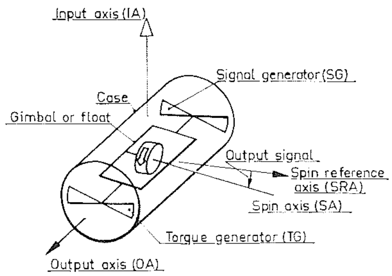

Fig. 1. Schematic diagram of a single-axis gyro.

2. Single axis gyros

The main features of a single-axis gyro are shown in Fig. 1. It consists of a gyrorotor supported by bearings in a gimbal which is supported in the case. The pivot axis of the gimbal is called the *output axis* (OA) of the gyro. The angle between the gimbal and the case is the *output signal* of the gyro. The gyro is provided with a *signal generator* (SG) which gives a signal proportional to the output signal. The gimbal is also provided with another device, the *torque generator* (TG), which makes it possible to apply a torque to the gimbal. The space between the gimbal and the case is often filled with a fluid. This serves to float the gimbal in order to decrease the friction torque in the gimbal bearings. The gimbal is therefore also called the *float*. This word will be used as being synonymous with gimbal even when the gimbal is not floated. In some applications the fluid is also used to introduce viscous damping between the gimbal and the case.

The *spin axis* (SA) of the gyro coincides with the axis of the rotor. The axis coincident with the spin axis when the output signal is zero is called the *spin reference axis* (SRA). The *input axis* of the gyro (IA) is orthogonal to the output axis and the spin axis. The axes OA, SRA and IA form a righthanded orthogonal coordinate set.

We will now give an equation which describes the performance of the single axis gyro. Introduce the following notations

J	the moment of inertia of the gyrorotor with respect to the spin axis
ω_0	the angular velocity of the gyrorotor
$H=J\cdot\omega_0$	the angular momentum of the gyrorotor
aJ	the moment of inertia of the gimbal, including the gyrorotor, with respect to the output axis
$Jm(t)$	the disturbing torque acting on the float of the gyro
$\varphi(t)$	the output signal of the gyro
$D=\dfrac{d}{dt}$	differential operator

$aJ\sigma(D)$ the transfer operator from the output signal to torque about the output axis of the gyro (the viscous torque introduced by the buoyancy fluid is also included in the operator $\sigma(D)$). $\sigma(D)$ is supposed to be a rational function of the differential operator D.

$\left.\begin{array}{l}\Omega_{IA}\\ \Omega_{UA}\\ \Omega_{SRA}\end{array}\right\}$ the components of the angular velocity of the gyro with respect to inertial space

Assuming that the inertia ellipsoid of the float is symmetric with respect to the output axis. Newton's second law of motion gives after linearization

$$a[D^2+\sigma(D)]\,\varphi(t)=\omega_0\Omega_{IA}(t)-aD\Omega_{UA}(t)-m(t) \qquad (2.1)$$

This equation is called the *signal equation* as it tells how the output signal reflects the motion of the case of the gyro. The derivation of this equation is left for the reader. (It is derived in section 4 under more general conditions than stated above).

It is desirable that the output signal depends only on the component of the angular velocity along the input axis. The term $aD\Omega_{UA}(t)$ causes what is referred to as the *output axis sensitivity* of the gyro. The last term in (2.1) is caused by the disturbances acting on the float, e.g. friction torque, mass unbalance torque, buoyancy unbalance torque, etc. The influence of these terms will be discussed later. Neglecting them the signal equation runs

$$\varphi(t)=\frac{\omega_0}{a[D^2+\sigma(D)]}\cdot\Omega_{IA}(t) \qquad (2.2)$$

The transfer function of all types of single-axis gyros can be represented by this equation. Different types of gyros correspond to different operators $\sigma(D)$. A few examples are given below.

Example 1. (The Integrating Gyro).
Choose

$$\sigma(D)=0$$

This means that there is no coupling from the output signal to torque acting on the float. The equation (2.2) gives,

$$\varphi=\frac{\omega_0}{a}\int^t\theta_{IA}(\tau)d\tau$$

where $\theta\hat{\imath}_A$ is the angle of rotation of the case of the gyro about the input axis. The output signal is thus proportional to the time integral of the angle of rotation of the case of the gyro which explains the name *integrating gyro*.

Example 2. (The Proportional Gyro).
Choose

$$\sigma(D)=\alpha D$$

This means that a torque proportional to the angular velocity of the float with respect to the case, is applied to the float. In practice this is done either by filling the space between the case and the float with a viscous liquid or by feeding the torquemotor of the gyro by the electronically differentiated output signal of the gyro. The equation (2.2) gives,

$$\varphi(t)=\frac{\omega_0}{a}\cdot\frac{1}{D(D+\alpha)}\Omega_{IA}=\frac{\omega_0}{a}\int^t e^{-\alpha(t-\tau)}\theta_{IA}(\tau)d\tau$$

The steady state output signal is thus proportional to the angle of rotation of the case, which explains why a gyro of this type is called a *proportional gyro*. The name "rate-integrating gyro" and the inconsistent abbreviation of this: "integrating gyro" is also found in the literature.

The sensitivity of the proportional gyro is

$$s=\frac{\omega_0}{\alpha a}$$

Gyros of this type are frequently used in high precision navigation platforms because of their very low drift-rates. Cf. DRAPER (1951), DRAPER, WRIGLEY, GROHE (1955) and DRAPER, WRIGLEY, HOVORKA (1960).

Example 3. (The Differentiating Gyro or the Rate Gyro).
Choose

$$\sigma(D)=\alpha D+\kappa$$

This means that the torque applied to the float is a linear combination of the output signal and its time-derivative. The term αD is necessary for a stable operation of the gyro. The term κ is in practice obtained by a mechanical spring between the case and the float or by feeding the torque-generator with a signal proportional to the output signal.

The equation (2.2) gives

$$\varphi(t)=\frac{\omega_0}{a[D^2+\alpha D+\kappa]}\Omega_{IA}(t)$$

In the steady state the output signal is thus proportional to the angular velocity of the case.

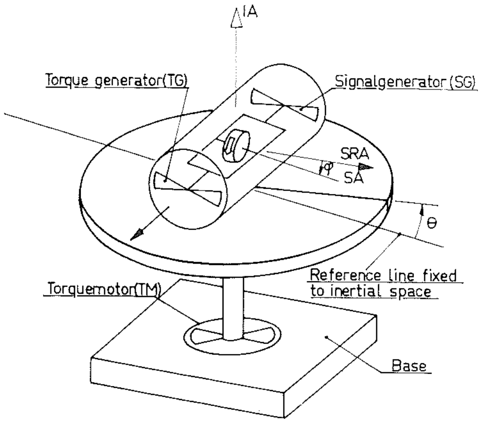

Fig. 2. Schematic diagram of a single-axis platform system.

3. Single axis platform systems

3.1. System description

In the single-axis case the stable element is suspended in such a way that it can rotate about an axis whose orientation is fixed with respect to inertial space. The orientation of the controlled member is defined by the angle $\theta(t)$. Stabilization of the platform means arranging a system in such a way that the angle $\theta(t)$ is a given function of time and of the position of the controlled member. When we restrict the discussion to inertial stabilized systems the prescribed angle is a constant. A simplified diagram of a single axis platform system is shown in Fig. 2.

The controlled member is provided with a single-axis gyro whose input axis is coincident with the axis of rotation of the controlled member. The system is provided with a torque motor which makes it possible to apply a torque to the controlled member.

The angular motions of the controlled member are sensed by the single-axis gyro. The output signal of the gyro is electronically processed and fed to the torque motor (TM) of the stable element and to the torque generator (TG) of the gyro. A signal flow diagram is shown in Fig. 3.

The problem is to determine how to do the electronic processing so that the controlled member maintains its orientation in spite of disturbances.

3.2. Analysis

We will start by deriving the equation of motion of the system. Introduce the notations

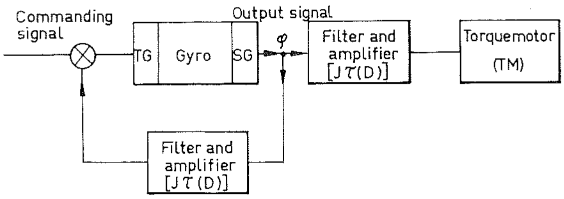

Fig. 3. Signal flow diagram of a signal-axis platform system.

bJ	the moment of inertia of the controlled member, including the gyro, with respect to the axis of rotation
$H=J\cdot\omega_0$	the angular momentum of the gyro
$JM(t)$	the disturbing torque acting on the controlled member
$JM_k(t)$	the control torque applied to the controlled member by the torquemotor (TM)
$\theta(t)$	the orientation of the controlled member with respect to inertial space
$\varphi(t)$	the output signal of the gyro
$D=\dfrac{d}{dt}$	differential operator

Applying Newton's second law of motion to the controlled member we get after linearization

$$bJD^2\theta(t)=JM(t)+JM_k(t)-HD\varphi(t) \tag{3.2.1}$$

The last term in (3.2.1) is due to the fact that a reaction torque is developed when the gyro produces an output signal. This term is referred to as *the primary reaction torque* of the gyro. The derivation of this equation is left for the reader. (It is derived under more general assumptions in section 4.)

The control torque, $M_k(t)$, is supposed to be a functional of the output signal of the gyro, $\varphi(t)$. We assume that it has the form

$$M_k(t)=-\tau(D)\cdot\varphi(t) \tag{3.2.2}$$

where $\tau(D)$ is a rational function of the differential operator $D=\dfrac{d}{dt}$.

The equations (2.1), (3.2.1) and (3.2.2) give

$$bD^2[1+Y_0(D)]\theta(t)=M(t)+\frac{bD}{\omega_0}Y_0(D)\cdot m(t) \tag{3.2.3}$$

where

$$Y_0(D)=\frac{\omega_0}{ab}\cdot\frac{\omega_0 D+\tau(D)}{D[D^2+\sigma(D)]} \tag{3.2.4}$$

This equation is referred to as the equation of motion of the controlled member with the servoloop closed. We will give a few examples which illustrate the properties of the system for different choices of the operators $\sigma(D)$ and $\tau(D)$.

Example 1

Choose

$$Y_0(D)=\frac{A_2}{D}$$

and suppose the disturbing torque acting on the float to be zero, i.e. $m(t)=0$. Equation (2.2.3) then gives

$$b[D^2+A_2D]\theta(t)=M(t)$$

An observer who analyses the system by applying torques to the controlled member will this find it mechanically equivalent to a rigid body whose moment of inertia is bJ and whose angular motions are damped with a moment proportional to its angular velocity with respect to inertial space. The coefficient of damping is A_2Jb. [$\mathrm{Nm}\cdot\mathrm{sec}\cdot\mathrm{rad}^{-1}$].

Example 2

Choose

$$Y_0(D)=\frac{A_3}{D^2}$$

and suppose that the disturbing torque acting on the gyrofloat is zero. Eq. (3.2.3) then gives

$$b[D^2+A_3]\theta(t)=M(t)$$

An observer who analyses the system by applying torques to the controlled member will thus find it mechanically equivalent to a rigid body whose moment of inertia is Jb and whose angular motions are spring-restrained to inertial space. The restraint is linear and the spring coefficient is A_3Jb [$\mathrm{Nm}\cdot\mathrm{rad}^{-1}$].

We will now define the concept of inertial stabilized platform system.

Definition 3.2.1

A single axis platform system is said to be inertial stabilized or stabilized with respect to inertial space if (3.2.2) *is stable with respect to the disturbance $M(t)$.*

(For a discussion of the concept of stability with respect to a disturbance we refer to MALKIN (1959). A necessary and sufficient condition is that all the roots of the equation $D^2[1+Y_0]=0$ have negative real-parts.)

Corollary

For an inertial stabilized platform system the function $Y_0(D)$ must at least have a double pole at the origin.

This means that for an inertial stabilized platform system the controlled member must at least be spring-restrained to inertial space. Compare example 2. Poles of $Y_0(D)$ at the origin of order higher than two means still tighter coupling between the controlled member and inertial space.

One obvious way of obtaining an inertial stabilized platform system is thus to choose

$$Y_0(D)=\frac{\omega_0^2}{D^2}+2\xi\frac{\omega_0}{D} \tag{3.2.5}$$

where the first term corresponds to spring-restraining the controlled member to inertial space and the second term is the damping necessary for a stable system. Introducing Y_0 into the (3.2.3) and Laplace-transforming we get

$$\theta(p)=\frac{1}{b[p^2+2\xi\omega_0 p+\omega_0^2]}M(p)+\frac{1}{\omega_0 p}\cdot\frac{2\xi\omega_0 p+\omega_0^2}{p^2+2\xi\omega_0 p+\omega_0^2}m(p)$$

The transformed variables are denoted by writing p for the argument. This equation shows how the controlled member reflects the disturbances $M(t)$ and $m(t)$. In many cases the specifications can be satisfied with a system of this type. This system will, however, be unnecessarily complicated. It is possible to obtain a system with almost the same properties with a less complicated instrumentation.

The disturbing torque acting on the float of the gyro has until now been neglected. The following theorem says something about their effects on the system.

Theorem 3.2.1

For a single axis inertial stabilized platform system the function

$$\frac{D[1+Y_0]}{Y_0}$$

has a root $D=0$.

The proof is obvious from the definition 3.2.1.

The theorem means that the transfer function from the disturbing torque acting on the gyrofloat to the angular deviation of the stable element has a pole at the origin. The system is thus very sensitive to low frequency disturbances acting on the float of the gyro. A constant disturbing torque gives in the steady state an angular deviation of the controlled member which increases linearly with time. This phenomenon is called the *drift* of the platform and is caused by the fact that a gyro responds to a torque on the float in the same way as to an angular velocity about the input axis. Compare the signalequation (2.1).

The theorem 3.2.1 also implies that the angular orientation of the stable element cannot be stable with respect to both the disturbances $m(t)$ and $M(t)$. (In principle we might have required stability with respect to $m(t)$. The system had then been unstable with respect to $M(t)$). The reason why we require stability with respect to $M(t)$ for an inertial stabilized system is that in practice the disturbances $M(t)$ are much greater than the disturbances $m(t)$.

If it is desired to have the stable element rotating in a prescribed manner with respect to inertial space, commanding signals are fed to the torque-generator of the gyro. For commanding signals of low frequencies the angle of rotation will then be proportional to the time-integral of the commanding signal. The inertial stabilized platform system is therefore often called an *integrating drive.*

3.3. Synthesis

The main features of the single axis system were derived in the previous section. We will now turn to the synthesis problem, i.e. we will determine the operators $\tau(D)$ and $\sigma(D)$. We must then consider

A. The character of the disturbances.
B. The sensitivity of the system to the disturbances.
C. The ability of the system to follow commanding signals.
D. The possibility of realizing the operators in physical components.

For the sake of convenience we have divided the disturbances into two groups, disturbing torque acting on the stable element, and disturbing torque acting on the float of the gyro.

The sensitivity of the system to the two groups of disturbances can be traded against each other. In the case of inertial stabilized systems the error caused by disturbing torques acting on the stable element can be made arbitrarily small by the proper choice of the transfer functions $\sigma(p)$ and $\tau(p)$. As a result the system is very sensitive to low frequency disturbing torques acting on the float of the gyro. If we instead choose to design a system in which the error due to a constant disturbing torque on the gyrofloat is finite in the steady state we will find that a constant disturbing torque on the controlled member in the steady state will give an error increasing linearly with time. The disturbing torques acting on the float of the gyro can however be made very small by proper design of the gyro. A good gyro may have a drift of about 0.01 °/hour.

In applications where it is desired to rotate the controlled member with respect to inertial space the system must of course respond fast enough to the commanding signals. The navigation frequencies are extremely low and they

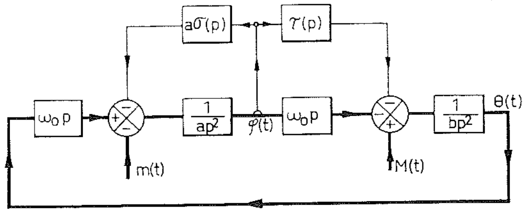

Fig. 4. Block diagram of a single-axis platform system.

will thus give very modest bandwidth requirements. In case of gear driven gimbals, the isolation of the controlled member from the base motions is only effective for frequencies lower than the bandwidth. For these systems the bandwidth must thus be greater than the highest frequency of the vibrations of the base. In direct driven gimbals there is an attenuation of the high frequency base motions due to the moment of inertia of the controlled member and the sensitivity of the system to disturbing torques can thus be traded against bandwidth. Cf. section 3.4.

The problem of determining $\sigma(D)$ and $\tau(D)$ is in many ways similar to the synthesis of servomechanisms. In order to facilitate the use of the standard methods for servomechanism synthesis we will give our problem a block-diagram representation. Laplace-transforming the equation (3.2.3) we get

$$\theta(p)=\frac{1}{bp^2[1+Y_0]}M(p)+\frac{1}{\omega_0 p}\cdot\frac{Y_0}{1+Y_0}m(p) \tag{3.3.1}$$

where

$$Y_0=\frac{\omega_0}{ab}\cdot\frac{\omega_0 p+\tau(p)}{p[p^2+\sigma(p)]} \tag{3.3.2}$$

and the transformed variables are denoted by writing p for the argument. These equations can be represented by the block-diagram of Fig. 4.

The synthesis problem is solved in the following way

1. Determine the open loop system function Y_0 or alternatively the closed loop system function

$$Y=\frac{Y_0}{1+Y_0}$$

consistent with the specifications.

2. Then determine the transfer functions $\sigma(p)$ and $\tau(p)$ consistent with the equation (3.3.2).

The first part to the synthesis is the classical problem on servomechanisms. Although no complete solution has yet been presented the problem is solved for certain classes of specifications in ordinary textbooks on automatic control. If the specifications on the system are complete the problem can be solved by optimization techniques. This is the case e.g. when the disturbing torques are stationary stochastic processes and the WIENER KOLMOGOROV theory can be utilized. When minimizing the RMS error the constraints given by the fixed components and the ability of the system to follow commanding signals must be considered. The technique for the handling of these auxiliary conditions are thoroughly treated by NEWTON, GOULD and KAISER (1957). However, in many situations the specifications on the system are incomplete, i.e. they do not uniquely determine the open loop system function Y_0.

The choice between the different possible transfer functions Y_0 is then governed by the complexity of their instrumentation. Solutions to this problem have been proposed by several authors, e.g. NICHOLS (JAMES-NICHOLS-PHILLIPS (1947), Ch. 4), EVANS (1948) and TRUXAL (1955).

These methods assume that the specifications are given in terms of error coefficients, bandwidth, and similar coefficients. The definition 3.2.1 of an inertial stabilized platform system gives the following condition on the error coefficients

Position error constant	$K_p = \infty$
Velocity error constant	$K_v = \infty$

The acceleration error constant is related to the spring coefficient of the restraint of the controlled member to inertial space by

$$k = Jb \cdot K_a \qquad [\mathrm{Nm} \cdot \mathrm{rad}^{-1}]$$

For an inertial stabilized platform system the choice of the closed loop transfer function is thus limited to the class of transfer functions giving zero velocity error. The order of magnitude of the constants for a platform in a high class navigation system are

Bandwidth	100 rad $\sec^{-1}$
Acceleration constant	$K_a = 10^3 - 10^6 \sec^{-2}$

There are applications with considerably lower bandwidth specifications. Notice however, that we can not carry out an effective synthesis from these specifications only. We must at least qualitatively know the distribution of the disturbing torques on various frequencies. Cf. section 3.4.

The second step in the synthesis, to determine $\sigma(p)$ and $\tau(p)$ when $Y_0(p)$ is given, has no unique solution. Either $\sigma(p)$ or $\tau(p)$ can be arbitrarily choosen, the other is then given by (3.3.2). In many applications there is no feedback around the gyro; the transfer function $\sigma(p)$ is then uniquely given by the choice of the gyro. Cf. section 2.

Also notice that it is possible to choose $\tau(p)=0$, which means that no gimbal torque motor is used. Disturbing torque acting on the controlled member is then counteracted by the precession of the gyro. The gyro thus serves the double purpose as sensing and actuating device. This means specifications on the gyro which are difficult to obtain with the technological means of today. A system of this type cannot withstand constant disturbing torque acting over long periods of time as the output signal of the gyro in that case will not be small.

3.4. Examples of synthesis

We will now give some examples on the synthesis of an inertial stabilized single axis platform system. In these examples we will use the synthesis method given by TRUXAL (1955). We start with the synthesis of a system where the primary reaction torque of the gyro can be neglected.

3.4.1. *Systems where the primary reaction torques are negligible*

Example 1

Synthesize a system with the specifications

Bandwidth	B
Acceleration constant	K_a

The primary reaction torque of the gyro is negligible, which means that

$$|\tau(p)| \gg \omega_0 |p| \qquad \text{for all actual frequencies}$$

Equation (3.3.2) then gives

$$Y_0 = \frac{\omega_0}{ab} \cdot \frac{\tau(p)}{p[p^2+\sigma(p)]} \tag{3.4.1}$$

Choose the closed loop system function

$$Y = \frac{\beta^2 p_1 p_2}{z_1} \cdot \frac{p+z_1}{(p^2+2\zeta\beta p+\beta^2)(p+p_1)(p+p_2)} \tag{3.4.2}$$

This choice is governed by

1. The excess of poles over zeros for Y must at least equal the excess (N) for Y_0. In this case we have $N=3$. (This implies that $\tau(p)\to 1$ for the highest frequency of interest. In some cases it is desirable to have $\tau(p)\to \frac{1}{p^n}$ at the highest frequency of interest. The corresponding modifications are left for the reader).

2. At least one zero is needed in order to get a system with an infinite velocity constant.

The bandwidth condition is satisfied by the choice of ζ and β. The zero z_1 is determined by the condition on the velocity constant, i.e.

$$\frac{1}{K_v}=\frac{2\zeta}{\beta}+\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{z_1}=0$$

We have then two constants p_1 and p_2 left for satisfying the condition on the acceleration constant. The remaining condition is chosen in such a way that the compensating network becomes as simple as possible.

The open loop system function is

$$Y_0=\frac{Y}{1-Y}=\frac{\beta^2 p_1 p_2}{z_1}\cdot\frac{p+z_1}{p^2(p+p_1')(p+p_2')} \tag{3.4.3}$$

where

$$p_1'+p_2'=2\zeta\beta+p_1+p_2$$

$$p_1'\cdot p_2'=\beta^2+p_1p_2+2\zeta\beta(p_1+p_2)$$

The acceleration constant is

$$K_a=\beta^2\cdot\frac{p_1\cdot p_2}{p_1'\cdot p_2'}$$

The transfer function from disturbing torque acting on the controlled member to the angular deviation of the controlled member is

$$Y_M=\frac{Y}{bp^2Y_0}=\frac{1}{b}\cdot\frac{(p+p_1')(p+p_2')}{(p^2+2\zeta\beta p+\beta^2)(p+p_1)(p+p_2)} \tag{3.4.4}$$

The equations (3.4.1) and (3.4.4) give

$$\tau(p)=\frac{ab\beta^2p_1p_2}{\omega_0 z_1}\cdot\frac{(p+z_1)(p^2+\sigma(p))}{p(p+p_1')(p+p_2')} \tag{3.4.5}$$

There are many possibilities for determining $\sigma(p)$ and $\tau(p)$ consistent with this equation.

a) Assume that no feedback is used around the gyro. The transfer function $\sigma(p)$ is then determined by the characteristics of the gyro. The gyro is supposed to be proportional, hence

$$\sigma(p)=ap$$

Cf. section 2. The equation (3.4.5) gives

$$\tau(p)=\frac{ab\beta^2 p_1 p_2}{\omega_0 z_1}\cdot\frac{(p+z_1)(p+\alpha)}{(p+p_1')(p+p_2')}$$

In order to get a simple transfer function $\tau(p)$ it is desirable to have $p_2'=\alpha$.

Given ζ, β, α and K_a we have 4 equations to determining the unknowns p_1, p_2, p_1' and z_1. If the system should be stable it must be required that the poles p_1 and p_2 are in the right plane. This gives the following conditions for the constants β, α and K_a

$$\begin{cases} \dfrac{\alpha}{\beta}>2\zeta \\ \\ \dfrac{K_a}{\beta^2}<1-2\zeta\dfrac{\beta}{\alpha} \end{cases} \tag{3.4.6}$$

If the specifications are consistent with these it is thus possible to chose $p_4=\alpha$. The transfer function $\tau(p)$ then becomes

$$\tau(p)=\frac{ab\beta^2 p_1 p_2}{\omega_0 z_1}\cdot\frac{p+z_1}{p+p_1'}$$

Introduce the following numerical values

$$\beta=100 \qquad rad\cdot sec^{-1}$$

$$K_a=10^3 \qquad sec^{-2}$$

$$\zeta=0.7$$

$$\alpha=600 \qquad sec^{-1}$$

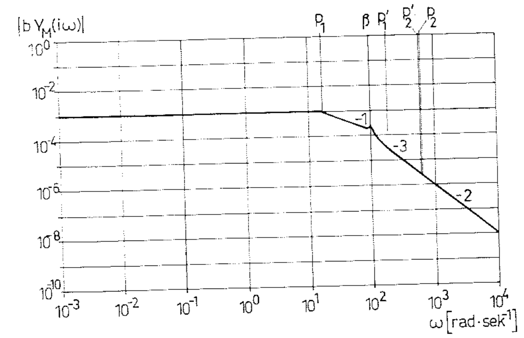

Fig. 5. Frequency response of the transfer function Y_M from disturbing torque acting on the controlled member to controlled member angular deviation for a system with $B=100$ rad $\cdot$ $\cdot$ sec^{-1}, $K_a=10^3$ sec^{-2}.

then

$$Y=6.65\times 10^6 \frac{p+16.8}{(p^2+140p+10\,000)(p+17.8)(p+628)}$$

$$Y_0=6.65\times 10^6 \frac{p+16.8}{p^2(p+186)(p+600)}$$

$$\tau(p)=6.65\times 10^6 \frac{ab}{\omega_0}\cdot\frac{p+16.8}{p+186}$$

$$Y_M=\frac{1}{b}\cdot\frac{(p+186)(p+600)}{(p^2+140p+10\,000)(p+17.8)(p+628)}$$

The frequency response of the transfer function Y_M is shown in Fig. 5.

b) It is now assumed that

$$\sigma(p)=\alpha p+\kappa$$

which can be obtained by using a rategyro or by having feedback around a proportional gyro. Cf. section 2. The equation (3.4.5) gives

$$\tau(p)=\frac{ab\beta^2 p_1 p_2}{\omega_0 z_1}\cdot\frac{(p+z_1)(p^2+\alpha p+\kappa)}{p(p+p_1')(p+p_2')}$$

In order to get a simple transfer function $\tau(p)$ it is desirable to have

$$\alpha=p_1' p_2'$$

$$\kappa=p_1'+p_2'$$

Given β, ζ, K_a, α and κ there is five variables (p_1', p_1', p_2, p_2' and z_1) to satisfy six equations. In general it is thus not possible to specify more than four of the constants. Under conditions similar to a) the system can be solved and we get

$$\tau(p)=\frac{ab\beta^2 p_1 p_2}{\omega_0 z_1}\cdot\frac{p+z_1}{p}$$

In this case the transfer function $\tau(p)$ includes an integration. The character of the over all system transfer functions are of course the same as in a).

Example 2

The specifications of the system considered in example 1 are subject to the restriction

$$\frac{K_a}{\beta^2}<1-2\zeta\frac{\beta}{\alpha}$$

The acceleration constant must thus at least be less than the square of the bandwidth. If higher values of the acceleration constant are required the overall system function cannot have the form given by (3.4.2). Instead the overall system function

$$Y(p)=\frac{\beta^2 p_1 p_2 p_3}{z_1 z_2}\cdot\frac{(p+z_1)(p+z_2)}{(p^2+2\zeta\beta p+\beta^2)(p+p_1)(p+p_2)(p+p_3)} \tag{3.4.7}$$

is chosen.

The condition on the velocity constant gives

$$\frac{1}{K_v}=\frac{2\zeta}{\beta}+\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}-\frac{1}{z_1}-\frac{1}{z_2}=0$$

The open loop system function is

$$Y_0=\frac{Y}{1-Y}=\frac{\beta^2 p_1 p_2 p_3}{z_1 z_2}\cdot\frac{(p+z_1)(p+z_2)}{p^2(p+p_1')(p+p_2')(p+p_3')} \tag{3.4.8}$$

where

$$p_1'+p_2'+p_3'=p_1+p_2+p_3+2\zeta\beta$$

$$p_1'p_2'+p_1'p_3'+p_2'p_3'=p_1p_2+p_1p_3+p_2p_3+2\zeta\beta(p_1+p_2+p_3)+\beta^2$$

$$p_1'p_2'p_3'=p_1p_2p_3+2\zeta\beta(p_1p_2+p_1p_3+p_2p_3)$$

The acceleration constant is then

$$K_a=\beta^2\frac{p_1p_2p_3}{p_1'p_2'p_3'}$$

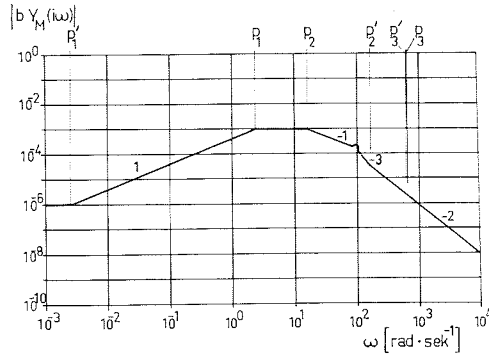

Fig. 6. Frequency response of the transfer function Y_M from disturbing torque acting on the controlled member to controlled member angular deviation for a system with $B=100$ rad · sec^{-1}, $K_a=10^6$ sec^{-2}.

Given ζ, β and K_a there are eight variables to satisfy five equations. It is thus possible to impose three other conditions. These are chosen in order to get a simple instrumentation in the same way as was demonstrated in the previous example. Notice that arbitrarily high values of the acceleration constant can be specified. The transfer function from disturbing torque acting on the controlled member to controlled member angular deviation is

$$Y_M = \frac{Y}{bp^2 Y_0} = \frac{(p+p_1')(p+p_2')(p+p_3')}{(p^2+2\xi\beta p+\beta^2)(p+p_1)(p+p_2)(p+p_3)} \tag{3.4.10}$$

Fig. 6 and Fig. 7 show the frequency response of this transfer function for the specifications $\beta=100$ rad · sec^{-1} $K_a=10^6$ rad · sec^{-2} and $\beta=10^{-2}$ rad · sec^{-1} $K_a=10^3$ rad · sec^{-2} respectively.

3.4.2. *Remark concerning the specifications*

By comparing Fig. 5, Fig. 6, and Fig. 7 we can conclude that the acceleration constant is not necessarily a proper measure of the sensitivity of the system to disturbing torques acting on the controlled member. To judge this it is necessary, at least qualitatively, to know the frequency distribution of the disturbing torques. A comparison of Fig. 5 and Fig. 7 tells e.g. that the influence of high frequency disturbing torques (>1 rad · sec^{-1}) can be greatly diminished by a reduction of the bandwidth.

3.4.3. *Systems where the primary reaction torques are not negligible*

As the platform system becomes smaller the primary reaction torque of the gyro can no longer be neglected. When the reaction torque is not negligible

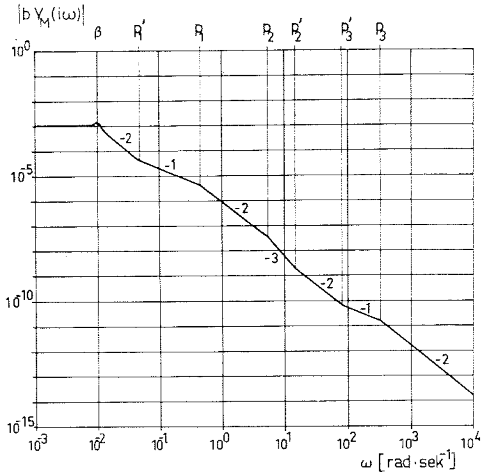

Fig. 7. Frequency response of the transfer function Y_M from disturbing torque acting on the controlled member to controlled member angular deviation for a system with $B=0.01$ rad · · $\sec^{-1}$, $K_a=10^3 \sec^{-2}$.

why should it not be used for stabilization purposes? In section 3.3 we found that it was not convenient to let the reaction torque of the gyro take care of all the stabilization but is it not possible to synthesize part of the open loop system function, e.g. the damping term of Y_0, by the primary reaction torque? These problems will be discussed in connection with some examples in the following.

Example 3

Synthesize a single axis platform system with the specifications

Bandwidth	ω_B
Acceleration constant	K_a

Also investigate the possibilities of utilizing the primary reaction torque of the gyro.

When the reaction torque is not negligible the term $\omega_0 p$ can not be neglected in comparison with $\tau(p)$. The complete equation for Y_0, (3.3.2), must thus be used.

The following closed loop system function is chosen

$$Y(p)=\frac{\beta^2 p_1}{z_1}\cdot\frac{p+z_1}{(p^2+2\zeta\beta p+\beta^2)(p+p_1)} \qquad (3.4.12)$$

This choice is governed by

1. The excess of poles over zeros for Y must at least equal the excess (N) for Y_0. In this case we have N=2. Notice that N is independent of the properties of $\tau(p)$. Cf. example 1.
2. At least one zero is needed in order to get a system with infinite velocity constant.

The bandwidth condition, which essentially determines the transient response, is satisfied by the choice of ζ and β. The zero z_1 is determined by the requirement for an infinite velocity constant, i.e.

$$\frac{1}{K_v}=\frac{2\zeta}{\beta}+\frac{1}{p_1}-\frac{1}{z_1}=0$$

The remaining constant p_1 is given by the specification on the acceleration constant.

The open loop system function is

$$Y_0=\frac{Y}{1-Y}=\frac{\beta^2 p_1}{z_1}\cdot\frac{p+z_1}{p^2(p+p_1')} \tag{3.4.13}$$

where

$$p_1'=p_1+2\zeta\beta$$

The acceleration constant is

$$K_a=\beta^2\cdot\frac{p_1}{p_1'}$$

The transfer function from disturbing torque acting on the controlled member to the angular deviation of the controlled member is

$$Y_M=\frac{Y}{bp^2Y_0}=\frac{1}{b}\,\frac{p+p_1'}{(p^2+2\zeta\beta p+\beta^2)(p+p_1)}$$

Given β, ζ and K_a we have two equations for determining p_1 and z_1. Solving these, we get

$$p_1=2\zeta\beta\cdot\frac{K_a}{\beta^2-K_a}$$

$$z_1=2\zeta\beta\cdot\frac{K_a}{\beta^2-K_a(1-4\zeta^2)}$$

If the transient behaviour of the system is not to be significantly affected by p_1 and z_1 then the quotient p_1/z_1 should not deviate too much from unity.

Put

$$\frac{p_1}{z_1}=1+\delta$$

The constant δ is the relative influence of p_1 and z_1 on the transient response. Cf. TRUXAL (1950). Requiring that $\delta<\delta_0$ and further that p_1 and z_1 are in the right half plane, the following inequality is obtained

$$1<\frac{\beta^2}{K_a}<1+\frac{4\zeta^2}{\delta_0} \tag{3.4.14}$$

which is the condition to be imposed on the specifications of a system with an over all system function according to (3.4.12).

A reasonable value is $\delta_0=0.25$. Putting $\zeta=0.7$ we get

$$0.11<\frac{K_a}{\beta^2}<1$$

Cf. (3.4.6).

When the open loop transfer function is determined there remains to determine $\tau(p)$ and $\sigma(p)$.

The equations (3.4.13) and (3.3.2) give

$$\tau(p)=\frac{ab\beta^2 p_1}{\omega_0 z_1}\cdot\frac{a_3p^3+a_2p^2+\sigma(p)(p+z_1)}{p(p+p_2)} \tag{3.4.15}$$

where

$$a_2=z_1-\frac{\omega_0^2 z_1 p_2}{ab\beta^2 p_1}$$

$$a_3=1-\frac{\omega_0^2 z_1}{ab\beta^2 p_1}$$

To give $\tau(p)$ a simple form it is desirable to have $a_3=0$.

Putting $a_3=0$ and we get

$$\frac{\omega_0^2}{ab\beta^2}=1+\delta \tag{3.4.16}$$

which is the condition to be satisfied if the damping coefficient should be synthesized by the primary reaction torque of the gyro. The condition means physically that the moment of inertia of the controlled member should match the angular velocity of the gyrorotor and the bandwidth of the system. In the following it is assumed that (3.4.16) is satisfied, which means $a_3=0$. We will now turn to the problem of determining $\tau(p)$ and $\sigma(p)$. Only one case will be discussed.

Assume that the gyro is proportional, i.e.

$$\sigma(p)=\alpha p$$

hence

$$\tau(p)=\omega_0\frac{p(z_1+\alpha-p_2)+\alpha z_1}{p+p_1'}$$

This is further simplified if we have

$$\alpha=p_1'-z_1=p_1+2\zeta\beta-z_1$$

then

$$\tau(p)=\omega_0(p_1'-z_1)\frac{z_1}{p+p_1'}$$

The above condition on the damping coefficient of the gyro can be achieved either by choosing a suitable filling fluid or by using feedback around the gyro.

Example 4

The choice of the over all system function (3.4.12) in example 3 restricts the synthesis to specifications where

$$1<\frac{\beta^2}{K_a}<1+\frac{4\zeta^2}{\delta_0}$$

If the specifications are not consistent with this inequality, another overall system function, which allows higher values of the acceleration constant, must be chosen. A possible choice is

$$Y(p)=\frac{\beta^2 p_1 p_2}{z_1 z_2}\cdot\frac{(p+z_1)(p+z_2)}{(p^2+2\zeta\beta p+\beta^2)(p+p_1)(p+p_2)} \tag{3.4.17}$$

The condition on the velocity constant gives

$$\frac{1}{K_v}=\frac{2\zeta}{\beta}+\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{z_1}-\frac{1}{z_2}=0$$

The open loop system function is

$$Y_0=\frac{\beta^2 p_1 p_2}{z_1 z_2}\cdot\frac{(p+z_1)(p+z_2)}{p^2(p+p_1')(p+p_2')} \tag{3.4.18}$$

where

$$p_1'+p_2'=p_1+p_2+2\zeta\beta$$

$$p_1'p_2'=p_1\cdot p_2+2\zeta\beta(p_1+p_2)+\beta^2-\frac{\beta^2 p_1 p_2}{z_1 z_2}$$

The acceleration constant is

$$K_a = \beta^2 \cdot \frac{p_1 p_2}{p_1' p_2'}$$

The transfer function from disturbing torque acting on the controlled member to the controlled member angular deviation is

$$Y_M(p) = \frac{Y}{bp^2 Y_0} = \frac{1}{b} \cdot \frac{(p+p_1')(p+p_2')}{(p^2+2\zeta\beta p+\beta^2)(p+p_1)(p+p_2)}$$

The transfer function $\tau(p)$ is

$$\tau(p) = \frac{ab}{\omega_0} \cdot \frac{\beta^2 p_1 p_2}{z_1 z_2} \cdot \frac{a_4 p^4 + a_3 p^3 + a_2 p^2 + \sigma(p)(p+z_1)(p+z_2)}{p(p+p_1')(p+p_2')}$$

where

$$a_4 = 1 - \frac{\omega_0^2}{ab\beta^2} \cdot \frac{z_1 z_2}{p_1 p_2}$$

$$a_3 = z_1 + z_2 - \frac{\omega_0^2}{ab\beta^2} \cdot \frac{z_1 z_2}{p_1 p_2} (p_1' + p_2')$$

$$a_2 = z_1 \cdot z_2 - \frac{\omega_0^2}{ab\beta^2} \cdot \frac{z_1 z_2}{p_1 p_2} \cdot p_1' \cdot p_2'$$

The transfer function $\tau(p)$ has a particularly simple form if $a_4 = 0$ i.e.

$$\frac{ab\beta^2}{\omega_0^2} = \frac{p_1 p_2}{z_1 z_2}$$

which is the condition to be satisfied if the damping term should be synthesized with the primary reaction torque of the gyro.

4. Three axis platform systems

4.1. System description

A simplified diagram of a three axis platform system is shown in Fig. 8. The controlled member is suspended for three degrees of freedom, e.g. by a system of gimbals. The suspension is arranged in such a way that it is possible to apply a control torque to the stable element. This is obtained by providing the gimbals with torque-motors.

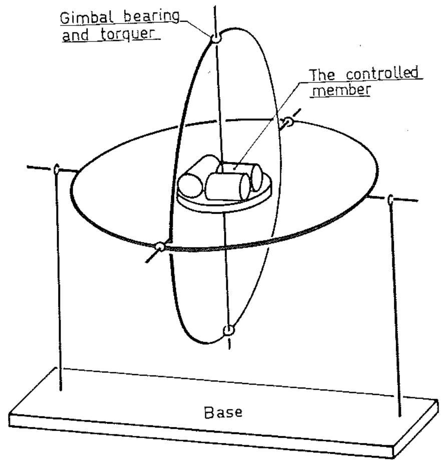

Fig. 8. Simplified diagram of a three-axis platform system.

The angular motions of the stable element are sensed by three single axis gyros, whose input axes do not lie in the same plane. The output signals of the gyros are processed electronically and fed to the gimbal torquemotors in such a way that the controlled member maintains the desired orientation in spite of the disturbances. In order to distribute the signals to the gimbal torquemotors it is necessary to know the mutual orientation of the gimbals. This information is obtained from resolvers on the gimbals.

We will assume that it is possible to command each torquemotor by signals from all gyros and also that the torquegenerator of each gyro can be controlled by all the output signals.

A signal flow diagram of one of the channels is shown in Fig. 9.

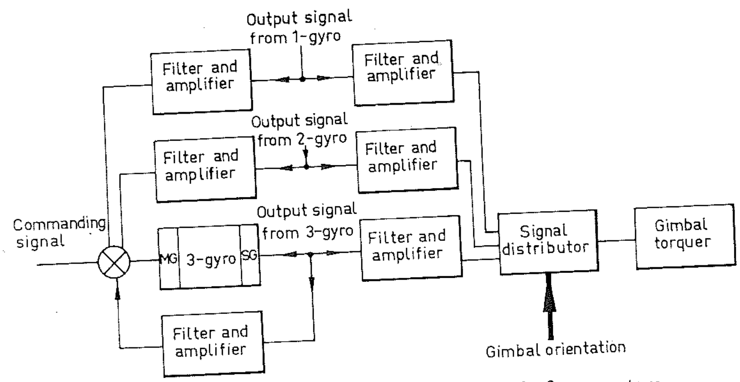

Fig. 9. Signal flow diagram of a three-axis platform system.

The problem is to determine how to do the electronic processing. The answer to this problem will be given in terms of the transfer functions from the output signals of the gyros to the torque applied to the controlled member and to the floats of the gyros. The torque applied to the controlled member will be given as components on axes fixed to the controlled member. When these components are determined only a coordinate transformation is required to obtain the torque to be applied by the gimbal torquers.

4.2. **Equations of motion**

We will now derive the equations which describe the performance of the system. To simplify the analysis we make the following assumptions

1. The controlled member and the gyros are rigid bodies.
2. The three gyros have the same mechanical properties.
3. The gyrorotor is symmetric with respect to the axis of rotation.
4. The center of mass of the gyros lies on the output axis.
5. The angular velocity of the gyrorotor with respect to the float is constant.

4.2.1. *Coordinate systems*

The center of mass of the controlled member, including the gyrofloats, is O. The gyros are denoted by 1, 2 and 3, and referred to as the "1-gyro" etc. The center of the m-gyro is denoted by $O^{(m)}$. Introduce the right-handed orthogonal coordinate sets

$Oy_1y_2y_3$ fixed to the controlled member

$O^{(m)}x_1^{(m)}x_2^{(m)}x_3^{(m)}$ fixed to the m-gyro, the x_1, x_2 and x_3-axes coincide with the input axis, output axis and spin reference axes respectively

$O^{(m)}z_1^{(m)}z_2^{(m)}z_3^{(m)}$ fixed to the float of the m-gyro the z_1-, z_2- and z_3-axes coincide with the input, output and spin axes respectively.

Cf. Fig. 10.

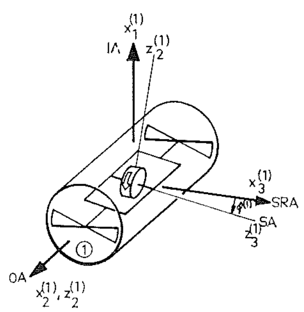

Fig. 10.

The following conventions are introduced in order to simplify the algebraic work.

(1) Latin indices used as subscripts will take all values from 1 to 3 unless the contrary is specified.

(2) If a Latin index is repeated in a term, it is understood that a summation with respect to that index over the range 1, 2, 3 is implied.

Introduce the Kronecker delta δ_{ij} and the permutation synbol ε_{ijk} defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases}$$

$$\varepsilon_{ijk} = \begin{cases} 1 \text{ if indices } ijk \text{ occur in cyclic order} \\ -1 \text{ if indices } ijk \text{ occur in acyclic order} \\ 0 \text{ if two indices are equal} \end{cases}$$

The position of the m-gyro on the controlled member is thus given by $O^{(m)}$, and the orientation of the m-gyro by the transformation from the y-set to the x-set

$$x_i^{(m)} = p_{ij}^{(m)} y_j \tag{4.2.1}$$

In the special case when the input axes of the gyros are orthogonal we can choose the coordinate systems in such a way that the input axis of the m-gyro is coincident with the y_m-axis. The orientation of the gyros is then determined by three angles $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$, called *the orientation angles* of the gyros, cf. Fig. 11. The transformation matrices $P^{(m)} = \{p_{ij}^{(m)}\}$ then become

$$\begin{cases} p_{1i}^{(m)} = \delta_{im};\ p_{im}^{(m)} = \delta_{1i} \\ p_{2,\,m+1}^{(m)} = \cos\theta^{(m)} \\ p_{2,\,m+2}^{(m)} = \sin\theta^{(m)} \\ p_{3,\,m+1}^{(m)} = -\sin\theta^{(m)} \\ p_{3,\,m+2}^{(m)} = \cos\theta^{(m)} \end{cases} \tag{4.2.2}$$

thereby defining

$$p_{i,\,m+3}^{(m)} = p_{im}^{(m)} \qquad m = 1, 2, 3$$

The transformation from the x-set to the z-set is a rotation around the output axis. Hence

$$z_i^{(m)} = r_{ij}^{(m)} x_j^{(m)} \tag{4.2.3}$$

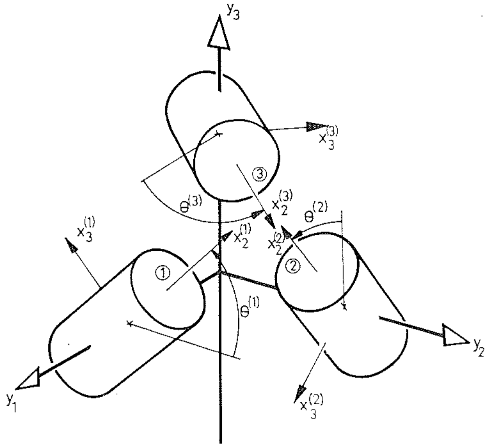

Fig. 11. Arrangement of the gyros for a system with orthogonal input axes.

where

$$\begin{cases} r_{11}^{(m)} = \cos \varphi^{(m)} \\ r_{13}^{(m)} = -\sin \varphi^{(m)} \\ r_{31}^{(m)} = \sin \varphi^{(m)} \\ r_{33}^{(m)} = \cos \varphi^{(m)} \\ r_{2i}^{(m)} = r_{i2}^{(m)} = \delta_{i2} \end{cases} \tag{4.2.4}$$

Combining (4.2.1) and (4.2.3) we get

$$z_i^{(m)} = q_{ij}^{(m)} y_j \tag{4.2.5}$$

where

$$q_{ij}^{(m)} = r_{is}^{(m)} p_{sj}^{(m)}$$

If the input axes of the gyros coincide with the y_i-axes the transformation matrices $Q^{(m)} = \{q_{ij}^{(m)}\}$ are

$$Q^{(1)} = \begin{pmatrix} \cos \varphi^{(1)} & \sin \varphi^{(1)} \sin \theta^{(1)} & -\sin \varphi^{(1)} \cos \theta^{(1)} \\ 0 & \cos \theta^{(1)} & \sin \theta^{(1)} \\ \sin \varphi^{(1)} & -\cos \varphi^{(1)} \sin \theta^{(1)} & \cos \varphi^{(1)} \cos \theta^{(1)} \end{pmatrix}$$

$$Q^{(2)} = \begin{pmatrix} -\sin \varphi^{(2)} \cos \theta^{(2)} & \cos \varphi^{(2)} & \sin \varphi^{(2)} \sin \theta^{(2)} \\ \sin \theta^{(2)} & 0 & \cos \theta^{(2)} \\ \cos \varphi^{(2)} \cos \theta^{(2)} & \sin \varphi^{(2)} & -\cos \varphi^{(2)} \sin \theta^{(2)} \end{pmatrix} \tag{4.2.6}$$

$$Q^{(3)} = \begin{pmatrix} \sin \varphi^{(3)} \sin \theta^{(3)} & -\sin \varphi^{(3)} \cos \theta^{(3)} & \cos \varphi^{(3)} \\ \cos \theta^{(3)} & \sin \theta^{(3)} & 0 \\ -\cos \varphi^{(3)} \sin \theta^{(3)} & \cos \varphi^{(3)} \cos \theta^{(3)} & \sin \varphi^{(3)} \end{pmatrix}$$

4.2.2. ***Analysis of the orientation of the gyros***

We will now introduce two quantities which depend on the orientation of the gyros and influence the dynamics of the system.

The total angular momentum of the gyros is

$$H \cdot \sqrt{s}$$

where

$$s = \sum_{m=1}^{3} \hat{x}_3^{(m)2} \tag{4.2.7}$$

The quantity s is called the *spinnumber* or the *spin* of the system. The significance of this quantity is shown in chapter 5. Another quantity of significance is the *output orientation number* l, which is defined as the triple scalar product

$$l = [\hat{x}_2^{(1)}, \hat{x}_2^{(2)}, \hat{x}_2^{(3)}] \tag{4.2.8}$$

The output axis orientation number can be interpreted geometrically as the volume of the parallelepiped with the output axes unit vectors $\hat{x}_2^{(1)}$, $\hat{x}_2^{(2)}$ and $\hat{x}_2^{(3)}$ as concurrent sides.

The spinnumber and the output axis orientation number obviously satisfy the inequalities

$$0 \leq s \leq 9$$

$$-1 \leq l \leq 1$$

If the input axes of the gyros are orthogonal, the orientation of the gyros is determined by the orientation angles $\theta^{(1)}$, $\theta^{(2)}$ and $\theta^{(3)}$. The arrangement of the gyros can then be classified by the triplet $\theta^{(1)}, \theta^{(2)}, \theta^{(3)}$. An arrangement is called *cyclic* if all angles $\theta^{(m)}$ are equal and *orthogonal* if they are multiples of $\frac{1}{2}\pi$.

The equations (4.2.1), (4.2.2), (4.2.7) and (4.2.8) give the following relationship between the spin number, the output axis orientation number and the orientation angles

$$s = (\sin\theta^{(1)} - \cos\theta^{(3)})^2 + (\sin\theta^{(2)} - \cos\theta^{(1)})^2 + (\sin\theta^{(3)} - \cos\theta^{(2)})^2 \tag{4.2.9}$$

$$l = \sin\theta^{(1)}\sin\theta^{(2)}\sin\theta^{(3)} + \cos\theta^{(1)}\cos\theta^{(2)}\cos\theta^{(3)} \tag{4.2.10}$$

In case of orthogonal input axes we have

$$0 \leq s \leq 6$$

$$-1 \leq l \leq 1$$

If we further require that the arrangement is orthogonal we get

$$1 \leq s \leq 5$$

There are 64 orthogonal arrangements which can be arranged in four groups with $s=1$, $l=0$; $s=3$, $l=-1$; $s=3$, $l=1$; $s=5$, $l=0$; respectively.

4.2.3. *The equation of motion of one of the gyros*

We will now derive the equation of motion of one of the gyros. Introduce the notations.

J	the moment of inertia of the gyrorotor
ω_0	the angular velocity of the gyrorotor
AJ_{kl}	the inertia matrix of the gyrofloat including the gyrorotor $AJ_{kl}=\int(\delta_{kl}\delta_{ij}-\delta_{kj}\delta_{il})z_i z_j dm$
$\boldsymbol{\Omega}$	the angular velocity of the controlled member with respect to inertial space
Ω_s	the component of $\boldsymbol{\Omega}$ on the y_s axis
$\varphi^{(m)}$	the output signal of the m-gyro
$D=\dfrac{d}{dt}$	differential operator
$JA_{22}\sigma_{ij}(D)$	the transfer function from the output signal of the j-gyro to torque acting on the float of the i-gyro. (The viscous damping due to the flotation fluid is included in $\sigma_{ij}(D)$).
$Jm^{(i)}$	the component, on the output axis, of the disturbing torque acting on the float of the i-gyro.

The angular velocity of the controlled member is

$$\boldsymbol{\Omega}=\Omega_s \hat{y}_s$$

The float of the m-gyro has the angular velocity $\boldsymbol{\omega}^{(m)}$.

$$\boldsymbol{\omega}^{(m)}=\omega_s^{(m)}\hat{z}_s^{(m)}$$

where

$$\omega_s^{(m)}=q_{st}^{(m)}\Omega_t+\dot{\varphi}^{(m)}\delta_{s2} \tag{4.2.11}$$

The angular momentum of the float of the m-gyro is $\boldsymbol{H}^{(m)}$. Hence

$$\boldsymbol{H}^{(m)}=JA_{rs}\omega_s^{(m)}\hat{z}_r^{(m)}+J\omega_0\hat{z}_3^{(m)}$$

Differentiating with respect to time we get

$$\dot{H}^{(m)} = J[A_{ks}\dot{\omega}_s^{(m)} + A_{js}\omega_s^{(m)}\omega_i^{(m)}\varepsilon_{ijk} + \omega_0\omega_i^{(m)}\varepsilon_{i3k}]\hat{z}_k^{(m)} \tag{4.2.12}$$

The equations (4.2.11) and (4.2.12) give

$$\begin{aligned}\dot{H}^{(m)} = J[&A_{k2}\ddot{\varphi}^{(m)} + \varepsilon_{23k}\dot{\varphi}^{(m)}\omega_0 + A_{ks}q_{st}^{(m)}\dot{\Omega}_t + q_{it}^{(m)}\varepsilon_{i3k}\Omega_t\omega_0 + \\ &+ A_{j2}\varepsilon_{2jk}\dot{\varphi}^{(m)}\dot{\varphi}^{(m)} + A_{j2}q_{ir}^{(m)}\varepsilon_{ijk}\Omega_r\dot{\varphi}^{(m)} + A_{js}q_{st}^{(m)}\varepsilon_{2ik}\Omega_t\dot{\varphi}^{(m)} + \\ &+ A_{ks}\dot{q}_{st}^{(m)}\Omega_t + A_{js}q_{st}^{(m)}q_{ir}^{(m)}\varepsilon_{ijk}\Omega_t\Omega_r]\hat{z}_k^{(m)}\end{aligned} \tag{4.2.13}$$

The component of $\dot{H}^{(m)}$ along the output axis is

$$\begin{aligned}\dot{H}_2^{(m)} = J[&A_{22}\ddot{\varphi}^{(m)} + A_{2s}q_{st}^{(m)}\dot{\Omega}_t - q_{1t}^{(m)}\Omega_t\omega_0 + A_{j2}q_{ir}^{(m)}\varepsilon_{ij2r}\Omega\dot{\varphi}^{(m)} + \\ &+ A_{2s}\dot{q}_{st}^{(m)}\Omega_t + A_{js}q_{st}^{(m)}q_{ir}^{(m)}\varepsilon_{ij2}\Omega_t\Omega_r]\end{aligned}$$

The torque acting on the float of the m-gyro has a component $\mathscr{M}_2^{(m)}$ along the output axis where

$$\mathscr{M}_2^{(m)} = -JA_{22}[\sigma_{m1}(D)\varphi^{(1)} + \sigma_{m2}(D)\varphi^{(2)} + \sigma_{m3}(D)\varphi^{(3)}] - Jm^{(m)}$$

The terms in the bracket are the viscous torque and the torque applied by the torquegenerator which is controlled by all the output signals. The last term is the disturbing torque acting on the float of the gyro.

Newton's second law of motion gives

$$\begin{aligned}A_{22}[D^2\varphi^{(m)} + \sum_{s=1}^{3}\sigma_{ms}(D)\varphi^{(s)}] &= \omega_0 q_{1t}^{(m)}\Omega_t - A_{2s}q_{st}^{(m)}\dot{\Omega}_t \\ &- A_{j2}q_{ir}^{(m)}\varepsilon_{ij2}\Omega_r\dot{\varphi}^{(m)} - A_{2s}\dot{q}_{st}^{(m)}\Omega_t - A_{js}q_{st}^{(m)}q_{ir}^{(m)}\varepsilon_{ij2}\Omega_t\Omega_r - m^{(m)}\end{aligned} \tag{4.2.14}$$

which is the equation of motion of one of the gyrofloats. This equation is called the *signal equation* as it tells how the output signal of one gyro reflects the angular motions of the controlled member.

The left member of (4.2.14) represents the dynamic properties of the gyros and the feedback from the signal generators to the torque generators of the gyros. The first term of the right member is dominant. The coefficients $q_{qt}^{(m)}$ form a matrix

$$Q = \{q_{1t}^{(m)}\} = \begin{pmatrix} \cos\varphi^{(1)} & \sin\varphi^{(1)}\sin\theta^{(1)} & -\sin\varphi^{(1)}\cos\theta^{(1)} \\ \sin\varphi^{(2)}\cos\theta^{(2)} & \cos\varphi^{(2)} & \sin\varphi^{(2)}\sin\theta^{(2)} \\ \sin\varphi^{(3)}\sin\theta^{(3)} & -\sin\varphi^{(3)}\cos\theta^{(3)} & \cos\varphi^{(3)} \end{pmatrix}$$

The diagonal elements are the desired elements in the sense that an output signal is obtained for the component of the angular velocity on the corresponding input axis. The dependance of the diagonal elements on $\varphi^{(i)}$ means that the sensitivity of the gyros is not constant. The nondiagonal elements of Q are due to the fact that, when a gyro signals, there is a component of the angular momentum of the gyro orthogonal to the spin reference axis. The gyro thus senses the component of angular velocity along the spin reference axis. This effect is referred to as *spin (reference) axis sensitivity*. It introduces a term in the signal equation of second order in $\varphi^{(i)}$ and Ω_j. The spin axis sensitivity can thus be decreased by keeping the output signals small.

The second term of the right member of (4.2.14) is due to the output axis sensitivity. This effect is linear and was already discussed in section 2.

The other terms in (4.2.14) are at least of the second order in $\varphi^{(i)}$ and Ω_j. The dominant second order term is the spin axis sensitivity as the term introduced by this effect is multiplied by ω_0 which is much greater than the other coefficients. (A typical value of ω_0 is 2000 rad $\cdot$ sec^{-1}.)

4.2.4. *The equation of motion of the controlled member*

We will now derive the equation of motion of the controlled member. Introduce

JC_{kl} the inertia matrix of the stable element (including three mass points, each equal to the mass of one gyrofloat, situated at the center of the floats) with respect to the y-set, i.e.

$$JC_{kl}=(\delta_{kl}\delta_{ij}-\delta_{kj}\delta_{il})\int y_i y_j dm$$

where the integration is carried out over the stable element and the three mass points.

$JB_{kl}(\varphi)$ the inertia matrix of the stable element with respect to the y-set, including the floats fixed in their actual positions, i.e.

$$JB_{kl}=(\delta_{kl}\delta_{ij}-\delta_{kj}\delta_{il})\int y_i y_j dm$$

where the integration is performed over the stable element and the gyrofloats.

$J\tau_{ij}(D)$ the transfer function from the output signal of the j-gyro to the y_i-component of the torque applied to the controlled member. (The $\tau_{ij}(D)$:*s* are assumed to be rational functions of the differential operator $D=\frac{d}{dt}$)

$JM_i(t)$ the y_i-component of the disturbing torque acting on the controlled member.

The angular momentum of the controlled member is

$$H = JC_{lk}\Omega_k\hat{y}_l + \sum_{m=1}^{3} H^{(m)}$$

Differentiating with respect to time we get

$$\dot{H} = J(C_{lk}\dot{\Omega}_k + C_{jk}\Omega_i\Omega_k\varepsilon_{ijl})\hat{y}_l + \sum_{m=1}^{3} \dot{H}^{(m)}$$

The equation (4.2.13) gives

$$\begin{aligned}\dot{H} = J[&A_{k2}q_{kl}^{(m)}\ddot{\varphi}^{(m)} + \delta_{k1}q_{kl}^{(m)}\dot{\varphi}^{(m)}\omega_0 + A_{ks}q_{st}^{(m)}q_{kl}^{(m)}\dot{\Omega}_t + \\ &+ q_{it}^{(m)}q_{kl}^{(m)}\varepsilon_{i3k}\Omega_t\omega_0 + A_{j2}q_{kl}^{(m)}\varepsilon_{2jk}\dot{\varphi}^{(m)}\dot{\varphi}^{(m)} + \\ &+ A_{j2}q_{ir}^{(m)}q_{kl}^{(m)}\varepsilon_{ijk}\Omega_r\dot{\varphi}^{(m)} + A_{js}q_{st}^{(m)}q_{kl}^{(m)}\varepsilon_{2ik}\Omega_t\dot{\varphi}^{(m)} + \\ &+ A_{ks}\dot{q}_{st}^{(m)}q_{kl}^{(m)}\Omega_t + A_{js}q_{st}^{(m)}q_{ir}^{(m)}q_{kl}^{(m)}\varepsilon_{ijk}\Omega_t\Omega_r]\hat{y}_l\end{aligned} \tag{4.2.14}$$

The torque acting on the platform is composed of components from the torque motors and disturbing torques JM.

Newton's second law of motion gives

$$\begin{aligned}B_{lm}(\varphi)\dot{\Omega}_m + B_{jk}(\varphi)\varepsilon_{ijl}\Omega_i\Omega_k + \sum_{m=1}^{3} [&A_{k2}q_{kl}^{(m)}\ddot{\varphi}^{(m)} + q_{1l}^{(m)}\dot{\varphi}^{(m)}\omega_0 + \\ &+ \tau_{lm}(D)\varphi^{(m)} + q_{it}^{(m)}q_{kl}^{(m)}\varepsilon_{i3k}\Omega_t\omega_0 + A_{j2}q_{kl}^{(m)}\varepsilon_{2jk}\dot{\varphi}^{(m)}\dot{\varphi}^{(m)} + \\ &+ A_{j2}q_{ir}^{(m)}q_{kl}^{(m)}\varepsilon_{ijk}\Omega_r\dot{\varphi}^{(m)} + A_{js}q_{st}^{(m)}q_{kl}^{(m)}\varepsilon_{2ik}\Omega_t\dot{\varphi}^{(m)} + \\ &+ A_{ks}\dot{q}_{st}^{(m)}q_{kl}^{(m)}\Omega_t] = M_l\end{aligned} \tag{4.2.15}$$

4.3. **Linear approximations of the equations of motion**

We will now linearize the equations of motion of the system in order to obtain a linear mathematical model of the three axis platform system. To judge whether this linear model is a faithful description of the system in any special case, it is necessary to analyse the complete equations.

For the inertia matrix $JB_{kl}(\varphi)$ of the stable element, including the floats with all moving parts fixed in their *actual* positions, we have

$$B_{kl}(\varphi) = C_{kl} + \sum_{m=1}^{3} A_{st}q_{sk}^{(m)}q_{tl}^{(m)}$$

(Notice that the elements $B_{ij}(\varphi)$ depend on the output signals of the gyros. However, if the inertia ellipsoids of the gyros are symmetric with respect to the output axes, the elements B_{ij} are constants.)

Introducing this in (4.2.14) we get

$$\dot{\boldsymbol{H}} = J(B_{lk}(\varphi)\dot{\Omega}_k + B_{jk}(\varphi)\Omega_i\Omega_k\varepsilon_{ijl})\hat{y}_l +$$

$$+ J\sum_{m=1}^{3} [A_{k2}q_{kl}^{(m)}\ddot{\varphi}^{(m)} + q_{1l}^{(m)}\dot{\varphi}^{(m)}\omega_0 +$$

$$+ q_{it}^{(m)}q_{kl}^{(m)}\varepsilon_{i3k}\Omega_t\omega_0 + A_{j2}q_{kl}^{(m)}\varepsilon_{2jk}\dot{\varphi}^{(m)}\dot{\varphi}^{(m)} +$$

$$+ A_{j2}q_{ir}^{(m)}q_{kl}^{(m)}\varepsilon_{ijk}\Omega_r\dot{\varphi}^{(m)} + A_{js}q_{st}^{(m)}q_{kl}^{(m)}\varepsilon_{2ik}\Omega_t\dot{\varphi}^{(m)} +$$

$$+ A_{ks}\dot{q}_{st}^{(m)}q_{kl}^{(m)}\Omega_t]\hat{y}_l$$

In many applications considerable effort is made in order to make the system behave linearly. The output signals of the gyros are usually restricted to small values. In a high precision navigation platform system the output signals are of the order of magnitude of a few seconds of an arc. Notice, however, that there are many situations where the linear model is insufficient.

4.3.1. *The linearized signal equation*

Neglecting all terms of (4.2.14) which are of second or higher order in $\varphi^{(i)}$ and Ω_j, we get

$$A_{22}D^2\varphi^{(m)} + \sum_{s=1}^{3}\sigma_{ms}(D)\cdot\varphi^{(m)} +$$

$$A_{2s}p_{st}^{(m)}D\Omega_t - p_{1t}^{(m)}\omega_0\Omega_t + m^{(m)} = 0 \tag{4.3.1}$$

This equation can be written in a more compact form by introducing the notations

$$\boldsymbol{\varphi} = \begin{pmatrix}\varphi^{(1)}\\ \varphi^{(2)}\\ \varphi^{(3)}\end{pmatrix}, \qquad \boldsymbol{\Omega} = \begin{pmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix}, \qquad \boldsymbol{m} = \begin{pmatrix}m^{(1)}\\ m^{(2)}\\ m^{(3)}\end{pmatrix} \tag{4.3.2}$$

$$\mathbf{V}(D) = \{v_{ij}\}$$

$$v_{ij} = \frac{1}{A_{22}}\omega_0 p_{1j}^{(i)} - A_{2s}p_{sj}^{(i)}D \tag{4.3.3}$$

The equation (4.3.1) then becomes

$$\mathbf{S}(D)\cdot\boldsymbol{\varphi}(t) = \mathbf{V}(D)\boldsymbol{\Omega}(t) - \boldsymbol{m}(t) \tag{4.3.4}$$

which is the linearized signal equation for the three axis system. This equation shows how the output signals reflects the angular motions of the controlled member and the disturbances.

If the input axes of the gyros are orthogonal the expression for $\mathbf{V}(D)$ can be further simplified. Introduce the unit matrix I, the matrices $\mathbf{L}$ and $\mathbf{N}$ defined by

$$\mathbf{L}=\begin{pmatrix} 0 & \cos\theta^{(1)} & \sin\theta^{(1)} \\ \sin\theta^{(2)} & 0 & \cos\theta^{(2)} \\ \cos\theta^{(3)} & \sin\theta^{(3)} & 0 \end{pmatrix} \tag{4.3.5}$$

$$\mathbf{N}=\begin{pmatrix} 0 & -\sin\theta^{(1)} & \cos\theta^{(1)} \\ \cos\theta^{(2)} & 0 & -\sin\theta^{(2)} \\ -\sin\theta^{(3)} & \cos\theta^{(3)} & 0 \end{pmatrix} \tag{4.3.6}$$

and their transponates $\mathbf{L}$ and $\mathbf{N}$ we get from (4.3.4)

$$\mathbf{V}(D)=\left(\frac{\omega_0}{A_{22}}-A_{21}D\right)\mathbf{I}-D\mathbf{L}-\frac{A_{23}}{A_{22}}D\mathbf{N} \tag{4.3.7}$$

If the inertia ellipsoid of the gyrofloat is symmetric with respect to the output axis, (4.3.7) is further reduced to

$$\mathbf{V}(D)=\frac{\omega_0}{A_{22}}\mathbf{I}-D\mathbf{L} \tag{4.3.8}$$

The first term is the desired term of $\mathbf{V}(D)$ and the second term is due to the output axis sensitivity of the gyros.

4.3.2. *The linearized equation of motion of the controlled member*

Neglecting all terms of (4.2.15) which are of second or higher order in $\varphi^{(i)}$ and Ω_j, we get

$$B_{lm}\dot{\Omega}_m+\omega_0\sum_{m=1}^{3}\left[p_{2t}^{(m)}\cdot p_{1l}^{(m)}-p_{1t}^{(m)}\cdot p_{2l}^{(m)}\right]\Omega_t+$$

$$+\sum_{m=1}^{3}\left[A_{k2}p_{kl}^{(m)}\ddot{\varphi}^{(m)}+p_{1l}^{(m)}\dot{\varphi}^{(m)}\omega_0+\tau_{lm}\varphi^{(m)}\right]=M_l \tag{4.3.9}$$

where

$$B_{ij}=B_{ij}(0)\,; \qquad \mathbf{B}=\{B_{ij}\}$$

The matrix $J\mathbf{B}$ is thus the inertia matrix of the controlled member *including all moving parts fixed in their null positions.* Using the notations of section 4.3.1 and further denoting

$$M=\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}$$

$$\mathbf{T}(D)=\{\tau_{ij}(D)\}$$

the equation (4.3.9) becomes

$$\mathbf{F}(D)\boldsymbol{\Omega}(t)=\boldsymbol{M}(t)-\mathbf{G}(D)\boldsymbol{\varphi}(t) \tag{4.3.10}$$

where

$$\mathbf{F}(D)=D\mathbf{B}+\omega_0(\mathbf{L}-\tilde{\mathbf{L}}) \tag{4.3.11}$$

$$\mathbf{G}(D)=\mathbf{T}(D)+(A_{12}D^2+\omega_0 D)\mathbf{I}+A_{22}D^2\tilde{\mathbf{L}}-A_{32}D^2\tilde{\mathbf{N}} \tag{4.3.12}$$

The equation (4.3.10) is referred to as *the linearized equation of motion of the controlled member*. The term $J\mathbf{F}(D)\boldsymbol{\Omega}(t)$ is the time-derivative of the angular momentum of the controlled member (notice the influence of the spin of the gyros), $J\boldsymbol{M}$ is the disturbing torque, and $J\mathbf{G}(D)\boldsymbol{\varphi}(t)$ is the control torque applied to the controlled member from the torque motors which are commanded by the gyros.

Eliminating the output signal between (4.3.4) and (4.3.10), we get

$$\mathbf{K}(D)\boldsymbol{\Omega}(t)=\boldsymbol{M}(t)+\frac{1}{A_{22}}\,\mathbf{G}(D)\mathbf{S}^{-1}(D)\boldsymbol{m}(t) \tag{4.3.13}$$

where

$$\mathbf{K}(D)=\mathbf{F}(D)+\mathbf{G}(D)\mathbf{S}^{-1}(D)\mathbf{V}(D) \tag{4.3.14}$$

The equation (4.3.14) is referred to as *the equation of motion of the controlled member with the servoloop closed.*

4.3.3. *Block diagram representations of the linearized equations*

Assuming the initial conditions to be zero and Laplace-transforming the equations (4.3.4), (4.3.10) and (4.3.14) we get

$$\mathbf{S}(p)\boldsymbol{\varphi}(p)=\mathbf{V}(p)\boldsymbol{\Omega}(p)-\boldsymbol{m}(p) \tag{4.3.15}$$

$$\mathbf{F}(p)\boldsymbol{\Omega}(p)=\boldsymbol{M}(p)-\mathbf{G}(p)\boldsymbol{\varphi}(p) \tag{4.3.16}$$

$$\mathbf{K}(p)\boldsymbol{\Omega}(p)=\boldsymbol{M}(p)-\frac{1}{A_{22}}\mathbf{G}(p)\mathbf{S}^{-1}(p)\boldsymbol{m}(p) \tag{4.3.17}$$

The signal equation and the equation of motion of the controlled member can be represented by the matrix block diagram of Fig. 12.

The equation of motion of the controlled member with the servoloop closed (4.3.17) can be represented by the block diagram of Fig. 13.

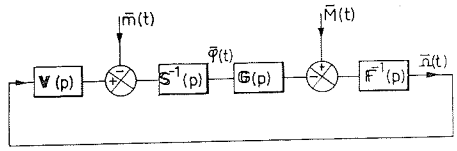

Fig. 12.

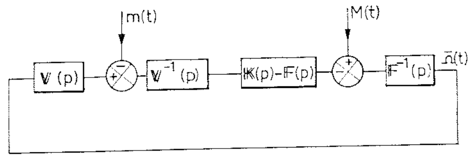

Fig. 13.

The matrix block diagrams show the global character of the system. If we are interested in the signal flow between the different channels we must study each component of the equations of motion. Fig. 14 shows a block diagram representation of one component of (4.3.17).

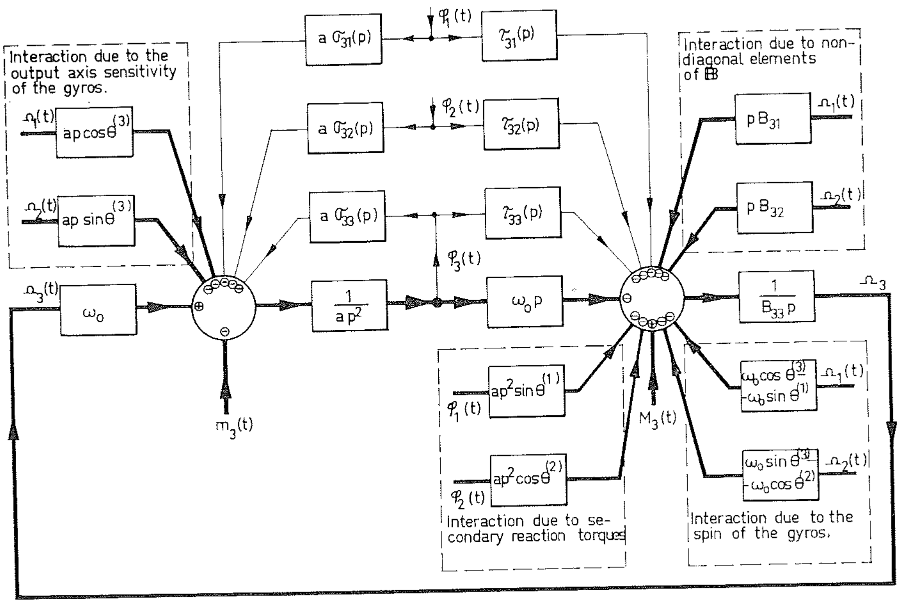

Fig. 14. Block diagram illustrating the third component of the equation of motion of the controlled member.

4.3.4. ***Single axis platform systems***

For a stable element with only one gyro, the equations (4.3.15), (4.3.16) and (4.3.17) are still valid if the matrices are interpreted in the following way

$$\mathbf{G}(p)=\begin{pmatrix} \omega_0 p+\sigma_{11}(p) \\ A_{22}p^2\cos\theta^{(1)}+\sigma_{21}(p) \\ A_{22}p^2\sin\theta^{(1)}+\sigma_{31}(p) \end{pmatrix}$$

$$\mathbf{S}(p)=p^2+\sigma_{11}(p)$$

$$\mathbf{V}(p)=\frac{1}{A_{22}}(\omega_0,\ -A_{22}p\cos\theta^{(1)},\ -A_{22}p\sin\theta^{(1)})$$

$$\mathbf{F}(p)=\begin{pmatrix} B_{11}p & B_{12}p+\omega_0\cos\theta^{(1)} & B_{13}p+\omega_0\sin\theta^{(1)} \\ B_{21}p-\omega_0\cos\theta^{(1)} & B_{22}p & B_{23}p \\ B_{31}p-\omega_0\sin\theta^{(1)} & B_{32}p & B_{33}p \end{pmatrix}$$

It is assumed that the input axis of the gyro coincides with the y_1-axis and that the inertia ellipsoid of the gyrofloat is symmetric with respect to the output axis. If the angular motions of the controlled member are restricted to rotations around the y_1-axis the equation (4.3.17) reduces to

$$\Omega_1(p)=\frac{1}{B_{11}p(1+Y_0)}M_1(p)+\frac{1}{\omega_0}\cdot\frac{Y_0}{1+Y_0}m_1(p) \tag{4.3.18}$$

where

$$Y_0=\frac{\omega_0}{A_{11}B_{11}}\cdot\frac{\tau_{11}(p)+\omega_0 p}{p[p^2+\sigma_{11}(p)]} \tag{4.3.19}$$

This proves the statements of section 3.2.

4.4. Preliminary analysis of three axis systems

4.4.1. *The concept of platform*

The mathematical model developed in the previous section will now be used to analyse the system. We start by introducing a new concept.

Definition 4.4.1

A *platform* is a physical object to which is attributed geometrical structure, mass distribution, and angular momentum. The geometrical structure and the mass distribution of the platform are equal to those of the controlled member. The angular momentum of the platform with respect to the point P is defined by

$$\frac{d}{dt}\boldsymbol{H}_P=\mathbf{K}_P(D)\boldsymbol{\Omega}(t)$$

where $\boldsymbol{\Omega}(t)$ is the angular velocity of the platform and

$$\mathbf{K}_P(D) = D\mathbf{B}_P + \omega_0(\mathbf{L} - \tilde{\mathbf{L}}) + \mathbf{G}(D)\mathbf{S}^{-1}(D)\mathbf{V}(D)$$

$J\mathbf{B}_p$ is the inertia matrix of the controlled member with respect to the point P. The other quantities above are defined in section 4.3.

The platform can be considered as a generalized rigid body. The elements of the "inertia matrix" of the platform can be arbitrary physically-realizable functions of the differential operator D, which offers many possibilities. Only a few of the many possible varieties of platforms have so far been subjected to experimental investigations. The inertial stabilized platform is one type which already has found several important applications. In the following we will only study this type.

Definition 4.4.2

A platform is said to be *inertial stabilized* if the angular orientation of the platform, relative inertial space, is stable with respect disturbing torques acting on the platform.

Corollary 1.

For an inertial stabilized platform the equation

$$\det p\mathbf{K}(p) = 0$$

has all the roots in the left half plane. Cf. MALKIN, (1959) p. 273.

Corollary 2.

For an inertial stabilized platform, the eigenvalues of $\mathbf{K}(p)$ have at least single poles for $p=0$.

Assuming that the eigenvalues of $\mathbf{K}(p)$ have single poles at the origin, we get

$$\lim_{p\to 0} p\mathbf{K}(p) = \mathbf{C}$$

This means physically that the platform is spring restrained to inertial space. The eigenvalues of $\mathbf{C}$ are the spring constants with respect to axes parallel to the eigenvectors of $\mathbf{C}$. For further comments on the physical interpretations we refer to the discussion of the single axis case in section 3.

Corollary 3.

For an inertial stabilized platform system the equation

$$\det\{p\mathbf{S}(p)\mathbf{G}^{-1}(p)\mathbf{K}(p)\} = 0 \tag{4.4.1}$$

has a root $p=0$.

4.4.2. *The stability of inertial stabilized platforms with respect to disturbing torques acting on the gyrofloats*

According to the definition 4.4.1 the angular deviation of the controlled member is stable with respect to disturbing torques acting on the platform. From corollary 3 it follows that a constant disturbing torque on a gyrofloat will give an angular deviation error which increases at least linearly with time. The *angular orientation* of the platform is thus unstable with respect to disturbing torques acting on the gyrofloats. We will now investigate what is required if the *angular velocity* of the controlled member should be stable with respect to torques acting on the gyrofloats. We start by discussing an example.

Example

Assume that the inertia ellipsoids of the gyrofloats are symmetric with respect to the output axes and that the servosystem from the output signals to the gimbal torquers are "perfect", meaning that

$$\varphi(t) \equiv 0 \quad \text{all} \quad t$$

Of course, this is not possible with physically-realizable transfer functions. In any case this example clearly shows the effect of the output axis coupling on the sensitivity of the system to disturbing torques acting on the gyrofloats. Introducing the above condition into the linearized signal equation (4.3.4), we get

$$A_{22}\mathbf{V}(D)\Omega(t) = \boldsymbol{m}(t)$$

If the input axes of the gyros are orthogonal, the characteristic equation becomes

$$\det \mathbf{V} = \left(\frac{\omega_0}{a}\right)^3 + \frac{\omega_0(s-3)}{2a}p^2 - lp^3 = 0 \tag{4.4.1}$$

The only possibility of avoiding characteristic roots with positive real part is by choosing an orientation of the gyros which gives

$$\begin{cases} l = 0 \\ s \geq 3 \end{cases} \tag{4.4.2}$$

If $s > 3$ two of the characteristic roots have vanishing real parts. To judge the stability we will thus have to consider the nonlinear terms in the signal equation. This is done in the special case when the orientation of the gyros given by

$$\theta^{(1)} = \theta^{(2)} = 0\,; \qquad \theta^{(3)} = -\frac{\pi}{2}$$

hence

$$l = 0\,; \qquad s = 5$$

The equation (4.2.14) then gives

$$\begin{cases} A_{22}\dot{\Omega}_2 = -\omega_0\Omega_3 + (A_{33}-A_{11})\cdot\dfrac{\omega_0\Omega_3^2}{\omega_0+2(A_{33}-A_{11})\Omega_3} + m^{(2)}(t) \\ \\ A_{22}\dot{\Omega}_3 = \omega_0\Omega_2 - (A_{33}-A_{11})\cdot\dfrac{\omega_0\Omega_2\Omega_3}{\omega_0+2(A_{33}-A_{11})\Omega_3} + m^{(3)}(t) \end{cases} \tag{4.4.3}$$

$$\Omega_1 = -\frac{\omega_0\Omega_3}{\omega_0+2(A_{33}-A_{11})\Omega_3}$$

$$\omega_0 > 2\,|A_{33}-A_{11}|\cdot|\Omega_3|$$

The time derivative of the Liapunov function

$$V = \Omega_2^2 + \Omega_3^2$$

with respect to the equation (4.4.3) is zero, which implies that the equation is stable with respect to initial value disturbances. However, this is not sufficient to make the angular velocity stable with respect to the disturbances $m^{(i)}(t)$.

Considering the fact that the output axis orientation number depends on the orientation of mechanical axes, it is impossible, of course, to obtain $l=0$. Neither is it possible to have the input axes of the gyros exactly orthogonal to each other. The characteristic equation (4.4.1) is then replaced by

$$\det \mathbf{V} = V_1\left(\frac{\omega_0}{a}\right)^3 + V_2\left(\frac{\omega_0}{a}\right)^2 p + V_3\left(\frac{\omega_0}{a}\right)p^2 + V_4 p^3 = 0 \tag{4.4.4}$$

where

$$V_1 = [\hat{x}_1^{(1)}, \hat{x}_1^{(2)}, \hat{x}_1^{(3)}]$$

$$V_2 = -[\hat{x}_2^{(1)}, \hat{x}_1^{(2)}, \hat{x}_1^{(3)}] - [\hat{x}_1^{(1)}, \hat{x}_2^{(2)}, \hat{x}_1^{(3)}] - [\hat{x}_1^{(1)}, \hat{x}_1^{(2)}, \hat{x}_2^{(3)}]$$

$$V_3 = [\hat{x}_1^{(1)}, \hat{x}_2^{(2)}, \hat{x}_2^{(3)}] + [\hat{x}_2^{(1)}, \hat{x}_1^{(2)}, \hat{x}_2^{(3)}] + [\hat{x}_2^{(1)}, \hat{x}_2^{(2)}, \hat{x}_1^{(3)}]$$

$$V_4 = -[\hat{x}_2^{(1)}, \hat{x}_2^{(2)}, \hat{x}_2^{(3)}]$$

The vector $\hat{x}_i^{(j)}$ is the unit vector along the $x_i^{(j)}$-axis defined in section 4.2.1.

The scalar triple products above can be interpreted geometrically as the volume of the parallelepiped which has the vectors for concurrent sides. Requiring that all the roots of (4.4.4) are in the left half plane, we get

$$V_i > 0 \qquad i = 1, 2, 3, 4$$

$$V_2 V_3 > V_1 V_4$$

which replaces the condition (4.4.2). In case of approximatively orthogonal input axes the above condition gives

$$\begin{cases} s > 3 + \delta_1 \\ -\delta_2 < l < \delta_3 \end{cases} \tag{4.4.5}$$

where the δ_i:s are small quantities depending on the deviation of the axes from the orthogonal positions. A similar result is obtained if it is also taken into account that the inertia ellipsoids of the gyros in practice cannot be perfectly symmetrical about the output axis. Assuming $A_{21} \neq 0$ and $A_{23} \neq 0$ conditions similar to (4.4.5), where δ_1, δ_2 and δ_3 now depend on A_{21} and A_{23}, are obtained.

It is thus possible to get characteristic roots, all of which have negative real parts. However, this will require accurate adjustement of the orientation of the gyros.

The conditions for the stability of the angular velocity of the controlled member with respect to disturbing torques acting on the gyrofloats will now be discussed. The linear model of the system will be utilized. The angular velocity of the controlled member is then given by

$$\boldsymbol{\Omega}(p) = \mathbf{K}^{-1}(p)\boldsymbol{M}(p) + \frac{1}{A_{22}}\mathbf{K}^{-1}(p)\mathbf{G}(p)\mathbf{S}^{-1}(p)\boldsymbol{m}(p) \tag{4.4.6}$$

The extension to the general case is obtained by a theorem of MALKIN (1959) which says that a stationary motion is stable with respect to a disturbance if all the characteristic roots of the linear approximation of the equation of motion have negative real parts. It is thus only in case of characteristic roots with vanishing real parts that the nonlinear terms must be studied. However, before doing so we should consider the fact that some characteristic roots depend on quantities which only have limited accuracy. In practice the system must thus be arranged in such a way that all the roots of the characteristic equation have negative real parts. We thus have the following condition.

Theorem 4.4.1

In order that the angular velocity of the controlled member is stable with respect to disturbing torques acting on the gyrofloats it is a necessary and sufficient condition that the equation

$$\det \mathbf{V}(p)[\mathbf{K}(p) - \mathbf{F}(p)]^{-1} = 0$$

has no roots in the right half plane.

If the arrangement of the gyros is chosen in such a way that

$$l = 0$$

$$s \geq 3$$

the equation det $\mathbf{V}(p)=0$ has no roots in the right half plane, it is then sufficient to require that function det $[\mathbf{K}(p)-\mathbf{F}(p)]^{-1}$ has no zeros in the right half plane, which is a very weak condition. If the above condition is not satisfied, the theorem implies that the functions det $\mathbf{V}(p)$ and det $[\mathbf{K}(p)-\mathbf{F}(p)]$ should have the same zeros in the right half plane. This is a very restrictive condition, a very important situation where it is satisfied is discussed in section 5.2.

In order to shorten the notations we introduce

Definition 4.4.3

An inertial stabilized platform system is said to be stable if the angular velocity of the controlled member is stable with respect to disturbing torques acting on the gyrofloats.

5. Influence of the cross couplings on the dynamics of the system

5.1. Introduction

The analysis of section 4.4 was a straight-forward generalization of the results obtained for the single axis case. Some problems specific to the three axis system will now be analysed. The three axis system can be regarded as three single axis systems with mutual interaction cf. Fig. 14. This approach is convenient if the interaction between the single axis channels is small. In that case the system can be analysed and synthesized essentially from a single axis approach which means a considerable simplification. On the contrary, if the interaction between the systems is large, the single axis approach will say very little about the three axis system. In this section the influence of the interaction, or crosscoupling, will be analysed in order to obtain criteria for the validity of the single axis approach. A linear model of the system will be used throughout this section. It is also assumed that,

1. The inertia ellipsoids of the gyro floats are symmetrical with respect to the output axes, i.e.

$$A_{ij}=O \qquad i\neq j$$

Put

$$A_{22}=a$$

2. The inertia ellipsoid of the controlled member is a sphere, i.e.

$$B_{ij}=b\cdot\delta_{ij}$$

3. The input axes of the gyros are mutually orthogonal.

The equations (4.3.8) and (4.3.11) then give

$$\mathbf{V}(p)=\frac{\omega_0}{a}\mathbf{I}-p\mathbf{L} \tag{5.1.1}$$

$$\mathbf{F}(p)=bp\mathbf{I}+\omega_0(\mathbf{L}-\tilde{\mathbf{L}}) \tag{5.1.2}$$

Example

Consider a system which satisfies 1, 2 and 3 and further

4. The torque-generator of each gyro is only controlled by the output signal of the same gyro. The control characteristics are the same for all gyros. Hence

$$\sigma_{ij}(p)=\sigma(p)\delta_{ij}$$

5. The m-component of the control torque applied to the controlled member is only commanded by the output signal of the m-gyro. The same feedback characteristics are used in all channels. Hence

$$\tau_{ij}(p)=\tau(p)\delta_{ij}$$

With these assumptions (4.3.2) and (4.3.12) run

$$\mathbf{S}^{-1}(p)=[p^2+\sigma(p)]^{-1}\mathbf{I}$$

$$\mathbf{G}(p)=[\tau(p)+\omega_0 p]\mathbf{I}+ap^2\tilde{\mathbf{L}}$$

The equation (4.3.14) then gives

$$\mathbf{K}(p)=bp[1+Y_0]\cdot\mathbf{I}-\frac{ab}{\omega_0}p^2Y_0\mathbf{L}+\omega_0\mathbf{L}$$

$$-\frac{\omega_0\sigma(p)}{p^2+\sigma(p)}\cdot\tilde{\mathbf{L}}-\frac{ap^3}{p^2+\sigma(p)}\tilde{\mathbf{L}}\cdot\mathbf{L} \tag{5.1.3}$$

where Y_0 is the open loop system function for one of the single axis loops, i.e.

$$Y_0=\frac{\omega_0}{ab}\cdot\frac{\tau(p)+\omega_0 p}{p[p^2+\sigma(p)]} \tag{5.1.4}$$

Cf. (3.3.2)

The fact that $\mathbf{K}(p)$ is not diagonal means that there are interactions between the channels. The second term of $\mathbf{K}(p)$ is due to the *output axis sensitivity* of the gyros. Cf. section 2.

When the gyro signals, it produces reaction torques. The component of the reaction on the input axis is called the *primary reaction torque*. This causes the term $\omega_0 p$ in the numerator of Y_0. The other components of the reaction torques are called *secondary reaction torques.* (These are usually considerably smaller then the primary reaction torque). The third and fourth terms of $\mathbf{K}(p)$ are due to secondary reaction torques. The last term of $\mathbf{K}(p)$ is due to a combination of output axis sensitivity and secondary reaction torques. Cf. Fig. 14 for a representation of the interaction in terms of a block diagram.

5.2. Analysis of systems where the interaction is entirely due to the output axis sensitivity of the gyros

5.2.1. *Problem statement*

There is a large class of systems in which the gyros act essentially as sensing devices with very small secondary reaction torques which are effectively masked by a small amount of Coulomb friction.

Many of the existing conventional platform systems with small gyros and comparatively large and heavy stable elements belong to this class of systems.

For systems in which the interaction between the channels is entirely due to the output axis sensitivity of the gyros, (5.1.3) reduces to

$$\mathbf{K}(p)=bp[1+Y_0]\mathbf{I}-\frac{ab}{\omega_0}p^2 Y_0\mathbf{L} \tag{5.2.1}$$

A block diagram of such a system is shown in Fig. 15. Also compare Fig. 14.

The stability conditions for systems of this type will now be further investigated.

Theorem 5.2.1

The angular velocity of the controlled member of an inertial stabilized platform with a $\mathbf{K}(p)$ *matrix according to* (5.2.1) *is stable with respect to disturbing torques acting on the gyrofloats if* Y_0 *has no zeros in the right half plane.*

Proof

The equations (5.1.1), (5.1.2) and (5.2.1) give

$$\mathbf{K}(p)-\mathbf{F}(p)=\frac{ab}{\omega_0}pY_0\mathbf{V}(p)$$

Hence

$$\mathbf{V}(p)[\mathbf{K}(p)-\mathbf{F}(p)]^{-1}=\frac{\omega_0}{abpY_0}$$

The system is then stable according to the theorem 4.4.1.

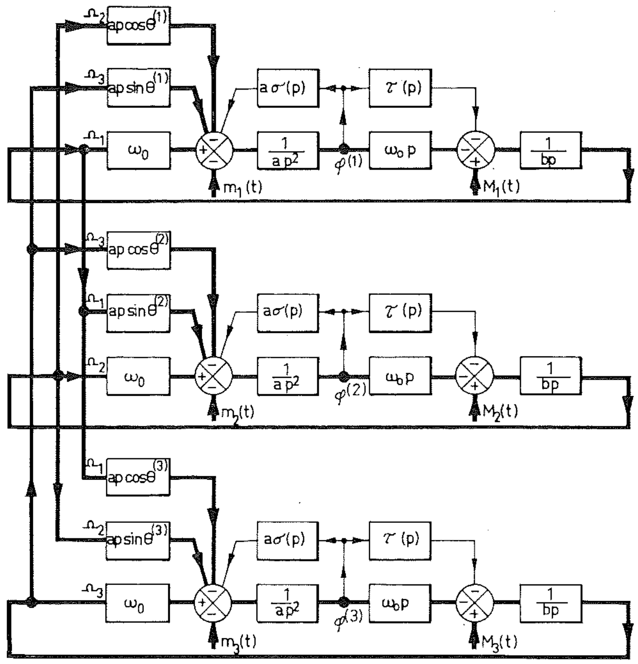

Fig. 15. Block diagram of a system where the only interaction is due to the output axis sensitivity of the gyros.

In order to judge the stability of systems of this type it is thus sufficient to ensure that the system is inertial stabilized. We thus have to analyse the characteristic equation.

$$\det p\mathbf{K}(p)=0 \tag{5.2.2}$$

The stability of this equation together with the weak condition on Y_0 in the theorem 5.2.1 then implies that the system is stable.

The characteristic equation can be reduced to

$$bp^2[1+Y_0]-t_i\cdot\frac{ab}{\omega_0}p^3Y_0=0 \tag{5.2.3}$$

where the t_i:s are the roots of the equation

$$t^3+\frac{s-3}{2}t-l=0 \tag{5.2.4}$$

s and l are the spinnumber and the output axis orientation number introduced in section 4.2.2. The numerical values of the roots of (5.2.4) for integral values of s and l are given in the table below.

s \ l	-1	0	$+1$
0	-1.477 $0.738 \pm i0.361$	0 ± 1.225	1.477 $-0.738 \pm i0.361$
1	-1.326 $0.662 \pm i0.563$	0 ± 1.000	1.326 $-0.662 \pm i0.563$
2	-1.165 $0.583 \pm i0.720$	0 ± 0.707	1.165 $-0.583 \pm i0.720$
3	-1 $0.500 \pm i0.865$	0 ± 0	1 $-0.500 \pm i0.865$
4	-0.836 $0.418 \pm i0.739$	0 $\pm i0.707$	0.836 $-0.418 \pm i0.739$
5	-0.682 $0.233 \pm i0.961$	0 $\pm i$	0.682 $-0.233 \pm i0.961$
6	-0.554 $0.277 \pm i1.315$	0 $\pm i1.225$	0.554 $-0.277 \pm i1.315$

If the single-axis channels are stable, the equation

$$p^2[1+Y_0]=0$$

has all the roots in the left half plane.

The condition $s=3$, $l=0$ is then sufficient for the stability of the three axis system. Some questions now arise. Is it possible to obtain a stable system if this condition is not satisfied? Although a system with $s=3$ and $l=0$ is stable, is it sufficiently damped to be of practical use? Before answering the questions we will recall the properties of the Y_0-functions. Y_0 is the open loop system function for one single axis loop. The properties of Y_0 for inertial stabilized systems were discussed in section 3.3. It was found that Y_0 should have at least a double pole at the origin. Possible analytical forms of Y_0 were also discussed in the sections 3.2 and 3.3. Cf. (3.2.5), (3.4.3), (3.4.8), (3.4.13) and (3.4.18).

5.2.2. *Stability conditions for systems with* $Y_0 = \dfrac{2\zeta\beta}{p} + \dfrac{\beta^2}{p^2}$

One of the possible open loop system functions is

$$Y_0 = \frac{2\zeta\beta}{p} + \frac{\beta^2}{p^2}$$

Cf. section 3.2. The stability condition for a system with this special open loop system function will now be analysed.

The characteristic equation of the system is

$$p^2+2\zeta\beta p+\beta^2-t_i\gamma(2\zeta p^2+\beta p)=0 \qquad i=1, 2, 3 \tag{5.2.5}$$

where the t_i:s are the roots of (5.2.4) and γ is the cross-coupling coefficient defined by

$$\gamma=\frac{a\beta}{\omega_0} \tag{5.2.6}$$

Cf. (5.2.3).
Notice that the crosscoupling coefficient increases with the bandwidth β.

Combining two of the equations (5.2.5) with complex conjugated t_i-values, an equation of the fourth degree with real coefficients is obtained. Applying the theorem of Hurwitz on this and the remaining second order equation, we obtain the following condition for stability

$$\begin{cases} 4\zeta^2-4\zeta\gamma(1+2\zeta^2)Re\{t_i\}+\gamma^2(1+8\zeta^2)(Re\{t_i\})^2-\gamma^3 2\zeta\,|\,t_i\,|^2 Re\{t_i\}\geq 0 \\ 2\zeta-\gamma Re\{t_i\}\geq 0 \qquad i=1, 2, 3 \end{cases}$$

In case of equality in the first of the equations the characteristic equation has two pure imaginary roots

$$p=\pm i\frac{2\zeta-\gamma Re\{t_i\}}{2\zeta\gamma Im\{t_i\}}\beta$$

The stability condition is obviously satisfied for all γ-values if

$$Re\{t_i\}\leq 0 \qquad i=1, 2, 3$$

i.e.

$$l=0$$

$$s\geq 3$$

Cf. (5.2.4)
If this condition is not satisfied the system is at least stable for sufficiently small values of the cross-coupling coefficient γ.

If either $l\neq 0$ and arbitrary s, or $l=0$ and $s<3$ the equation (5.2.4) has at least one root in the right half plane. Let t_0 be the root in the first quadrant or on the real axis. The condition of stability gives

$$\begin{cases} (2\zeta-\gamma Re\{t_0\})^2(1-2\zeta\gamma Re\{t_0\})-2\zeta\gamma^3(Im\{t_0\})^2 Re\{t_0\}\geq 0 \\ 2\zeta-\gamma Re\{t_0\}\geq 0 \end{cases}$$

The inequalities are satisfied if

$$\gamma Re\{t_0\}\leq f(\zeta, \alpha_0) \qquad (5.2.7$$

where

$$f(\zeta, \alpha_0) = \min(2\zeta, z_0)$$

and z_0 is the smallest positive root of the equation.

$$\begin{cases} 2\zeta(1+\alpha_0)z^3 - (1+8\zeta^2)z^2 + 4\zeta(1+2\zeta^2)z - 4\zeta^2 = 0 \\ \alpha_0 = \left(\dfrac{Im\{t_0\}}{Re\{t_0\}}\right)^2 \end{cases}$$

Systems with $l > 0$ and arbitrary s, or $l = 0$ and $s < 3$ have

$$Im\{t_0\} = 0$$

hence

$$\alpha_0 = 0$$

The function $f(\zeta, \alpha_0)$ then reduces to

$$f(\zeta, 0) = \min\left(2\zeta, \frac{1}{2\zeta}\right)$$

A graph of the function $f(\zeta, 0)$ is given in Fig. 16.

Summarizing the stability conditions for a system with

$$Y_0 = \frac{2\zeta\beta}{p} + \frac{\beta^2}{p^2}$$

we get

I. Systems with $l < 0$ are stable if

$$\gamma Re\{t_0\} \leq f(\zeta, \alpha_0)$$

II. Systems with $l = 0$ and $s < 3$ are stable if

$$\gamma Re\{t_0\} \leq \min\left(2\zeta, \frac{1}{2\zeta}\right)$$

III. Systems with $l = 0$ and $s \geq 3$ are stable for all values of γ

IV. Systems with $l > 0$ are stable if

$$\gamma Re\{t_0\} \leq \min\left(2\zeta, \frac{1}{2\zeta}\right)$$

If the cross-coupling coefficient γ is sufficiently small the system discussed is thus stable, independent of the orientation of the gyros. The upper limit of γ for a stable system is given by the equation (5.2.7).

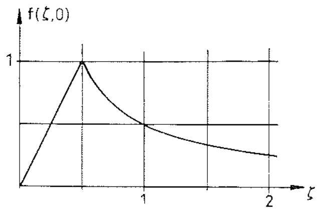

Fig. 16.

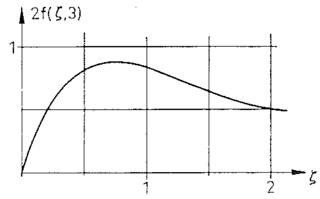

Fig. 17.

Example

Give the stability conditions for systems with the following arrangements of the gyros

$$A.\ \left[0,\ 0,\ \frac{\pi}{2}\right]$$

$$B.\ [\pi,\ \pi,\ \pi]$$

$$C.\ [0,\ 0,\ 0]$$

$$D.\ \left[\frac{\pi}{2},\ \frac{\pi}{2},\ \pi\right]$$

The s, l and t_0 numbers are obtained from the equations (4.2.9). (4.2.10) and (5.2.4). We get

$$A.\ s=1,\ l=0,\ t_0=1$$

$$B.\ s=3,\ l=-1,\ t_0=\tfrac{1}{2}+i\tfrac{1}{2}\sqrt{3}$$

$$C.\ s=3,\ l=1,\ t_0=1$$

$$D.\ s=5,\ l=0,\ t_0=0$$

According to II and IV the systems A and C are stable if

$$\gamma \leq f(\zeta,\ 0)=\min\left(2\zeta,\ \frac{1}{2\zeta}\right)$$

The system B is stable if

$$\gamma \leq 2 \cdot f(\zeta,\ 3)$$

where

$$f(\zeta,\ 3)=\min\,[2\zeta,\ z_0]$$

and z_0 the smallest positive root of the equation

$$8\zeta z^3-(1+8\zeta^2)z^2+4\zeta(1+2\zeta^2)z-4\zeta^2=0$$

Fig. 17 shows a graph of the function $2f(\zeta,\ 3)$.
The system D is always stable.

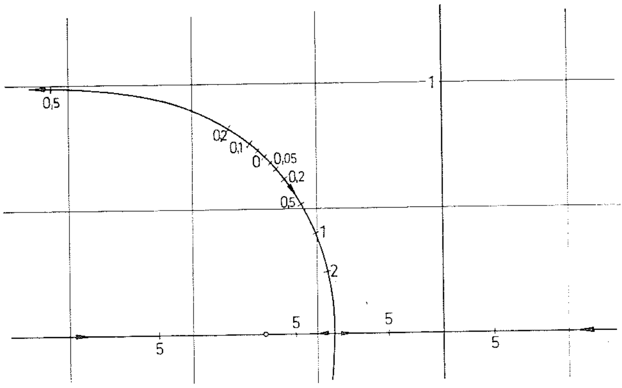

Fig. 18. Root locus with respect to the cross coupling coefficient γ for the equations

$$p^2+1.41p+1-\gamma t_i(1.41p^2+p)=0 \qquad i=1, 2, 3$$

of a system with $l=0$; $s=1$.

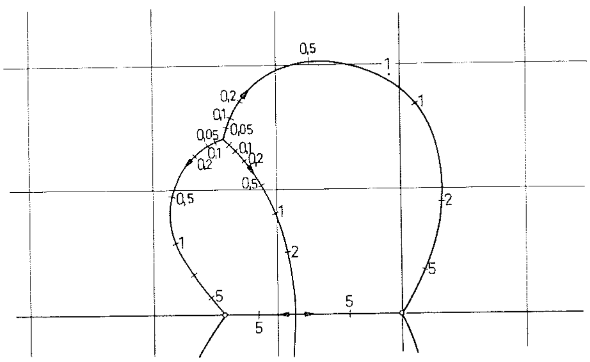

Fig. 19. Root locus with respect to the cross coupling coefficient γ for the equations

$$p^2+1.41p+1-\gamma t_i(1.41p^2+p)=0 \qquad i=1, 2, 3$$

of a system with $l=-1$; $s=3$.

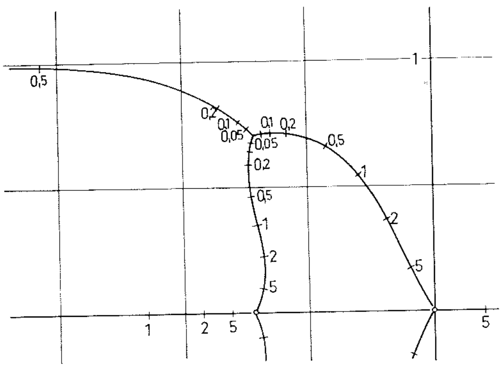

Fig. 20. Root locus with respect to the cross coupling coefficient γ for the equations

$$p^2+1.41p+1-\gamma t_i(1.41p^2+p)=0 \qquad i=1, 2, 3$$

of a system with $l=+1$; $s=3$.

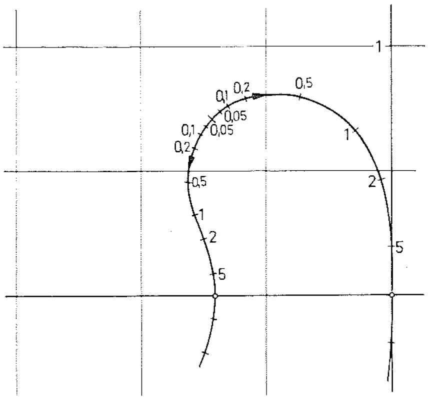

Fig. 21. Root locus with respect to the cross coupling coefficient γ for the equations

$$p^2+1.41p+1-\gamma t_i(1.41p^2+p)=0 \qquad i=1, 2, 3$$

of a system with $l=0$; $s=5$.

Although a system is stable if the equation (5.2.7) is satisfied it may be too oscillative for practical use. To judge this we have to solve the characteristic equation. This is most conveniently carried out with the graphical method of EVANS. This method gives directly the root locus of the characteristic equation with respect to the coupling coefficient.

Figs. 18–21 show the root loci for the characteristic equations of the systems treated in the example.

5.2.3. ***Stability conditions for systems with arbitrary Y_0***

In practical applications it is necessary to consider open-loop system functions which are considerably more complicated than the one discussed in section 5.2.2. The analysis, can, however, be carried out in a straight-forward way following the scheme of section 5.2.2, using the reduction of the characteristic equation given in section 5.2.1. In case of complex open-loop system functions Y_0, the algebraic conditions are rather difficult to handle. We will in general find that, if the single axis loops are stable, the output axis coupling will not cause instability irrespective of the orientation of the gyros, if the crosscoupling coefficient γ is sufficiently small and if certain restrictions are placed on the open loop system function, Y_0.

Some possible restrictions on the open-loop system function which can be formulated in terms of frequency response are given below. Let M_p denote the maximum value of the closed-loop gain i.e.

$$M_p=\max_{0<\omega<\infty}\left|\frac{Y_0(i\omega)}{1+Y_0(i\omega)}\right|$$

Assume that

$$\max_{p\in C}\left|Y_0(p)\right|<K\cdot\frac{\beta}{|p|}$$

where C is a contour composed of the part of the imaginary axis

$$|Im p|\geq K\left(1+\frac{1}{M_p}\right) \tag{5.2.8}$$

and a semicircle on the imaginary axis as diameter, to the left of it. Both M_p and K can be easily obtained from a frequency response plot of Y_0.

Theorem 5.2.1

A sufficient condition for the stability of the characteristic equation (5.2.3) *under the above condition on Y_0 is*

$$\gamma<\frac{1}{\max|t_i|}\cdot\frac{1}{K(1+M_p)} \tag{5.2.9}$$

Proof

On the contour C', consisting of the imaginary axis with a semicircle on this line as diameter enclosing the right half plane, we have

$$|p^2(1+Y_0)|>\gamma\left|t_i\frac{p^3}{\beta}Y_0\right|$$

According to Rouche's theorem the characteristic equation of the system (5.2.3) then has the same number of zeros in the right half plane as the equation

$$p^2(1+Y_0)=0$$

But this equation has no roots in the right half plane since the single-axis loops are stable, which proves the theorem.

In order to get a complete picture of the effect of the output axis sensitivity coupling on the dynamics of the system, the characteristic equation must be solved. For small values of the crosscoupling coefficient, the following expansion is useful.

$$p_i=p_0+\gamma t_i\frac{p_0}{\beta}\left[\frac{1}{\frac{d\log Y_0}{dp}}\right]_{p=p_0}+O(\gamma^2)$$

where p_0 is a characteristic root of a single axis channel and t_i the corresponding roots of the equation (5.2.4). For larger values of the crosscoupling coefficient, the characteristic equation is conveniently solved by the graphical method of Evans, even in case of equations of a high degree.

Example

Consider a system with

$$s = 3$$

$$l = -1$$

$$Y_0 = 1.07\frac{\beta^2}{p^2} \cdot \frac{(p+0.0467\beta)}{(p+1.46\beta)}$$

The characteristic equation of the system is of the 9th degree. According to section 5.2.1 it can be reduced to

$$(p^2+1.41\beta p+\beta^2)(p+0.05\beta)-1.1\, t_i\gamma\beta p(p+0.0467)=0 \qquad i=1,2,3$$

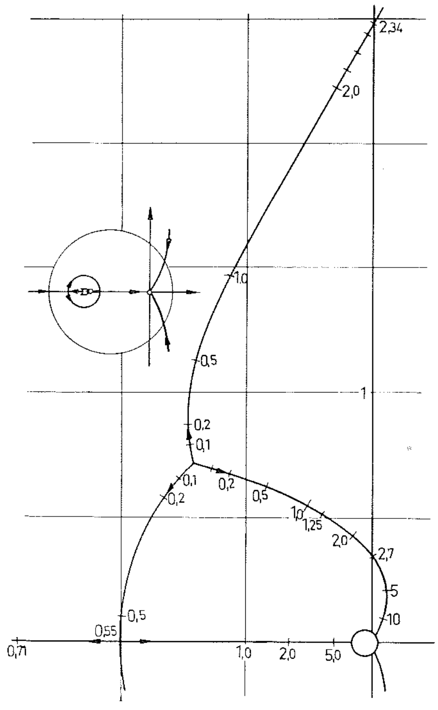

Fig. 22. Root locus with respect to the cross coupling coefficient γ for the equations

$$(p^2+1.41p+1)(p+0.05)+1.1\, t_i\gamma(p^2+0{,}0467p)=0 \qquad i=1,2,3$$

of a system with $l=-1$, $s=3$.

where the t_i:s are the roots of the equation (5.2.4) i.e.

$$t_1 = -1$$

$$t_2 = 0.5 + i0.865$$

$$t_3 = 0.5 - i0.865$$

and γ the crosscoupling coefficient

$$\gamma = \frac{a\beta}{\omega_0}$$

The root locus of the characteristic equation is shown in Fig. 22.

5.3. Example of the interaction caused by the secondary reaction torques

If the secondary reaction torques are not negligible, the zeros of the functions det $\mathbf{V}(p)$ and det $[\mathbf{K}(p)-\mathbf{F}(p)]$ in the right half plane do not coincide except in very special cases. In general, it must then be required that the orientation of the gyros will be chosen in such a way that $l=0$ and $s \geq 3$. The function det $\mathbf{V}(p)$ then has no zeros in the right-half plane. With some very weak additional conditions (det $[\mathbf{V}(p)-\mathbf{F}(p)]^{-1}$ shall have no zeros in the right half plane), the system is then stable if it is inertial stabilized. To judge this, it is sufficient to consider the characteristic equation

$$\det \mathrm{p}\ \mathbf{K}(\mathrm{p}) = 0$$

which can be done by the methods of the previous section. Let it suffice to give an example.

Example

Consider a system where the arrangement of the gyros is

$$\theta^{(1)} = 0;\ \theta^{(2)} = \frac{\pi}{2},\ \theta^{(3)} = -\frac{\pi}{4} \text{ i.e. } l=0,\ s=3. \text{ Further assume}$$

$$B_{ij} = b \cdot \delta_{ij}$$

$$A_{ij} = 0 \qquad i \neq j$$

$$A_{22} = a$$

$$\tau_{ij}(p) = \tau(p)\delta_{ij}$$

$$\mathbf{S}(p) = \frac{a}{\omega_0}[p^2 + \sigma(p)] \cdot \mathbf{V}(p)$$

The equation (4.3.14) then gives

$$\mathbf{K}(p)=Y_1\mathbf{I}+Y_2\mathbf{L}-Y_3\tilde{\mathbf{L}}$$

where

$$Y_1=bp[1+Y_0]$$

$$Y_2=\omega_0$$

$$Y_3=\frac{\omega_0}{a}\cdot\frac{\sigma(p)}{p^2+\sigma(p)}$$

The characteristic equation of the system can then be reduced to

$$\det p\mathbf{K}(p)=pY_1[p^2Y_1^2+3p^2Y_2Y_3]=0$$

Assuming

$$Y_0=\frac{2\zeta\beta}{p}+\frac{\beta^2}{p^2}$$

and

$$\sigma(p)=\alpha p$$

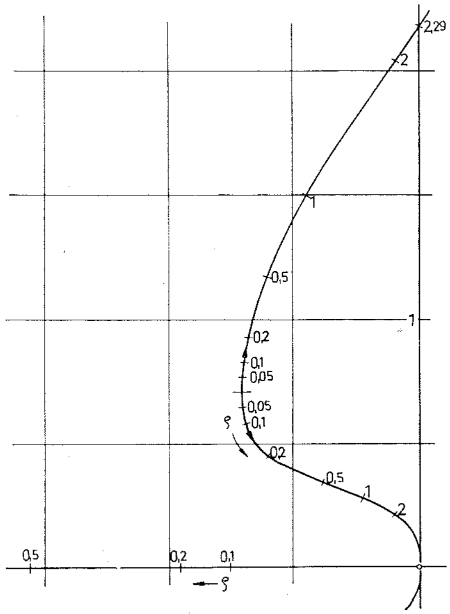

Fig. 23. Root locus with respect to the cross coupling coefficient γ for the equation $(p^2+1.41p+1)^2(p+0.71)+2.13P^2p^2=0$

we get

$$(p^2+2\zeta\beta p+\beta^2)^2(p+\alpha)+3\rho^2\alpha\beta^2 p^2=0$$

where ρ is the cross-coupling coefficient

$$\rho=\frac{\omega_0}{b\beta}$$

which can be interpreted physically as the quotient between the angular momentum of the gyro and that of the controlled member. Notice that ρ decreases with increasing bandwidth.

The above equation can be conveniently solved by the graphical method of EVANS. The result is shown in Fig. 23.

6. Sensitivity of the system to disturbances

6.1. Introduction

There are many reasons why the controlled member should deviate from its desired orientation. In order to obtain a complete picture of the deviation, we have to consider the details of the suspension, the motion of the carrying vehicle, the termal distribution in the gimbals, their elastic deformation etc.

For the sake of convenience we have divided the disturbances into two groups

$M(t)$ the disturbing torque acting on the controlled member

$m(t)$ the disturbing torque acting on the gyrofloats

The main object of the preliminary synthesis of the single axis system was the elimination of the disturbing torques acting on the controlled member. Doing so we had to compromise between the ability of the system to follow commanding signals and the sensitivity to disturbing torques $M(t)$.

The sensitivity of the system to disturbing torques acting on the controlled member can essentially be judged from a single axis analysis. Cf. fig. 5, 6 and 7. It has been shown that the angular orientation of the stable element of an inertial stabilized platform is unstable with respect to disturbing torques acting on the floats of the gyros. A constant disturbing torque on one gyrofloat gives in steady state an indication error increasing with time. In some cases it was also necessary to use special arrangement of the gyros if a constant disturbing torque acting during a finite time interval should not give an indication error increasing exponentially with time. Cf. section 4.4.

If the disturbances are given, the indication error can be calculated from the system equations. If the character of the disturbances are known only in statistical terms, the statistical character of the indication error can sometimes be determined. This problem will now be further discussed.

6.2. **The single axis case**

In the single axis case the indication error is given by

$$\theta(p)=\frac{1}{bp^2[1+Y_0]}M(p)+\frac{1}{\omega_0 p}\cdot\frac{Y_0}{1+Y_0}m(p)$$

The error obtained for deterministic disturbances is thus easily computed. If the disturbing torque M(t) is a stationary stochastic process, the indication error will tend to a stationary process. However, this is usually not the case for errors due to disturbing torque acting on the gyrofloat.

Assume that the disturbing torque acting on the gyrofloat is a stationary gaussian stochastic process with zero average. The power spectrum of the disturbing torque is supposed to be $\Phi_m(\omega)$.

The angular velocity of the controlled member is then a stationary gaussian process with zero average whose power spectrum is

$$\Phi_\Omega(\omega)=\frac{1}{\omega_0^2}\cdot\frac{Y_0(i\omega)}{1+Y_0(i\omega)}\cdot\frac{Y_0(-i\omega)}{1+Y_0(-i\omega)}\cdot\Phi_m(\omega)$$

The corresponding covariance function is

$$R_\Omega(\tau)=\int_{-\infty}^{\infty}\Phi_\Omega(\omega)e^{i\omega\tau}d\omega$$

The angular deviation of the controlled member

$$\theta(t)=\int_0^t\Omega(\tau)d\tau$$

is also a gaussian process. However, this process is not asymptotic stationary (unless $\Phi_\Omega(\omega)\underset{\omega\to 0}{=}O(\omega^2)$). The amplitude distribution of $\Phi(t)$ at a fixed time is $N(0, \sigma(t))$ where

$$\sigma^2(t)=\int_0^t(t-s)R_\Omega(s)ds \tag{6.2.1}$$

Notice that most of the high frequency part of $\Phi_m(\omega)$ can be eliminated if it is possible to reduce the bandwidth of the servosystem.

The random drift of a single axis platform system due to stationary disturbing torques could thus be specified by the covariance function $R_m(\tau)$ of the disturbing torque acting on the gyrofloats.

6.3. **The three axis case**

We will start by introducing a quantity which is a suitable description of the controlled member.

Introduce a coordinate set $O\xi_i\xi_2\xi_3$ fixed to inertial space and initially coincident with the y-set. The transformation of the ξ-set on the y-set is

$$y = \mathbf{C}(t) \cdot \boldsymbol{\xi} \tag{6.3.1}$$

where

$$\mathbf{C}(0) = \mathbf{I}$$

The orientation of the controlled member is thus completely determined by the transformation matrix $\mathbf{C}(t)$. According to Euler's theorem of a rigid body, an orthogonal transformation can be interpreted as a rotation around the eigenvector of the transformation matrix. The angle of rotation $\theta(t)$ is used to specify the angular deviation of the controlled member. The angle $\theta(t)$ is related to the matrix $\mathbf{C}(t)$ by the relation

$$\theta(t) = \arccos \tfrac{1}{2}[Tr\,\mathbf{C}(t) - 1] \tag{6.3.2}$$

leaving ambiguity to the sign of $\theta(t)$. Tr $\mathbf{C}$ is the trace of the matrix $\mathbf{C}$. We obtain the following equation for $\mathbf{C}(t)$

$$\mathbf{C}(t) = \mathbf{I} + \int_0^t \boldsymbol{\Omega}(t')\mathbf{C}(t')dt' \tag{6.3.3}$$

where

$$(\boldsymbol{\Omega})_{jk} = \Omega_i \varepsilon_{ijk}$$

and $\boldsymbol{\Omega}(t)$ the angular velocity of the stable element.

Introduce the matrix sequence

$$\mathbf{C}_0 = \mathbf{I}$$

$$\mathbf{C}_n(t) = \mathbf{I} + \int_0^t \boldsymbol{\Omega}(t')\mathbf{C}_{n-1}(t'\,dt'$$

This sequence converges in norm to the solution of (6.3.3) at least when $||\Omega(t)||$ is bounded in an interval including $(0, t)$. As

$$\boldsymbol{\Omega} + \tilde{\boldsymbol{\Omega}} = 0$$

the solution $\mathbf{C}$ is an orthogonal matrix.

With (6.3.2), (6.3.3) and (4.4.6), it is thus possible to compute the indication error if the disturbances $\boldsymbol{M}(t)$ and $\boldsymbol{m}(t)$ are given. Similarly, the statistical properties of the indication error can be derived from those of the disturbances. An example is given below.

Example

The equation (6.3.3) will now be approximated. The approximation will be used to estimate the variance of the indication error due to disturbing torques acting on the gyrofloats.

The equation (6.3.3) gives

$$\|\mathbf{C}-\mathbf{C}_3\| = \sum_{\nu=4}^{\infty} \frac{(\|\Omega\| t)^{\nu}}{\nu!}$$

Taking the vector norm $\|x\|$ as $\max\limits_{t\varepsilon(0,\tau)} \left[\sum_1^3 x_i^2\right]^{\frac{1}{2}}$, we get

$$a=\|\Omega\|=\max_{t\varepsilon(0,\tau)}[\Omega_1^2(t)+\Omega_2^2(t)+\Omega_3^2(t)]^{\frac{1}{2}}$$

further is

$$|Tr\mathbf{C}-Tr\mathbf{C}_2|\leq 3\sum_{\nu=4}^{\infty}\frac{(at)^{\nu}}{\nu!} \qquad 0\leq t\leq\tau$$

The equation (6.3.2) thus has the solution

$$\theta^2(t)=e_i(t)\cdot e_i(t)+\varepsilon \tag{6.3.4}$$

where

$$e_i(t)=\int_0^t \Omega_i(\tau)d\tau$$

and the error ε is given by

$$|\varepsilon|<2\cos at-(at)^2+\sum_{\nu=4}^{\infty}\frac{(at)^{\nu}}{\nu!} \tag{6.3.5}$$

Assuming that the components of the angular velocity are independent, stationary, stochastic processes with the same autocorrelation function $R_{\Omega\Omega}(\tau)$, we get for the variance of the angular deviation of the controlled member

$$E\theta^2(t)=3\int_0^t(t-s)R_{\Omega}(s)ds+\varepsilon \tag{6.3.6}$$

where the error ε is given by (6.3.5) cf. (6.2.1).

6.4. **Errors caused by the spin axis sensitivity**

Throughout the analysis we have used a linear model of the system. The system is usually arranged in such a way that it behaves almost linearly. The output signals of the gyros are e.g. restricted to very small values. The dominant nonlinear terms in the signal equation are the terms due to the spin axis sensitivity of the gyros. Cf. section 4.2.3. Including these the signal equation runs

$$\mathbf{S}(D)\varphi(t)=\left[\frac{\omega_0}{a}Q(\varphi)-D\mathbf{L}\right]\Omega(t)-\boldsymbol{m}(t)$$

The equation of motion of the system can be solved by iteration. The iteration process converges for $t<\tau$ at least if the disturbances $\boldsymbol{M}(t)$ and $\boldsymbol{m}(t)$ are finite. Hence

$$\Omega(t)=\Omega^{(0)}(t)+\Omega^{(1)}(t)+\ \ldots$$

where

$$\mathbf{K}(D)\Omega^{(0)}(t)=\boldsymbol{M}(t)+\frac{1}{A_{22}}\mathbf{G}(D)\mathbf{S}^{-1}(D)\boldsymbol{m}(t)$$

$$\mathbf{K}(D)\Omega^{(1)}(t)=\frac{1}{A_{22}}\mathbf{G}(D)\mathbf{S}^{-1}(D)\cdot\boldsymbol{m}^{(1)}(t)$$

and

$$m_1^{(1)}(t)=\left(\frac{\omega_0}{a}\right)^2(\sin\theta^{(1)}\Omega_2-\cos\theta^{(1)}\Omega_3)\cdot\frac{1}{D^2+\sigma(D)}\cdot\Omega_1$$

$$+\frac{\omega_0}{a}\sin\theta^{(1)}\cdot\cos\theta^{(1)}\left[\Omega_3\cdot\frac{D}{D^2+\sigma(D)}\cdot\Omega_3-\Omega_2\cdot\frac{D}{D^2+\sigma(D)}\Omega_2\right]$$

$$+\frac{\omega_0}{a}\left[\cos^2\theta^{(1)}\Omega_3\cdot\frac{D}{D^2+\sigma(D)}\Omega_2-\sin^2\theta^{(1)}\Omega_2\cdot\frac{D}{D^2+\sigma(D)}\cdot\Omega_3\right]$$

etc.

Even if the components of the angular velocity are independent, the average value of the second term will in general not be zero. The influence of the spin axis sensitivity will thus in the first approximation be equivalent to a disturbing torque on the gyrofloats with a non-zero average. However the coefficient of the second term $\sin\theta^{(i)}\cos\theta^{(i)}$ will be zero if the arrangement of the gyros is orthogonal, i.e. $\theta^{(i)}=n\cdot\frac{\pi}{2}$.

7. Synthesis of inertial stabilized platform systems

As all problems have their individual character, it is impossible, of course, to give detailed schemes covering all possibilities. Let it therefore suffice to give the main lines of a synthesis procedure.

1. Choose a matrix $\mathbf{K}(p)$ which satisfies the specifications.
2. Design a system which has the $\mathbf{K}(p)$-matrix obtained above.
3. Check if it is possible to change the $\mathbf{K}(p)$-matrix in order to simplify the instrumentation without overriding the specifications.

For inertial stabilized systems the first step consists of choosing a $\mathbf{K}(p)$-matrix which gives a sufficiently tight coupling between the controlled member and inertial space. If it is also intended to rotate the system with respect to inertial space, we have to compromise between the ability to follow commanding signals. With the specifications usually given, there is no unique solution to the problem. The choice between the different possible solutions is governed by instrumentational considerations.

It is favourable to choose a diagonal $\mathbf{K}(p)$-matrix

$$\mathbf{K}(p)=p[1+Y_0]\mathbf{I} \tag{7.1}$$

where Y_0 is determined from the synthesis of a single axis channel cf. sections 3.3 and 3.4.

For the second step in the synthesis we start with the equation

$$\mathbf{K}(p)=\mathbf{F}(p)+\mathbf{G}(p)\mathbf{S}^{-1}(p)\mathbf{V}(p) \tag{7.2}$$

The $\mathbf{K}(p)$-matrix is obtained in the first step. This equation gives 9 equations for determining the 18 transfer functions $\sigma_{ij}(p)$ and $\tau_{ij}(p)$, the orientation angles $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ and the comparents of the inertia matrices $\mathbf{A}$ and $\mathbf{B}$. The problem is thus highly undetermined and it is thus possible to impose several other conditions.

Example

Suppose that the gyros, their orientation, the controlled member, and all fl$_{ij}(p)$:*s* are given. The equations (4.3.12) and (4.3.14) give

$$\mathbf{T}(p)=\mathbf{G}(p)-(A_{12}p^2+\omega_0 p)\mathbf{I}-A_{22}p^2\tilde{\mathbf{L}}+A_{32}p^2\tilde{\mathbf{N}}$$

$$\mathbf{G}(p)=[\mathbf{K}(p)-\mathbf{F}(p)]\mathbf{V}^{-1}(p)\mathbf{S}(p)$$

It is obvious from this example that $\mathbf{T}(p)$ will have nondiagonal elements even if $\mathbf{K}(p)$ is diagonal. If these nondiagonal elements are not mechanized, the system will not have the $\mathbf{K}(p)$ matrix given by the first step. The main problem of the third step is to analyse if this is consistent with the specifications. Special attention must also be paid to the stability conditions if the $\mathbf{K}(p)$ matrix is not diagonal. It is also necessary to analyse the order of magnitude of the angular velocities of the controlled member and the output signals of the gyros to judge if the linear model is appropriate.

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