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Threshold Saturation for Spatially Coupled Turbo-like Codes over the Binary Erasure Channel

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Abstract—In this paper we prove threshold saturation for spatially coupled turbo codes (SC-TCs) and braided convolutional codes (BCCs) over the binary erasure channel. We introduce a compact graph representation for the ensembles of SC-TC and BCC codes which simplifies their description and the analysis of the message passing decoding. We demonstrate that by few assumptions in the ensembles of these codes, it is possible to rewrite their vector recursions in a form which places these ensembles under the category of scalar admissible systems. This allows us to define *potential functions* and prove threshold saturation using the proof technique introduced by Yedla *et al.*

I. INTRODUCTION

Low-density parity-check (LDPC) convolutional codes [1], also known as spatially coupled LDPC (SC-LDPC) codes [2] have received a great deal of attention in the recent years as a result of their excellent performance under iterative decoding. In particular, it has been shown that the threshold of a belief propagation (BP) decoder improves to the threshold of an optimal maximum a-posteriori (MAP) decoder. This remarkable phenomenon is called threshold saturation.

Spatial coupling is not limited to LDPC codes. Braided convolutional codes (BCCs) are a class of spatially coupled (SC) codes introduced in [3]. Recently, the authors introduced spatially coupled turbo codes (SC-TCs) [4], as the SC counterparts of parallel [5] and serially [6] concatenated convolutional codes. In [4], [7]–[9], we investigated threshold saturation for SC parallel concatenated codes (SC-PCCs), SC serially concatenated codes (SC-SCCs) and BCCs over the binary erasure channel (BEC). We derived closed-form density evolution (DE) equations for SC-TCs and BCCs and investigated their decoding thresholds. Our numerical results suggest that threshold saturation occurs for SC-PCCs, SC-SCCs and BCCs.

In this paper, we formally prove threshold saturation for SC-TCs and BCCs over the BEC. Our proof relies on the proof technique based on potential functions, recently proposed by Yedla *et al.* [10], [11]. We introduce a compact graph representation to describe PCC, SCC and BCC ensembles. Similar to a protograph [12], the compact graph makes it easier to illustrate the analysis of the message passing decoding. We then demonstrate that by few assumptions, the DE recursions of SC-TCs [4], [8] and BCCs [7], [8], can be rewritten in a form that corresponds to the recursion of a scalar admissible

system as in [10]. This makes it possible to derive suitable potential functions for TCs and uncoupled BCCs. Finally, we prove threshold saturation for SC-TCs and BCCs following the same lines as the proof in [10] for SC-LDPC codes.

II. COMPACT GRAPH REPRESENTATION

It is possible to analyze message passing decoding algorithms in an efficient way by the use of factor graphs [12]. However the conventional factor graph of codes with convolutional component codes, such as PCCs, SCCs and BCCs, gets very large as the length of the component codes increases. In this section we introduce a more compact graph representation, in which each trellis is represented by a single factor node and each collection of variables of the same type is represented by a single variable node. We use this compact graph representation to obtain DE equations and describe the spatially coupled ensembles.

Fig. 1(a) shows the conventional factor graph representation of a rate $R = 1/3$ PCC. This code is built from two rate-1/2 recursive systematic encoders, referred to as upper and lower encoders; we call the corresponding trellises upper and lower trellises and denote them by T_U and T_L , respectively. The information sequence at time slot t is denoted by \mathbf{u}_t and is a vector of N bits $\mathbf{u}_t = (u_1, u_2, \dots, u_N)$. The information sequence \mathbf{u}_t and its reordered copy are encoded by the upper and lower encoder to produce parity sequences \mathbf{v}_t^U and \mathbf{v}_t^L , respectively.

Fig. 1(b) shows the compact graph representation of this code. Each of the sequences \mathbf{u}_t , \mathbf{v}_t^U and \mathbf{v}_t^L are represented by a single black circle (variable node). Thus, each variable node in the compact graph corresponds to a number of code symbols. The trellises are replaced by squares (factor nodes) which are labeled by their length¹. The permutation is represented by a line that crosses the edge which connects \mathbf{u}_t to T_L in order to emphasize that a reordered copy of \mathbf{u}_t is used in T_L . The transmitted code sequence is $\mathbf{v} = (\mathbf{u}_t, \mathbf{v}_t^U, \mathbf{v}_t^L)$.

Fig. 1(c) depicts the compact graph representation of a rate $R = 1/4$ SCC built from two rate-1/2 recursive systematic convolutional encoders, referred to as inner and outer encoders. We call the corresponding trellises inner and outer trellises and denote them by T_I and T_O , respectively. At time t , the information sequence \mathbf{u}_t , of length N , is encoded by

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¹The length of a trellis is equal to the length of each of the sequences which are connected to that trellis.

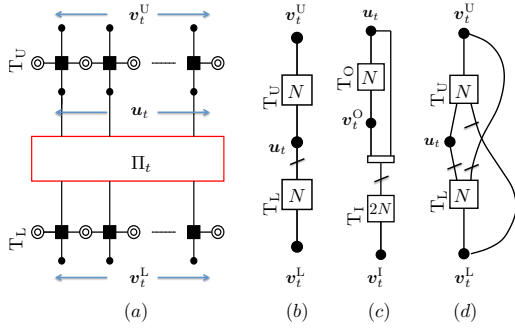


Fig. 1. (a) Conventional factor graph of a PCC, compact graph of a (b) PCC, (c) SCC, (d) BCC.

the outer encoder to produce the parity sequence v_t^O . Then, u_t and v_t^O are multiplexed and reordered to create a sequence \tilde{v}_t^O whose length is $2N$. The sequence \tilde{v}_t^O is encoded by the inner encoder to produce the parity sequence v_t^L . The transmitted code sequence is $v = (u_t, v_t^O, v_t^L)$.

BCCs consist of two rate-2/3 component convolutional encoders. As for PCCs, we call the component encoders upper and lower and the corresponding trellises as upper and lower trellises and denote them by T_U and T_L , respectively. BCCs are inherently SC but we can define uncoupled BCCs by tailbiting a chain of a coupled code with coupling length $L = 1$. Fig. 1(d) shows the compact graph of uncoupled BCCs. At time t , the parity sequences of T_U and T_L are denoted by v_t^U and v_t^L , respectively. The information sequence u_t and a reordered copy of v_t^L are used in T_U to produce v_t^U . Likewise, some reordered copies of u_t and v_t^U are used in T_L to produce v_t^L . The transmitted code sequence is $v = (u_t, v_t^U, v_t^L)$.

III. DENSITY EVOLUTION AND SCALAR ADMISSIBLE SYSTEM

In this section, we first define a scalar admissible system. Then we show that by few assumptions in the ensembles of TCs and BCCs, it is possible to rewrite their DE recursions in a form which corresponds to the recursion of a scalar admissible system.

Definition 1 ([10]): A scalar admissible system (f, g) , is defined by the recursion

$$x^{(i)} = f(g(x^{(i-1)}); \varepsilon), \quad (1)$$

where $f : [0, 1] \times [0, 1] \rightarrow [0, 1]$ and $g : [0, 1] \rightarrow [0, 1]$ satisfy the following conditions.

- f is increasing in both arguments $x, \varepsilon \in (0, 1]$;
- g is increasing in $x \in (0, 1]$;
- $f(0; \varepsilon) = f(x; 0) = g(0) = 0$;
- f and g have continuous second derivatives.

A. Parallel Concatenated Codes

Consider the PCC in Fig. 1(b). To formulate the DE equations as we obtained in [4], let $p_{U,s}^{(i)}$ and $p_{L,s}^{(i)}$ denote the extrinsic erasure probabilities to u_t from T_U and T_L , respectively. Consider transmission over the BEC with erasure probability ε . The erasure probabilities to T_U from u_t and v_t^L ,

in the i th iteration, are $\varepsilon \cdot p_{L,s}^{(i-1)}$ and ε , respectively. Thus the DE update for T_U is given by

$$p_{U,s}^{(i)} = f_{U,s} \left(q_L^{(i-1)}, \varepsilon \right), \quad (2)$$

where

$$q_L^{(i-1)} = \varepsilon \cdot p_{L,s}^{(i-1)} \quad (3)$$

and $f_{U,s}$ denotes the transfer function of T_U for the information bits. Similarly, the DE update for T_L can be written as

$$p_{L,s}^{(i)} = f_{L,s} \left(q_U^{(i-1)}, \varepsilon \right) \quad (4)$$

where

$$q_U^{(i-1)} = \varepsilon \cdot p_{U,s}^{(i-1)}. \quad (5)$$

The DE equations for PCCs in (2)-(5) involve different edges and hence form a vector recursion. However, considering identical T_U and T_L nodes (i.e., identical component encoders), it follows that $f_{U,s} = f_{L,s} \triangleq f_s$. Therefore, $p_{U,s}^{(i)} = p_{L,s}^{(i)} \triangleq x^{(i)}$. Now, using this and by substituting (3) into (2) and (6) into (5), the DE can be written as a scalar recursion,

$$x^{(i)} = f_s(\varepsilon x^{(i-1)}, \varepsilon), \quad (6)$$

where the initial condition is $x^{(0)} = 1$. Consider $f(x; \varepsilon) = f_s(\varepsilon \cdot x, \varepsilon)$ and $g(x) = x$. We show in the following that these two functions meet the conditions in Definition 1. Therefore, the recursion (6) is a recursion of a scalar admissible system. The function $g(x) = x$ is a simple function and it is easy to show that it satisfies all conditions in Definition 1. $f(x; \varepsilon)$ is a transfer function of a BCJR decoder with a-priori information x and channel erasure probability ε . In the following lemma we show that it satisfies the conditions in Definition 1.

Lemma 1: Consider a terminated convolutional code where all distinct input sequences have distinct coded sequences. For such a system, the transfer function $f(x; \varepsilon)$ of a BCJR decoder with a-priori probability x and channel erasure probability ε , or any convex combination of such transfer functions, satisfies all conditions in Definition 1.

Proof: The BCJR decoder is an optimal APP decoder. Consider two BECs with erasure probabilities ε_1 and ε_2 , with $\varepsilon_1 < \varepsilon_2$. The BEC with erasure rate ε_2 , can be seen as the concatenation of two BECs with erasure rates ε_1 and ε' , where $\varepsilon' = 1 - \frac{1-\varepsilon_2}{1-\varepsilon_1}$. The data processing inequality implies that $f(x; \varepsilon_1) < f(x; \varepsilon_2)$. This means that the erasure probability at the output of the BCJR decoder is monotone and increases with ε . When $\varepsilon = 0$, the input sequence can be recovered perfectly from the received sequence, as there is a one-to-one mapping of input sequences to coded sequences. This means $f(x; 0) = 0$. It is also possible to prove that $f(x_1; \varepsilon) < f(x_2; \varepsilon)$ for $x_1 < x_2$ (not shown due to lack of space).

Finally, $f(x; \varepsilon)$ is a rational function and its poles are outside the interval $x, \varepsilon \in [0, 1]$ (otherwise we may get infinite output erasure probability for a finite input erasure probability), so it has continuous first and second derivatives in the interval $x, \varepsilon \in [0, 1]$. ■

B. Serially Concatenated Codes

Consider the SCC in Fig. 1(c). We define by $p_{I,s}^{(i)}$ the extrinsic erasure probability from T_I to u_t and v_t^O . Likewise, let $p_{O,s}^{(i)}$ and $p_{O,p}^{(i)}$ denote the extrinsic erasure probabilities from T_O to u_t and v_t^O in the i th iteration, respectively. Consider the transmission over the BEC with erasure probability ε . The erasure probabilities from u_t and v_t^O to T_O in the i th iteration both are equal to

$$q_I^{(i-1)} = \varepsilon \cdot p_{I,s}^{(i-1)}. \quad (7)$$

Thus, the DE equations for T_O can be written as

$$p_{O,s}^{(i)} = f_{O,s} \left(q_I^{(i-1)}, q_I^{(i-1)} \right) \quad (8)$$

$$p_{O,p}^{(i)} = f_{O,p} \left(q_I^{(i-1)}, q_I^{(i-1)} \right), \quad (9)$$

where $f_{O,s}$ and $f_{O,p}$ denote the transfer functions of T_O for the input and parity bits, respectively.

The input sequence of the inner encoder consists of u_t and v_t^O , so that the erasure probability $q_O^{(i)}$ that comes to T_O through the set of these two variable nodes is the average of the extrinsic erasure probabilities from u_t and v_t^O , i.e.,

$$q_O^{(i)} = \varepsilon \cdot \frac{p_{O,s}^{(i)} + p_{O,p}^{(i)}}{2}. \quad (10)$$

Let $f_{I,s}$ denote the transfer function of T_I for the input bits. The DE equations for T_O can be written as

$$p_{I,s}^{(i+1)} = f_{I,s} \left(q_O^{(i)}, \varepsilon \right). \quad (11)$$

Equations (7) to (11) show that the DE for SCCs in Fig. 1(c) is a vector recursion. However, for identical T_O and T_I , it follows $f_{I,s} = f_{O,s} \triangleq f_s$ and $f_{I,p} = f_{O,p} \triangleq f_p$. Using this and $q_I^{(i-1)} \triangleq x^{(i)}$, by substituting (8)-(11) into (7), the DE recursion can be written as

$$x^{(i+1)} = \varepsilon \cdot f_s \left(\varepsilon g(x^{(i)}), \varepsilon \right), \quad (12)$$

where

$$g(x^{(i)}) = \frac{f_s \left(x^{(i)}, x^{(i)} \right) + f_p \left(x^{(i)}, x^{(i)} \right)}{2}, \quad (13)$$

and the initial condition is $x^{(0)} = 1$.

Consider $f(x; \varepsilon) = \varepsilon \cdot f_s(x, \varepsilon)$ and

$$g(x) = \frac{f_s(x, x) + f_p(x, x)}{2}.$$

According to Lemma 1, these two functions meet the conditions in Definition 1 and we can conclude that the DE recursion of SCCs in (12) is a recursion of a scalar admissible system.

C. Braided Convolutional Codes

Consider the BCC in Fig. 1(d). Let $p_{U,k}^{(i)}$ and $p_{L,k}^{(i)}$, $k = 1, 2, 3$, denote the extrinsic erasure probabilities from T_U and T_L in the i th iteration, through their k th connected edge, respectively. The exact DE equations can be written as [7]

$$p_{U,1}^{(i)} = f_{U,1} \left(\varepsilon \cdot p_{L,1}^{(i-1)}, \varepsilon \cdot p_{L,3}^{(i-1)}, \varepsilon \cdot p_{L,2}^{(i-1)} \right) \quad (14)$$

$$p_{U,2}^{(i)} = f_{U,2} \left(\varepsilon \cdot p_{L,1}^{(i-1)}, \varepsilon \cdot p_{L,3}^{(i-1)}, \varepsilon \cdot p_{L,2}^{(i-1)} \right) \quad (15)$$

$$p_{U,3}^{(i)} = f_{U,3} \left(\varepsilon \cdot p_{L,1}^{(i-1)}, \varepsilon \cdot p_{L,3}^{(i-1)}, \varepsilon \cdot p_{L,2}^{(i-1)} \right), \quad (16)$$

where $f_{U,k}$ denotes the transfer function of T_U for the k th connected edge. Likewise, the DE equations for T_L can be written by swapping $p_{U,k}^{(i)}$ and $p_{L,k}^{(i)}$ for $k = 1, 2, 3$.

Similarly to PCCs and SCCs, the DE equations of a BCC form a vector recursion. In order to modify this recursion to scalar form, in the first step consider identical upper and lower factor nodes. It follows $f_{U,k} = f_{L,k} \triangleq f_k$ and $p_{U,k}^{(i)} = p_{L,k}^{(i)} \triangleq x_k$ for $k = 1, 2, 3$. Then we can rewrite the DE equations of T_U as

$$x_1^{(i+1)} = f_1 \left(\varepsilon \cdot x_1^{(i)}, \varepsilon \cdot x_3^{(i)}, \varepsilon \cdot x_2^{(i)} \right) \quad (17)$$

$$x_2^{(i+1)} = f_2 \left(\varepsilon \cdot x_1^{(i)}, \varepsilon \cdot x_3^{(i)}, \varepsilon \cdot x_2^{(i)} \right) \quad (18)$$

$$x_3^{(i+1)} = f_3 \left(\varepsilon \cdot x_1^{(i)}, \varepsilon \cdot x_3^{(i)}, \varepsilon \cdot x_2^{(i)} \right). \quad (19)$$

According to the above equations, the DE recursion is still in vector form. To rewrite it in scalar form, one alternative is to consider identical component encoders with a time-varying trellis, such that all three transfer functions are equal. For example, by periodically changing the order of symbols along trellis branches, this function becomes the average of the transfer functions f_1, f_2, f_3 , $f_{ave} = \frac{f_1+f_2+f_3}{3}$. By the above assumption, $x_1 = x_2 = x_3 \triangleq x$. Using this in (17)-(19) we can simplify the DE recursion as

$$x^{(i+1)} = f_{ave}(\varepsilon \cdot x^{(i)}, \varepsilon \cdot x^{(i)}, \varepsilon \cdot x^{(i)}). \quad (20)$$

Considering $f(x; \varepsilon) = f_{ave}(\varepsilon \cdot x, \varepsilon \cdot x, \varepsilon \cdot x)$, $g(x) = x$ and Lemma 1, (20) is the recursion of a scalar admissible system.

IV. SINGLE SYSTEM POTENTIAL

Since the DE recursion of TCs and BCCs can be written as the recursion of a scalar admissible system, we can derive the corresponding potential functions [10].

Definition 2: For a scalar admissible system, the potential function $U(x; \varepsilon)$ is defined by

$$\begin{aligned} U(x; \varepsilon) &= \int_0^x (z - f(g(x); \varepsilon))g'(z)dz \\ &= xg(x) - G(x) - F(g(x); \varepsilon), \end{aligned} \quad (21)$$

where $F(x; \varepsilon) = \int_0^x f(z; \varepsilon)dz$ and $G(x) = \int_0^x g(z)dz$.

Proposition 1: The potential function has the following characteristics.

- $U(x; \varepsilon)$ is strictly decreasing in $\varepsilon \in (0, 1]$.
- An $x \in [0, 1]$ is a fixed point of the recursion (1) iff it is a stationary point of the potential function.

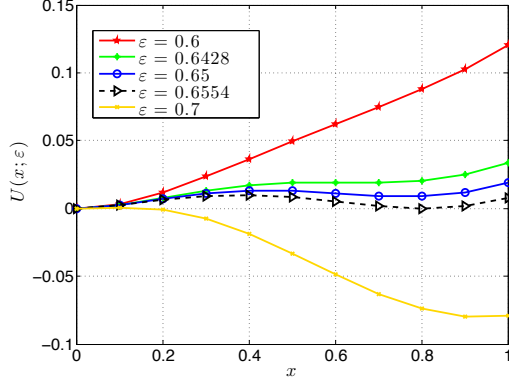


Fig. 2. The potential function of PCC with generator matrix $\mathbf{G} = (1, 5/7)$ in octal notation

Definition 3: If the DE recursion is the recursion of a BP decoder, the BP threshold is [10]

$$\varepsilon^{\text{BP}} = \sup \left\{ \varepsilon \in [0, 1] \mid U'(x; \varepsilon) > 0, \forall x \in (0, 1] \right\}.$$

According to Definition 3, for $\varepsilon < \varepsilon^{\text{BP}}$, the potential function has no stationary point and its derivative is always larger than zero for $x \in (0, 1]$.

Definition 4: For $\varepsilon > \varepsilon^{\text{BP}}$, the minimum unstable fixed point is $u(\varepsilon) = \sup \{ \tilde{x} \in [0, 1] \mid f(g(x); \varepsilon) < x, x \in (0, \tilde{x}) \}$. Then, the potential threshold is [10]

$$\varepsilon^* = \sup \left\{ \varepsilon \in [0, 1] \mid u(x) > 0, \min_{x \in [u(x), 1]} U(x; \varepsilon) > 0 \right\}.$$

The potential threshold depends on functions $g(x)$ and $f(x; \varepsilon)$. Since at least one of these functions depends on the component encoders, ε^* also depends on the component encoders.

Example 1: Consider a rate-1/3 PCC in Fig. 1(a) with identical component encoders with generator matrix $\mathbf{G} = (1, 5/7)$ in octal notation. Its potential function is

$$U(x; \varepsilon) = x^2 - G(x) - F_s(x; \varepsilon) = \frac{x^2}{2} - F_s(x; \varepsilon),$$

where $F_s(x; \varepsilon) = \int_0^x f_s(\varepsilon \cdot z, \varepsilon) dz$ and $G(x) = \int_0^x g(z) dz = \frac{x^2}{2}$.

The potential function of this code is shown in Fig. 2. As it is illustrated in the figure, $\varepsilon = 0.6428$ is the maximum channel erasure probability for which the derivative of the potential function is greater than zero and the potential function has no stationary point for $x \in (0, 1]$. Thus, $\varepsilon = 0.6428$ is the BP threshold of this code (see Definition 3). The potential threshold is $\varepsilon^* = 0.6554$ (see the black line in Fig. 2). These results match with our numerical results in [4].

V. COUPLED SYSTEM AND THRESHOLD SATURATION

Theorem 1: Consider a spatially coupled system defined by the following recursion at time t ,

$$x_t^{(i+1)} = \frac{1}{1+m} \sum_{k=0}^m f \left(\frac{1}{1+m} \sum_{j=0}^m g(x_{t+j-k}^{(i)}); \varepsilon \right). \quad (22)$$

For a large enough coupling memory and $\varepsilon < \varepsilon^*$, the only fixed point of the recursion is $x = 0$.

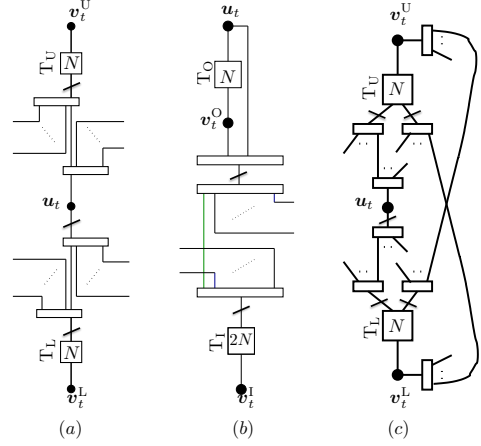


Fig. 3. Compact graph of (a) SC-PCCs, (b) SC-SCCs, (c) BCCs

Proof: The proof follows from [10]. \blacksquare

In the following we demonstrate that the recursion of SC-PCCs, SC-SCCs and BCCs correspond to the recursion in (22).

A. Spatially Coupled Parallel Concatenated Codes

Fig. 3(a) shows the compact graph of SC-PCCs at time t for coupling memory m . The sequence corresponding to the input variable node at this time, \mathbf{u}_t , is divided into $m+1$ sequences, $\mathbf{u}_{t,j}$, $j = 0, \dots, m$. At time t , the sequences $\mathbf{u}_{t-j,j}$, $j = 0, \dots, m$ are multiplexed and reordered. The resulting sequence is used as the input to T_U at time t . Likewise, a reordered copy of the sequence corresponding to the input variable node at the current time slot, \mathbf{u}'_t , is divided into $m+1$ sequences $\mathbf{u}'_{t,j'}$, $j' = 0, \dots, m$. At time t , the sequences $\mathbf{u}_{t-j',j'}$, $j' = 0, \dots, m$ are multiplexed and reordered. The resulting sequence is used as the input to T_L at time t . In other words, \mathbf{u}_t is connected to the set of T_U s and the set of T_L s at time slots t to $t+m$. Consider identical T_U s and T_L s. Due to the symmetric coupling structure, both erasure probabilities that come to \mathbf{u}_t at time t are equal and denoted by $x_t^{(i)}$. Following the compact graph of SC-PCCs, the erasure probability to T_U through its first edge is the average of the erasure probabilities from $\mathbf{u}_{t'}$, $t' = t-m, \dots, t$. Therefore, the update of T_U and T_L at time t is

$$f_{s,t} \left(\frac{\varepsilon}{m+1} \cdot \sum_{j=0}^m x_{t-j}^{(i)}, \varepsilon \right),$$

where $f_{s,t}$ is the transfer function of T_U and T_L at time t for the information bits. The erasure probability that comes to \mathbf{u}_t at time slot t through each of the incoming edges is the average of the erasure probabilities that come from the set of T_U s or T_L s at time slots t to $t+m$. The recursion equation at time slot t can then be written as

$$x_t^{(i+1)} = \frac{1}{1+m} \sum_{k=0}^m f_{s,t+k} \left(\frac{\varepsilon}{m+1} \cdot \sum_{j=0}^m x_{t-j+k}^{(i)}, \varepsilon \right). \quad (23)$$

The recursion (23) is identical to the recursion in (22). Thus, according to Theorem 1, for channels with erasure probability $\varepsilon < \varepsilon^*$, the only fixed point of recursion (23) is zero.

B. Spatially Coupled Serially Concatenated Codes

Fig. 3(b) shows the compact graph of SC-SCCs at time t for coupling memory m . Similarly to uncoupled SCCs, at time t , \mathbf{u}_t and \mathbf{v}_t^O are multiplexed and reordered to produce the sequence $\tilde{\mathbf{v}}_t^O$. $\tilde{\mathbf{v}}_t^O$ is randomly divided into $m+1$ sequences $\tilde{\mathbf{v}}_{t,j}^O$, $j = 0, \dots, m$. T_1 at time t receives a sequence which is built from sequences $\tilde{\mathbf{v}}_{t-j,j}^O$, $j = 0, \dots, m$, and reordered.

Consider identical T_O and T_1 and denote the erasure probability from T_1 to the set of \mathbf{u}_t and \mathbf{v}_t^O by x_t^i at time t and iteration i . Then, the message from this set to the set of T_1 s is $g(x_t^i)$, where $g(x)$ is obtained in (13). Following the edges which are connected to T_1 at time t , the erasure probability that T_1 receives through its first edge is

$$\frac{\varepsilon}{m+1} \cdot \sum_{j=0}^m g(x_{t-j}^i).$$

The update of T_1 can be written as

$$f_{s,t} \left(\frac{\varepsilon}{m+1} \cdot \sum_{j=0}^m g(x_{t-j}^i), \varepsilon \right),$$

where $f_{s,t}$ is the transfer function of T_1 and T_O at time t for their input bits.

Both \mathbf{u}_t and \mathbf{v}_t^O receive equal erasure probabilities from the set of T_1 s. This erasure probability is the average of the erasure probabilities from T_1 at time slots t to $t+m$. The DE recursion can then be written as

$$x_t^{(i+1)} = \frac{1}{1+m} \sum_{k=0}^m \varepsilon \cdot f_{s,t+k} \left(\frac{\varepsilon}{m+1} \cdot \sum_{j=0}^m g(x_{t-j+k}^i), \varepsilon \right),$$

where function g is given in (13). This recursion is identical to the recursion in (22). Therefore, we can conclude that for $\varepsilon < \varepsilon^*$ the only fixed point of the recursion of SC-SCCs is zero and threshold saturation occurs.

C. Braided Convolutional Codes

Fig. 3(c) shows the compact graph of BCCs at time t for coupling memory m . This ensemble of BCCs is slightly different from the ensembles we introduced in [8]. To couple the code with memory m , each of the sequences corresponding to \mathbf{u}_t , a reordered copy of the information sequence, $\tilde{\mathbf{u}}_t$, \mathbf{v}_t^U and \mathbf{v}_t^L is divided into $m+1$ sequences and denoted by $\mathbf{u}_{t,j}$, $\tilde{\mathbf{u}}_{t,j}$, $\mathbf{v}_{t,j}^U$ and $\mathbf{v}_{t,j}^L$ for $j = 0, \dots, m$, respectively. At time t , sequences $\mathbf{u}_{t-j,j}$ for $j = 0, \dots, m$, are multiplexed and reordered. The resulting sequence is used as the first input to T_U . Likewise, the sequences $\mathbf{v}_{t-j,j}^L$ for $j = 0, \dots, m$, are multiplexed and reordered. The resulting sequence is used as the second input of T_U . The sequences $\tilde{\mathbf{u}}_{t,j}$ for $j = 0, \dots, m$, are multiplexed and reordered and used as the first input of T_L . Likewise, the sequences $\mathbf{v}_{t-j,j}^U$ for $j = 0, \dots, m$, are multiplexed and reordered and the resulting sequence is used as second input of T_L .

Consider identical component encoders at time t . The erasure probabilities to \mathbf{u}_t , \mathbf{v}_t^U and \mathbf{v}_t^L are equal due to the symmetric coupling structure and denoted by $x_t^{(i)}$. Following

the compact graph, the erasure probabilities to T_U through all its incoming edges are equal and are given by the average of the erasure probabilities from $\mathbf{u}_{t'}$ s, $t' = t-m, \dots, t$, $q_t = \frac{\varepsilon}{1+m} \sum_{i=0}^m x_{t-j}^{(i)}$. Thus, the erasure probabilities from each of the factor nodes to their outgoing edges are equal to $f_{ave,t}(q_t, q_t, q_t)$, where $f_{ave,t}$ is the transfer function of T_U and T_L at time t for all edges. Finally the recursion at time slot t is

$$x_t^{(i+1)} = \frac{1}{1+m} \sum_{k=0}^m f_{ave,t+k}(q_{t+k}, q_{t+k}, q_{t+k}). \quad (24)$$

As (24) is identical to (22), according to Theorem 1, for channels with erasure probability $\varepsilon < \varepsilon^*$, the only fixed point of (24) is equal to zero.

VI. CONCLUSIONS

We considered three families of spatially-coupled turbo-like codes with identical component encoders whose density evolution recursions can be analyzed using the coupled scalar recursion framework of [10]. Then, based on this framework, we proved threshold saturation for these code ensembles over the BEC. For a more general case (different component encoders), the analysis is significantly more complicated and requires the coupled vector recursion framework of [11].

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