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# On the accuracy of benchmark tables and graphical results in the applied mechanics literature\*

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## Abstract

Converged normalized stress intensity factors for a matrix crack interacting with an elastic cylinder are presented. The new results differ from previously published results in several examples. The need for better error analysis in computational fracture mechanics is emphasized.

## Introduction

The purpose of this note is to initiate a discussion of the accuracy of benchmark tables and graphical results presented in the applied mechanics literature. Accurate benchmark results are essential in the development of new software. Programming errors easily occur. If one can not find at least three digit accurate results, for standard non-trivial setups, to verify against, many errors will go unnoticed.

Stress intensity factors are frequently tabulated and presented in graphs. These factors are considered difficult to compute, even though the underlying physical problem often is well-conditioned. The chief difficulties are to find and to implement efficient numerical algorithms and to assess the accuracy of the final result. There are many pitfalls. Finding the correct branch of the square root of complex numbers in the context of computing weight functions is just one example of a non-standard task which may occur and where even the properties of the compiler must be taken into account. Also, the orientation of coordinate systems and the various normalization factors and symbols used by different authors may cause confusion. Not surprisingly, many of the numerical results presented in the literature are of questionable quality. Convergence studies are seldom, if ever, presented. We believe that there is a particular need to reexamine previously published results in this area.

In order to illustrate the points made above we consider an example involving two papers presenting results for normalized stress intensity factors of a matrix crack in the presence of an elastic cylinder: one classic paper by Erdogan, Gupta, and Ratwani [1], and

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a recent paper by Cheeseman and Santare [2]. In the latter paper the authors validate their algorithm by comparing with results from the former paper. “Good agreement” is noted, but the statement is not supported by numerical results.

## Results

We simply recompute some results of Erdogan, Gupta, and Ratwani [1] and Cheeseman and Santare [2] using an algorithm based on a pair of integral equations for the crack and inclusion problem developed by Helsing and Peters [3]. The integral equations, number (48) and number (49) in Helsing and Peters, are of Fredholm’s second kind with compact operators. This allows for stable convergence. The integral equations are solved using a Nyström scheme with composite quadrature on a uniform mesh. We use 16-point Gauss-Legendre quadrature on all quadrature panels except for the two panels containing the crack tips. There we use Gauss-Jacobi quadrature. Great care is devoted to avoiding roundoff error throughout the code. The setups under investigation are depicted in Figure 1. The shear moduli of the matrix and of the cylinder are  $\mu_1 = 1$  and  $\mu_2 = 23$ . The Poisson’s ratios of matrix and of the cylinder are  $\nu_1 = 0.35$  and  $\nu_2 = 0.30$ . The two-dimensional bulk modulus  $\kappa$ , used in Helsing and Peters, is  $\kappa = \mu/(1 - 2\nu)$ . This bulk modulus should not be mixed up with the “kappa” used by many authors including Erdogan, Gupta, and Ratwani. The latter “kappa” corresponds to the quantity  $\varkappa = 3 - 4\nu$ , in Muskhelishvili’s notation.

Our new, converged, results do not always agree with the previously published results. This can be seen in Table 1 and in Figure 2. In many cases the results differ considerably, in digits and also in signs (for the secondary factors  $k_{21}$  and  $k_{22}$ ). It is hazardous to speculate in the reasons for this discrepancy. One thing is certain, however. Our results have converged stably. See Figure 3 for an example where the relative error for a stress intensity factor settles on a level of  $10^{-15}$  as the mesh is refined.

## Discussion

This note stresses the need for more error analysis in computational fracture mechanics. An algorithm may be correct in a mathematical sense. The results it produces on a computer may still be wrong if the problem is not properly resolved, if the algorithm is unstable, or if there is a bug in the code or in the compiler.

It is difficult to prove, rigorously, that a numerical solution to a non-trivial problem is accurate to a certain number of digits. Accurate benchmarks can, in our opinion, best be established by the agreement of several calculations performed by independent investigators. To this end, the presentation of numerical results in terms of numbers is essential. Graphs alone are not sufficient. The presentation of convergence studies, further, helps remove doubts about underresolution and instability. We encourage the inclusion of this type of information in forthcoming papers. We challenge other scientists in computational mechanics to confirm or disprove our new numerical results.

## References

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- [2] Cheeseman, B. A. and Santare, M. H., 2000, "The Interaction of a curved crack with a circular elastic inclusion," *Int. J. Fracture*, **103**, pp. 259-277.
- [3] Helsing, J. and Peters, G., 1999, "Integral equation methods and numerical solutions of crack and inclusion problems in planar elastostatics," *SIAM J. Appl. Math.*, **59**, pp. 965-982.

## Figures and Tables

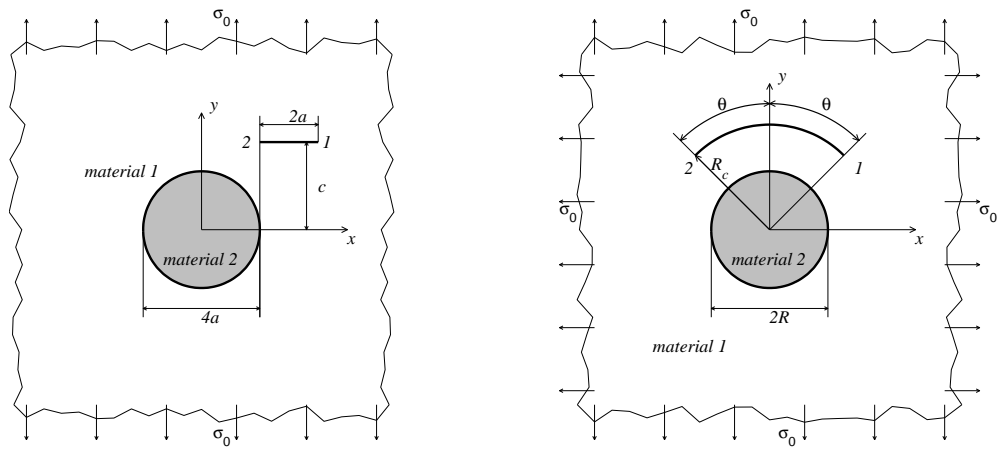


Figure 1: Left, a straight crack outside an inclusion under uniaxial tension. This is the setup of Erdogan, Gupta and Ratwani [1] corresponding to their Table 3. Right, an arc-shaped crack outside a circular inclusion under biaxial tension. This is the setup of Cheeseman and Santare [2] corresponding to their Figure 8.

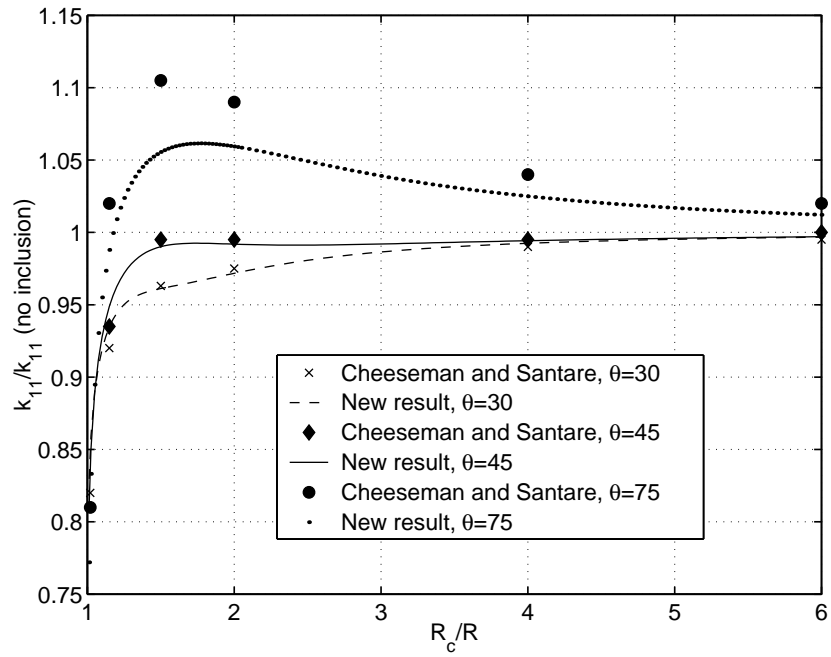


Figure 2: Normalized Mode I stress intensity factors of the setup in Figure 8 in Cheeseman and Santare [2] (the right image of our Figure 1) versus dimensionless distance for a circular arc-shaped crack interacting with an inclusion.

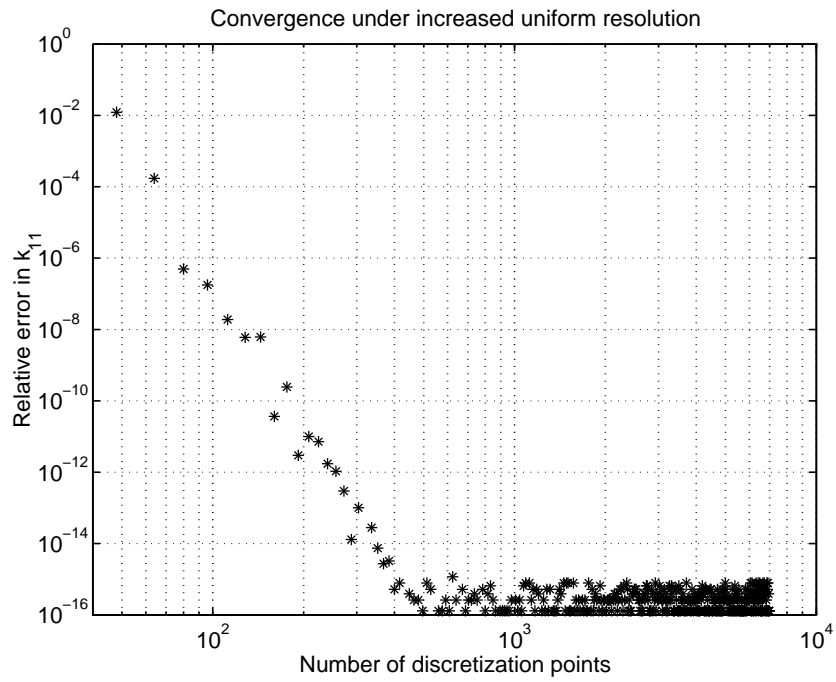


Figure 3: Convergence of the stress intensity factor  $k_{11}$  of Erdogan, Gupta, and Ratwani [1] for  $c = 2a$  in the left image of our Figure 1. The mesh is uniformly refined. The number of discretization points is  $N$ . Double precision arithmetic is used. The reference value  $k_{11} = 0.84973394741770513$  is computed with 592, or more, discretization points in quadruple precision arithmetic. Relative errors smaller than machine epsilon are displayed as  $1.11 \cdot 10^{-16}$ .

Table 1: Comparison between our new results and those of Table 3 in Erdogan, Gupta, and Ratwani [1] for the geometry of the left image in our Figure 1. The relations between the shear moduli of the inclusion and the matrix is  $\mu_2 = 23\mu_1$ . The normalized mode  $i$  stress intensity factor at crack tip  $j$  is denoted as  $k_{ij}$ .

$c/a$	$k_{11}^{\text{EGR}}$	$k_{11}^{\text{new}}$	$k_{12}^{\text{EGR}}$	$k_{12}^{\text{new}}$	$k_{21}^{\text{EGR}}$	$k_{21}^{\text{new}}$	$k_{22}^{\text{EGR}}$	$k_{22}^{\text{new}}$
0.3	0.784	0.790	0.225	0.235	-0.004	-0.023	0.072	0.073
0.5	0.792	0.797	0.341	0.347	-0.006	-0.037	0.101	0.102
1.0	0.817	0.817	0.613	0.613	-0.005	-0.067	0.057	0.061
1.5	0.839	0.833	0.763	0.755	0.008	-0.074	-0.007	0.012
2.0	0.860	0.850	0.845	0.830	0.034	-0.058	-0.021	0.018
3.0	0.905	0.897	0.953	0.936	0.089	-0.004	-0.001	0.067
4.0	0.951	0.947	1.014	1.003	0.117	0.032	0.002	0.079
8.0	1.020	1.022	1.043	1.043	0.088	0.032	-0.026	0.032