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Sparse Semi-Parametric Chirp Estimation

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***Abstract*—In this work, we present a method for estimating the parameters detailing an unknown number of linear chirp signals, using an iterative sparse reconstruction framework. The proposed method is initiated by a re-weighted Lasso approach, and then use an iterative relaxation-based refining step to allow for high resolution estimates. The resulting estimates are found to be statistically efficient, achieving the Cramér-Rao lower bound. Numerical simulations illustrate the achievable performance, offering a notable improvement as compared to other recent approaches.**

I. Introduction

MANY forms of everyday signals, ranging from radar to biomedical and seismic signals, and human speech, may be well modeled as periodic signals with instantaneous frequencies (IF) that varies slowly over time [1]. Such signals are often modeled as linear chirps, i.e., periodic signals with an IF that changes linearly over time. Given the prevalence of such signals, much effort has gone into formulating efficient estimation algorithms of the start frequency and rate of development, and then, in particular, for signals only containing a single (complex-valued) chirp-component. Here, one noteworthy method is the phase unwrapping algorithm presented by Djuric and and Kay [2]; further development of this method can be found in e.g. [3]. Other methods presented for single component estimation are, for example, based on Kalman filtering [4], [5], or sample covariance matrix estimates [6].

Recent work has in larger extent focused on also identifying multi-component chirp-signals such as the maximum likelihood technique presented in [7], and the fractional Fourier transform method [8]–[10]. Others have used the time-frequency distribution, i.e., Wigner-Ville, reassigned spectrogram, or a Gabor dictionary as a rough initial estimate, which then may be refined using image processing techniques to fit a linear chirp model [11]–[13]. The latter method seems to render good estimates, although it typically requires rather large data sets to do so. Clearly, the nonparametric methods have the advantage of computational efficiency, but also suffer from the poor resolution inherent to the Fourier transform. The parametric methods on the other hand have good performance and resolution, but require a priori knowledge of the number of components in the signal. Furthermore, it is not uncommon that one need to have good initial estimates to be able to use parametric methods, otherwise the algorithm might suffer from convergence problems.

In this work, we propose a semi-parametric algorithm for estimation of the parameters detailing a multi-component linear chirp signal, where the number of components is assumed to be unknown. The algorithm requires very few samples to get an accurate estimate of the parameters. Hence, even if the signal is comprised of non-linear chirp components, the signal can be divided into short segments for which the chirp rate is approximately linear. We demonstrate the performance of the proposed method on both real and simulated data, and compare the results with the corresponding Cramér-Rao lower bound (CRLB), as well as to competing algorithms. In this paper, scalars will be denoted with lower case symbols, e.g. x, whereas vectors will be denoted with bold lower case, $\mathbf{x}$. Matrices will be denoted with bold upper case letter, $\boldsymbol{X}$. Furthermore, $(\cdot)^T$, $\Re$, and $\Im$ will be used to denote the transpose, the real part, and the imaginary part, respectively.

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II. Signal Model

Consider the signal model

$$y(t) = \sum_{k=1}^{K} a_k e^{i2\pi\phi_k(t)} + e(t), \quad t = t_0, \ldots, t_{N-1} \tag{1}$$

where K denotes the unknown number of components, N the number of available samples, a_k the complex valued amplitude, $\phi_k(t)$ the time dependent frequency, and $e(t)$ as additive noise term, here assumed to be white and Gaussian distributed. Furthermore, the chirp frequency is assumed to be linear, at least under short time intervals, such that it may be well modelled as

$$\phi_k(t) = f_k^0 t + \frac{r_k}{2} t^2 \tag{2}$$

yielding an instantaneous frequency function

$$\phi_k'(t) = f_k^0 + r_k t \tag{3}$$

where f_k^0 and r_k denotes the starting frequency and the frequency rate, i.e., the slope of the chirp, respectively, for chirp component k. The considered problem consists of primerily estimating K, f_k^0, and r_k, but in the process also the phase shifts $\varphi_k \triangleq \angle a_k$ and the amplitudes $\alpha_k \triangleq |a_k|$. One may express (1) in a more compact manner, using matrix notation

$$\mathbf{y} = \boldsymbol{D}\mathbf{a} + \mathbf{e} \tag{4}$$

where

$$\mathbf{y} = \left[\begin{array}{ccc} y(t_0) & \ldots & y(t_{N-1}) \end{array} \right]^T \tag{5}$$

$$\mathbf{a} = \left[\begin{array}{ccc} a_1 & \ldots & a_K \end{array} \right]^T \tag{6}$$

$$\boldsymbol{D} = \left[\begin{array}{ccc} \mathbf{d}_1 & \ldots & \mathbf{d}_K \end{array} \right] \tag{7}$$

$$\mathbf{d}_k = \left[\begin{array}{cc} 1 \ldots & e^{i2\pi\phi_k(t_{N-1})} \end{array} \right]^T \tag{8}$$

III. Algorithm

The proposed algorithm initially estimates the frequency starting point and the frequency rate. Since the number of components in the signal is unknown, one may create the dictionary containing $P \gg K$ candidate chirps, thus approximating (1) with

$$\mathbf{y} \approx \boldsymbol{D}\mathbf{a} \tag{9}$$

where, $\boldsymbol{D}$ is an $N \times P$ dictionary matrix, and $\mathbf{a}$ the corresponding amplitudes. Solving (9) using ordinary least squares, if feasible, would yield a non-sparse solution, i.e., most of the indexes of $\mathbf{a}$ would be non-zero. By instead forcing the solution to have only a few non-zero elements, one may instead find a solution which indicates which dictionary elements that are most dominating in the signal. This may be achieved by solving

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & ||\mathbf{y} - \boldsymbol{D}\mathbf{x}||_2^2 \\ \text{subject to} \quad & ||\mathbf{x}||_0 < \rho \end{aligned} \tag{10}$$

where $||\mathbf{z}||_0$ denotes the number of non-zero elements in $\mathbf{z}$. Even though the cost function in (10) is convex, $||\mathbf{x}||_0$ is not, and the optimization problem has been shown to be NP-hard. To be able to solve (10), one has to relax the constraint, which is commonly done by considering the ℓ_1-norm instead of $||\mathbf{x}||_0$. Given this relaxation, the problem transforms to solving a Lasso optimization problem

$$\underset{\mathbf{x}}{\text{minimize}} \quad ||\mathbf{y} - \boldsymbol{D}\mathbf{x}||_2^2 + \lambda ||\mathbf{x}||_1 \tag{11}$$

where λ is a tuning parameter, controlling the sparsity of the solution. The solution obtained from (11) will depend on the grid structure of $\boldsymbol{D}$, i.e., if the true components are not contained in the dictionary, the components that are the closest to the true chirps will be activated, thus the corresponding indexes in $\mathbf{x}$ will be non-zero. Therefore, the solution attained from (11) will be biased in accordance to the chosen grid structure of $\boldsymbol{D}$. To avoid this bias, we introduce a nonlinear least squares (NLS) search to further increase the resolution. Let the residual from (11) be

$$\mathbf{r} = \mathbf{y} - \boldsymbol{D}\mathbf{x} \tag{12}$$

Each chirp is then iteratively updated by first adding one chirp to the residual formed in (12), conducting a NLS search for the parameter estimates, removing the found chirp using (12), and then adding the next chirp. When all chirps have been updated, one may continue updating the residual with the newly refined estimates. After a few iterations, the final estimates are found.

A. Reweighted LASSO

In the above algorithm, the user has to select a value for λ. A too high value of λ will suppress chirps with small amplitudes, whereas a too small value of λ will not manage to suppress the noise, which will lead to an overestimation of K. The value of λ is commonly chosen through cross-validation [14] or, by some data dependent heuristics. In this paper, we propose a simple heuristic for choosing λ. One may consider λ as a threshold for the energy for which signals that may be in

Algorithm 1 Chirp algorithm

```
Initiate w_i = 1, for i = 1, ..., P
for ℓ = 1, ... do
    Solve (15)
    Update (14)
end for
Compute (12)
for j = 1,... do
    for k = 1, ..., K̂ do
        z = r + D(·, I_K̂(k))^(j) x(I_K̂(k))^(j)
        Use NLS to update D(·, I_K̂(k))^(j) and x(I_K̂(k))^(j)
    end for
end for
```

the solution of (11). With this insight, one may then find it reasonable that a suitable choice of λ should somehow include the total energy of the signal, e.g. measured in the ℓ_2-norm $||\mathbf{y}||_2^2$. Herein, we suggest the following choice

$$\lambda = \frac{||\mathbf{y}||_2^2}{2N} \tag{13}$$

This suggests that a chirp is allowed in the solution of (11) if it contains more energy than half of the mean energy in the entire signal. This has empirically been shown to provide a reliable choice of λ, at least when N is around $15 - 30$. To further increase the robustness to the choice of λ, we propose a re-weighted Lasso approach, based on the techniquew introduced in [15]. In the re-weighted approach, one solves the minimization iteratively where, at every iteration, a weight matrix $\boldsymbol{W}$ with weights $w_1, \ldots, w_P$ on the diagonal and zeros elsewhere, is used. The diagonal elements in $\boldsymbol{W}$ are updated as

$$w_p^\ell = \frac{1}{|x_p^{\ell-1}| + \epsilon}, \quad p = 1, \ldots, P \tag{14}$$

where the superfix ℓ denotes the iteration, and $\epsilon > 0$ a tuning parameter, which prevents the solution from diverging. At each iteration the following minimization is solved

$$\underset{\mathbf{x}}{\text{minimize}} \quad ||\mathbf{y} - \boldsymbol{D}\mathbf{x}||_2^2 + \lambda ||\boldsymbol{W}^\ell \mathbf{x}||_1 \tag{15}$$

The algorithm is outlined in Algorithm 1 where $\boldsymbol{D}(\cdot, k)$ and $\mathbf{x}(k)$ denote the k:th column and the k:th index of the matrix $\boldsymbol{D}$ and the vector $\mathbf{x}$, respectively. Furthermore, let the number of non-zero elements in the solution from (11) be $\hat{K}$ and let the corresponding indexes in $\mathbf{x}$ be the index set $\boldsymbol{I}_{\hat{K}}$. It should be noted that the reweighted Lasso approach introduces another tuning parameter ϵ. In this paper, we have set ϵ to be

$$\epsilon = \frac{N}{||\mathbf{y}||_2^2} \tag{16}$$

which is in accordance to the discussion in [15], and has been empirically shown to be reliable for N around 15-30.

IV. Numerical Results

In this section, we set out to test the algorithm on real and simulated data, as well as comparing it to other chirp

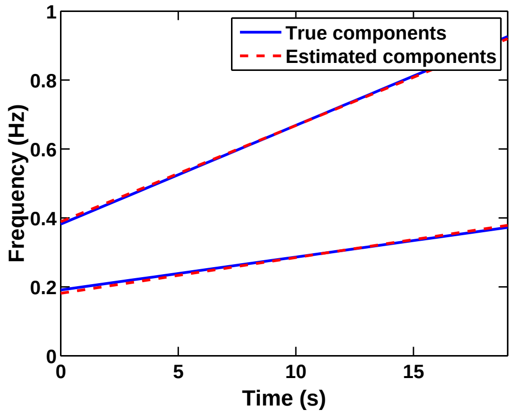

Fig. 1. The figure shows the ture (solid) and the estimated (dashed) IF.

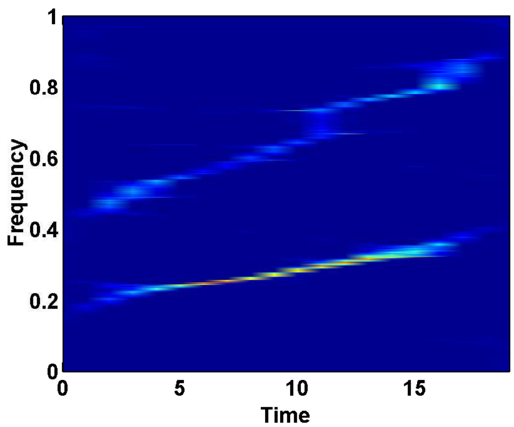

Fig. 2. The figure shows the estimated time-frequency distribution of chirp signals using reassigned spectrogram. The ground truth is shown in Figure 1.

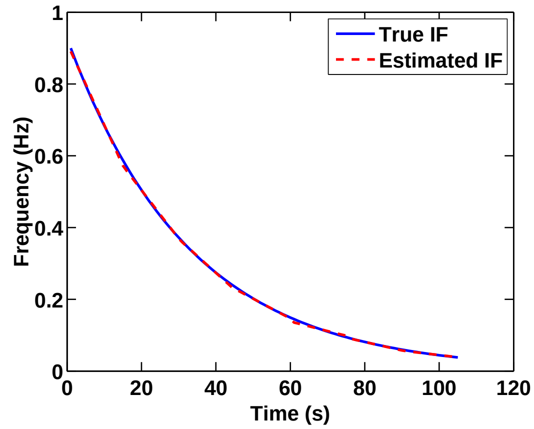

Fig. 3. The estimated chirp in dashed lines compared to the true chirp.

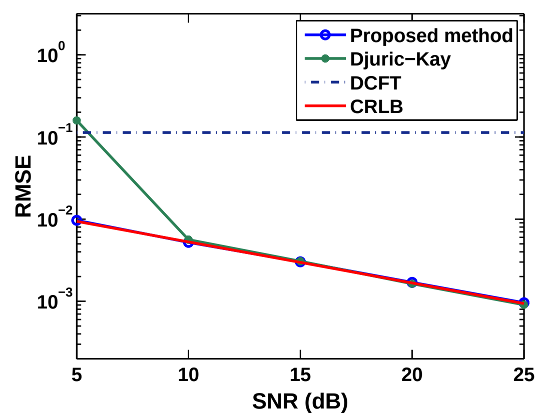

Fig. 4. Performance of the proposed method as compared with the CRLB, the Djuric-Kay method and the DCFT method, when estimating the starting frequency of one single chirp

algorithms and the CRLB. In this paper, we define the signal to noise ratio (SNR) as

$$\mathrm{SNR} = 10\log_{10}\left(\frac{P}{\sigma^2}\right) \tag{17}$$

where P is the power of the signal and σ^2 is the variance of the Gaussian noise. In the first example, we simulate a uniformly sampled signal with length $N = 20$ containing two chirp components, as depicted in Figure 1, which were corrupted with a white Gaussian noise with SNR=10 dB. The resulting estimates from the proposed method and the reassigned spectrogram [16] are shown in Figures 1 and 2, respectively. The reassigned spectrogram shows the two chirp components, but the estimates are blurred, as well exhibiting jumps in the frequencies. On the other hand, the proposed method manages to find the chirp components with out any such ambiguities.

We continue by showing how the proposed method may be used in tracking a non-linear chirp. In this example, we simulated an exponential chirp component defined as

$$\phi(t) = \left(\frac{r^t - 1}{\log(r)}\right) f_0, \tag{18}$$

where f_0 and r are parameters determining the starting frequency and the exponential rate of change. The signal, containing $N = 105$ samples, was divided in 7 equally sized sections, such that each segment may be reasonably well modelled as a linear chirp.The signal was corrupted by white Gaussian noise with SNR 20 dB. The proposed algorithm was applied on each section and the resulting chirp estimate is depicted in Figure 3, where it is clearly shown how the proposed method manages to estimate the evolving paramaters of the non-linear chirp.

Next, we examine the estimation performance of the proposed method as a function of SNR, as compare to the CRLB. The simulated signal contains one single chirp component with starting frequency $f^0 = 0.6/\pi$ and frequency rate $r = 0.03/\pi$, amplitude $\alpha = 1$, and a uniformly distributed random phase $\varphi \in U(-\frac{1}{2}, \frac{1}{2})$, which was randomized for each simulation. The sample length is set to $N = 20$. The parameters are estimated using the proposed method, where λ and ϵ were selected as suggested in Section III-A, the discrete chirp fourier transform algorithm (DCFT) [8], and the algorithm presented by Djuric and Kay in [2]. For each SNR, 1000 Monte-Carlo simulations were conducted and the resulting

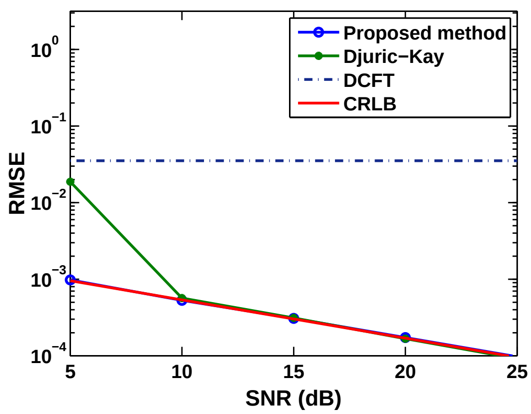

Fig. 5. Performance of the proposed method as compared with the CRLB, the Djuric-Kay method and the DCFT method, when estimating the frequency rate of one single chirp.

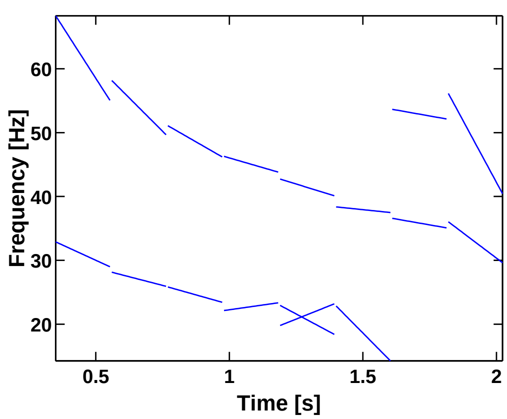

Fig. 6. Showing the time-frequency estimation of the bat signal using the proposed algorithm.

Root Mean Squared Error (RMSE) are shown in Figures 4 and 5. The proposed method only estimated the wrong number of components in 1 out of 1000 at the SNR=5 dB level. For the other SNR levels, the order estimations were without any errors. To assert a fair comparison, the simulations where the proposed method estimated the wrong model order were removed from all methods, and are thus not included in the RMSE graphs. As is clear from Figures 4 and 5, that the proposed method, with out any prior knowledge about the number of chirps, manages to attain the CRLB, ass well as outperforming the Djuric-Kay algorithm even though the Djuric-Kay algorithm has been allowed oracle model order information. Furthermore, the DCFT algorithm is stuck to the initial grid, which seggests why it does not manage to improve when the SNR increases. We proceed by examining the performance on multicomponent chirp signals. Since the competing methods did not manage to perform on multicomponent data, we only show the results for the proposed method as compared to the CRLB. Figures 7 and 8 depict the results of a Monte-Carlo simulation where 1000 simulations were used for each SNR-level. The starting frequency of the chirps were $f_1^0 = 0.6/\pi$ and $f_2^0 = 1.2/\pi$, and the slope rates were

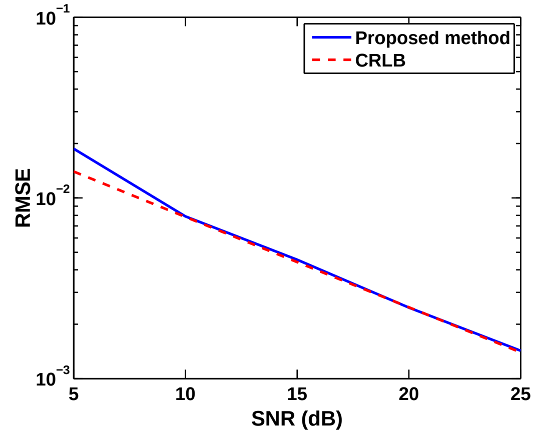

Fig. 7. Performance of the proposed method when estimating the starting frequencies of two non-crossing linear chirps, as compared to the Cramér Rao Lower Bound.

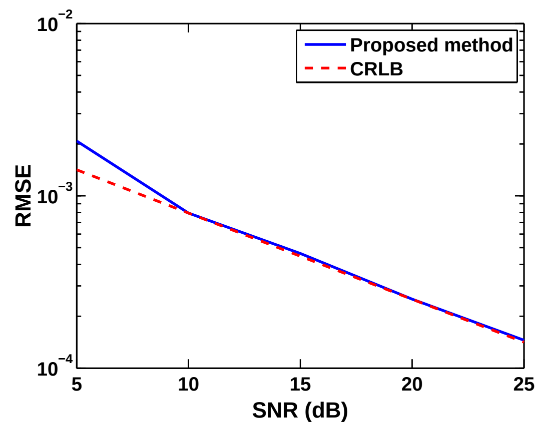

Fig. 8. Performance of the proposed method when estimating the frequency rates of two non-crossing linear chirps, as compared to the Cramér Rao Lower Bound.

$r_1 = 0.03/\pi$ and $r_2 = 0.09/\pi$. The amplitudes were set to one and the phase were drawn as $\varphi \in U[-1/2, 1/2]$ at each simulation. Once again, λ and ϵ were chosen as described in Section III-A. As one may note from Figures 7 and 8, the proposed method follows the CRLB for SNR levels greater or equal to 10. It should also be noted that the proposed method only estimated the wrong model order 26 times out of the 1000 simulations, and this only happened for the SNR=5 dB case. Again, these simulations were removed from both the proposed methods RMSE.

We continue by showing the performance on real data, containing sounds from bats [17]. Audio sources, such as music and speech, are often modeled as harmonics. Thus, it should be expected that the sound from a bat should contain a harmonic structure. Figure 6 shows the estimated chirps for a bat signal containing 240 samples divided into 8 equally long sections. The tuning parameters were selected as suggested in Section III-A. Figure 6 shows the expected harmonic structure, where two over-tones are visible.

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References

[1] P. Flandrin, "Time-Frequency and Chirps," in *Wavelet Applications VIII*, 2001.

[2] P. Djurić and S. Kay, "Parameter Estimation of Chirp Signals," *IEEE Transactions on Acoustics Speech and Signal Processing*, vol. 38, pp. 2118–2126, 1990.

[3] O. Besson, M. Ghogho, and A. Swami, "Parameter Estimation for Random Amplitude Chirp Signals," *IEEE Transactions on Signal Processing*, vol. 47, no. 12, pp. 3208–3219, 1999.

[4] I. Rusnak and L. Peled-Eitan, "New Approach to Estimation of Chirp Signal with Unknown Parameters," in *IEEE International Conference on Microwaves, Communications, Antennas and Electronics Systems (COMCAS)*, 2013.

[5] J. Gal, A. Campeanu, and I. Nafornita, "The Estimation of Chirp Signals Parameters by an Extended Kalman Filtering Algorithm," in *10th International Symposium on Signals, Circuits and Systems (ISSCS)*, 2011.

[6] B. Völcker and B. Ottersten, "Chirp Parameter Estimation from a Sample Covariance Matrix," *IEEE Transactions on Signal Processing*, vol. 49, no. 3, pp. 603–612, March 2001.

[7] R. M. Liang and K. S. Arun, "Parameter Estimation for Superimposed Chirp Signals," in *5th IEEE Int. Conf. on Acoustics, Speech, and Signal Processing*, 1992.

[8] X. Xia, "Discrete Chirp-Fourier Transform and Its Application to Chirp Rate Estimation," *IEEE Transactions on Signal Processing*, vol. 48, pp. 3122–3133, 2000.

[9] A. Brodzik, "On the Fourier Transform of Finite Chirps," *IEEE Signal Processing Letters*, vol. 13, no. 9, pp. 541–544, September 2006.

[10] D. Peacock and B. Santhanam, "Multicomponent Subspace Chirp Parameter Estimation Using Discrete Fractional Fourier Analysis," in *Proceedings of the IASTED International Conference Signal and Image Processing*, 2011.

[11] J. Xiao and P. Flandrin, "Multitaper Time-Frequency Reassignment for Nonstationary Spectrum Estimation and Chirp Enhancement," *IEEE Transactions on Signal Processing*, vol. 55, no. 6, pp. 2851–2860, June 2007.

[12] J. Guo, H. Zou, X. Yang, and G. Liu, "Parameter Estimation of Multi-component Chirp Signals via Sparse Representation," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 47, no. 3, pp. 2261–2268, July 2011.

[13] B. Wang and J. Huang, "Instantaneous Frequency Estimation of Multi-Component Chirp Signals in Noisy Enviroments," *Journal of Marine Science and Applications*, vol. 6, no. 4, pp. 13–17, December 2007.

[14] P. G. Bühlmann and S. van de Geer, *Statistics for High-Dimensional Data*, Springer Series in Statistics. Springer, 2011.

[15] E. J. Candes, M. B. Wakin, and S. Boyd, "Enhancing Sparsity by Reweighted l_1 Minimization," *Journal of Fourier Analysis and Applications*, vol. 14, no. 5, pp. 877–905, Dec. 2008.

[16] F. Auger and P. Flandrin, "Improving the Readability of Time-Frequency and Time-Scale Representations by the Reassignment Method," *IEEE Transactions on Signal Processing*, vol. 43, pp. 1068–1089, 1995.

[17] R. G. Baraniuk and D. L. Jones, "A Signal Dependent Time Frequency Representation: Optimal Kernel Design," *IEEE Transactions on Signal Processing*, vol. 41, pp. 1589–1602, April 1993.