Prices and Pareto Optima

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Abstract. We provide necessary conditions for Pareto optimum in economies where tastes or technologies may be nonconvex, nonsmooth, and affected by externalities. Firms can pursue own objectives, much like the consumers. Infinite-dimensional commodity spaces are accommodated. Public goods and material balances are accounted for as special instances of linear restrictions.

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1. Introduction

Economic theory, so preoccupied with prices and efficient allocations, naturally reserves prime positions to price-taking (competitive) behavior and to Pareto optima. Some theoretical reassurance - or comfort - derives, in this regard, from the *first welfare theorem*, saying that competitive equilibria, involving locally non-satiated consumers, are generally Pareto efficient [31], [35]. Additional reassurance, still theoretical though, comes with the *second welfare theorem,* telling that every Pareto optimum - under important convexity assumptions, and modulo appropriate redistribution of resources - can be sustained as a competitive equilibrium [1], [11], [12].

These results may have inspired some faith in market organization of economies (and possibly rendered pursuit of own profit or utility more respectable). The justification of such faith hinges, however, upon quite stringent assumptions, including convexity.[1] Given *nonconvex* tastes or technologies, existence of competitive equilibrium cannot generally be guaranteed. Thus, the link between price-taking equilibria and Pareto optima is in general only a one-way passage.[2] Externalities may also overthrow the two welfare theorems.

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[1] The assumptions run against various observable phenomena such as indivisibilities, scale economies, and risk-loving behavior. So, there are ample reasons to relax them [8], [40], [47].

[2] Broadly speaking, this corresponds to the situation in mathematical programming. There optimal solutions to sufficiently smooth problems must *locally* be supported by "prices" (dual vectors or Lagrange multipliers). Conversely, granted convexity the existence of a dual optimal solution suffices for *global* support - and for decomposition (decentralization, relaxation) of the program.

These facts do not preclude though that Pareto optima can be described by prices (and related to so-called pseudo- or quasi-equilibria [4], [21], [35]). To indicate how, consider the problem to maximize social welfare $W(u)$ subject to $u = [u_k(z_k)]$ and $\sum_{k \in K} z_k = e$. Consumer $k \in K$ derives utility $u_k(z_k) \in \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ from commodity bundle z_k; *total endowment* is e; and $W : \bar{\mathbb{R}}^K \to \bar{\mathbb{R}}$ is increasing. If $z = (z_k)$ is optimal, provided data be smooth and finite-valued near z, a "price" p must satisfy

$$p = \frac{\partial}{\partial u_k} W(u) \nabla u_k(z_k) \text{ for all } k \in K.$$

Reasonably, $\lambda_k := 1/\frac{\partial}{\partial u_k} W(u) > 0$, whence $\nabla u_k(z_k) = \lambda_k p$ for all $k \in K$. This shows that when substituting one good for another, in optimum, all agents apply the same exchange rate, namely the corresponding price ratio. Also, if consumer k were to maximixe a concave $u_k(\cdot)$ within "realized budget" $b_k := p \cdot z_k$, his *marginal utility of income* would equal λ_k. So, upon endowing k with budget b_k and replacing the welfare criterion with $W(u) := \sum_{k \in K} u_k/\lambda_k$, the optimal solution $z = (z_k)$ becomes a *Walrasian equilibrium* [39].

It's also useful to view these matters in terms of production. To that end consider the problem

$$\sup \left\{ \sum_{k \in K} \pi_k(z_k) : \sum_{k \in K} z_k = e \right\} =: \Pi(e),$$

where $\pi_k(z_k) \in \bar{\mathbb{R}}$ now is construed as the profit of division k in concern K, being allocated the part z_k of the total endowment e. (As just argued, $\pi_k = u_k/\lambda_k$ serves as one instance.) For any optimal allocation $z = (z_k)$ there should, under differentiability, exist a "*price*"

$$p = \nabla \pi_k(z_k) = \nabla \Pi(e) \text{ for all } k.$$

Indeed, supposing smoothness, as long as $k \mapsto \nabla \pi_k(z_k)$ isn't constant, the integrated concern should shift production factors away from sites where marginal productivity is inferior. Further, p had better report the *shadow price* $\nabla \Pi(e)$ imputed to scarce resources. Again marginal substitution rates should equal corresponding exchange rates, and be construed as price ratios.[3]

But *nonsmooth* or *non-convex* functions do complicate these matters and require more versatile tools.[4] Then, to define the said rates we shall proceed as follows. At a Pareto optimum assemble all better-than-actual choices into a product set, residing in the space of consumption-production profiles. That product set must be separable from the set of feasible allocations. That is, minor displacements should suffice to pull the two objects locally apart. Therefore, according to a modern extremal principle, their normal cones must be positively dependent at the Pareto optimum. Such

[3] Otherwise, at least two divisions would benefit by bilateral exchange. This observation points to *cooperative production games* and *core solutions* thereof; see [14], [15], [16].

[4] While Samuelson (1947), Debreu (1954), Negishi (1960) already accommodated non-smooth data, studies of non-convex items came later; see [4], [5], [8], [28], [29].

dependence furnishes a nonzero vector of marginal prices which capture the old idea of equal substitution rates.

A more "conventional" approach, not pursued here, first replaces preferences and production possibilities, at Pareto optimum, with conical approximations. The convex hull of those cones cannot span the entire commodity space (or a dense subset thereof). Consequently, the polar cones must intersect away from the origin. Any nonzero element in that intersection describes Pareto optimum by a necessary condition (sufficient under appropriate convexity).

Such an approach was first taken by Guesnerie [19] with subsequent extensions by Kahn, Vohra [28], [29], Yun [46] and others. The achievements of these studies were of two main sorts. *Mathematically*, it meant progress to dispense with smoothness - and to emphasize the importance of nonconvex separation theorems [6], [17]. That enterprise, and its sharpening of tools, reflects modern developments of differential analysis in which local approximation is done with cones instead of hyperplanes. Not totally inviting, in this regard, was the need to be or become familiar with a plethora of applicable cones.[5] *Economically*, the said studies did well in confirming received neoclassical results, saying that marginal rates of substitution should be equal across consumers and firms. And in the same vein, present public goods, it was shown that the valuation of such items complied with Samuelson's additive rule [18], [45].

This paper, also concerned with characterization of Pareto optima, has several purposes and novelties. We shall

- allow externalities;[6]
- avoid approximations in commodity space;[7]
- spare the reader much knowledge about cones;[8]
- use a new concept of $\partial-$compactness;
- divorce arguments about separation from those related to regularity of sets;
- treat consumers and firms symmetrically;
- incorporate material balances and public goods by means of linear constraints.

To achieve these aims the paper is planned as follows. Section 2 describes a general setting and recalls notions of Pareto optima. Section 3 specifies the economy. Section 4, being the heart of the paper, provides new results on Pareto optima. Main mathematical arguments are collected in several lemmas of independent interest.

2. A General Setting

There is a nonempty finite set K, consisting of economic agents engaged in consumption, production, and exchange of perfectly divisible goods. Bundles of these goods

[5] Useful cones were introduced by Bouligand, Clarke, Dubovickii-Miljutin, Ioffe, Mordukhovich, Rockafellar, and others [9], [22], [36], [42]. See [2] for a nice overview.

[6] Those could occur in preferences or production. We thus accommodate consumers displaying say, altruism or envy - and firms affected by such phenomena as congestion or pollution.

[7] We shall not replace original data, represented as sets, locally by cone approximations.

[8] In fact, those objects will here be taken as primitives, exemplified by well studied prototypes.

all belong to a so-called commodity space E, here taken to be real Banach with norm $\|\cdot\|$.[9] We shall refer to vectors $z \in Z := E^K$ as *profiles.* Elements in $E^{K \setminus k}$ will be denoted by z_{-k}. On the product space $Z = E^K$ we use the norm $\|z\| := \sum_{k \in K} \|z_k\|$.

• *Each agent* $k \in K$ has a *preference relation* $\mathcal{P}_k \subset Z \times E$. When the k-th component of $z \in Z$ is z_k, we interpret $z_k' \in \mathcal{P}_k(z) \subset E$ to mean that k *strictly prefers* z_k' to z_k (whence $z_k' \neq z_k$) in the circumstance z_{-k} imposed upon him by others.[10]

Agent k also has a *feasibility correspondence* $z \longrightarrow Z_k(z)$ to care about. That correspondence maps profiles z into subsets $Z_k(z)$ of E, and it constrains his feasible choices. With apologies for slight abuse of notation, we have $Z_k(z) = Z_k(z_{-k})$. The interpretation is that, given the exogenously imposed scenario z_{-k}, agent k can only contemplate choices $z_k \in Z_k(z_{-k})$.[11]

It is convenient to assemble preference and feasibility into *one* object. So, we define $P_k(z) := \mathcal{P}_k(z) \cap Z_k(z)$. *No* hypothesis is, for now, made concerning P_k, be it in terms of *ordering* (monotonicity, non-satiation, desirability, completeness, transitivity...), *topology* (openness, connectedness, continuity...), or *curvature* (convexity, strict convexity...). But we shall require, of course, that the *domain* $domP_k := \{z : P_k(z) \neq \varnothing\}$ be nonempty. And, quite naturally, letting *cl* denote closure, we posit the reflexivity condition that $z_k \in clP_k(z)$ whenever the k-th component of $z \in domP_k$ equals z_k. Examples of such preferences are given in infinite dimentional spaces by Bellaassali and Jourani [3].

• There are *material balances,* and restrictions on the consumption of various goods, that cannot be violated. These constraints are accounted for by requiring

$$Lz = 0. \tag{1}$$

L is a continuous linear operator with closed range which maps Z into another Banach space $\mathcal{E}$. We shall not elaborate on the origin or nature of L here, but return to this issue in the next section.

Definition 1. (Pareto optimality) *A profile z is called a feasible allocation, or simply an **allocation**, if $z_k \in Z_k(z)$ for all $k \in K$, and $Lz = 0$.*

*An allocation z is said to be **weakly Pareto optimal** if no other profile z' satisfying $Lz' = 0$ is such that $z_k' \in P_k(z)$ for all $k \in K$. It will be declared **strongly Pareto optimal** when no other profile z' has $Lz' = 0$ and $z_k' \in clP_k(z)$ for all $k \in K$*

[9] Even more general spaces could be - and have been - considered [12], [29], but we find the Banach setting largely sufficient and most tractable. Anyway, the reader not appreciating so much generality can construe E as finite-dimensional Euclidean.

[10] Some agent, say a firm, might come without objectives. We shall, however, without loss, equip each $k \in K$ with a preference $\mathcal{P}_k \subset Z \times E$. For example, the specification $\mathcal{P}_k(z) := E \setminus z_k$ could be one extreme and quite innocuous choice. This remedy provides additional generality, and it renders the role of consumers and firms more symmetric.

[11] Thus, in microeconomic jargon a *consumer* (*producer*) would have a consumption (production) correspondence.

with $z'_k \in P_k(z)$ *for at least one* k.

Remarks: There are, of course, local notions of Pareto optima. All subsequent statements about optimality also hold in the local sense.

Arguments below indicate that *nonlinear* coupling constraints could well be considered. But the classical legacy of welfare economics accommodates only linear or affine instances (1).

Suppose $Lz' = 0$ and $z'_k \in clP_k(z), \forall k$, with $z'_k \in P_k(z)$ for at least one k. If, on each such occasion, there exists a modification $\Delta z' \in KerL$ such that $z'_k + \Delta z'_k \in P_k(z), \forall k$, then any weak Pareto optimum must be strong.

Our general setting invites questions about existence of Pareto optima. Concerning that issue, after defining $z' \in clP(z) \Leftrightarrow z'_k \in clP_k(z), \forall k$, one would search for Pareto optima among maximal points of clP on $KerL$. If clP is transitive on $KerL$ with at least one compact value, then a maximal point exists. Also, present a social welfare function W, of individualistic Bergson-Samuelson type, which maps utility profiles $u(z) := [u_k(z)] \in \bar{\mathbb{R}}^K$ monotonically into $\bar{\mathbb{R}}$, then Pareto optimality relates to the exercise of maximizing $z \mapsto W(u(z))$. We shall henceforth take a Pareto optimum as given and rather inquire about its nature. Before that inquiry we spell the above construction better out in an economic context.

3. THE ECONOMY

Suppose here that the agent set is the disjoint union $I \cup J =: K$ of two nonempty finite sets I and J, consisting of consumers (individuals) $i \in I$ and firms (job-shops) $j \in J$. We refer to pairs $(x, y) \in E^I \times E^J$ as *consumption-production profiles,* $x_i \in E$ being the consumption of individual i and $y_j \in E$ the production of job-shop j.

• *Goods* might - and typically do - come in two categories, namely *private* and *public.* Consumption of the first type is exclusive, i.e., what is taken by any one individual becomes unavailable for all others. By contrast, a good is public if its consumption is constant across all individuals. To reflect that dichotomy the space should be defined as a product $E := E^{priv} \times E^{pub}$ of two underlying commodity spaces. The effect is, of course, that any profile (x, y) must have x_i^{pub} constant across I. Mathematically, this amounts to a *linear restriction* on the set of admissible profiles, and we shall prefer to treat it as such. Before formalizing this feature we first consider joint constraints related to resource availability.

• There are *material balances* that cannot be violated. Typically, society at large has available an aggregate, clearly specified "resource endowment" $e \in E$. It then faces a "budget restriction"

$$\left(\sum_{i \in I} x_i^{priv}, \xi \right) \leq \sum_{j \in J} y_j + e. \tag{2}$$

Here ξ is the common consumption of public goods, if any, and $\leq$ denotes a vector ordering on E defined, as usual, via a nonempty, closed, convex, pointed cone $E_+ \subset E$ by $e' \leq e \Longleftrightarrow e - e' \in E_+$. Condition (2) captures that total consumption cannot exceed what production and endowment jointly have furnished. With no loss of generality, but at considerable notational gain, we shall take the order cone E_+ to be degenerate, i.e., we let $E_+ = \{0\}$ and proceed with equality in (2). Indeed, if necessary, introduce a fictitious firm having the constant production correspondence $z \longrightarrow Z_j(z) := -E_+$ that reflects disposal (pure waste) of surplus if any. Suppose this firm has already been included with a suitable preference $\mathcal{P}_j$.

For still more simplification, again purely notational, we shall add the overall endowment e to *one* producer. That is, single out one firm j and ascribe it a new correspondence $z \longrightarrow P_j(z) + \{e\}$. Alternatively, we could add yet another firm having the constant, singleton correspondence $z \longrightarrow clP_j(z) = \{e\}$ if $z_j = e$, and $clP_j(z) = \varnothing$ otherwise. Once again, suppose one (or the other) of those remedies has already been implemented.

The advantage of these modifications, once made, will be to have $e = 0$ and all coupling constraints linear. To wit, define $L : Z \to E^{priv} \times (E^{pub})^I$ by

$$(Lz)^{priv} := \sum_{i \in I} x_i^{priv} - \sum_{j \in J} y_j^{priv}, \qquad (Lz)_i^{pub} := x_i^{pub} - \sum_{j \in J} y_j^{pub} \text{ for all } i, \tag{3}$$

and impose (1). This transcription motivates us to assume henceforth that the economy is subject to the coupling constraint (1) where the continuous linear operator L maps Z into another Banach space $\mathcal{E}$, for example $E^{priv} \times (E^{pub})^I$. It appears natural to posit that L be surjective. In general we shall contend, however, with L having closed range.

4. MARGINAL PRICES

Following the classical lead of Hicks [20], Lange [33], and Samuelson [43] we seek a differential description of Pareto efficiency. But data are here nonsmooth. Therefore we must accommodate *generalized derivatives* called *subdifferentials.* These objects are introduced next, together with *distance functions* and *normal cones.* The approach is axiomatic.

Definition 2. (Axioms for subdifferentials) *A **subdifferential** ∂ on a Banach space E operates on functions $f : E \to \mathbb{R} \cup \{+\infty\}$ and vectors $v \in E$. For any $f, g : E \to \mathbb{R} \cup \{+\infty\}$ the following must hold at each $v \in E$:*

- *$\partial f(v)$ is contained in the topological dual E^*, and $\partial f(v) = \varnothing$ when $f(v) = +\infty$;*
- *$\partial f(v)$ coincides with the subdifferential of convex analysis when f is convex;*
- *$0 \in \partial f(v)$ when f is locally minimal at v;*
- *$\partial f(v) = \partial g(v)$ when f and g coincide around v;*
- *$\partial(f + g)(v) \subset \partial f(v) + \partial g(v)$ when f is lower semicontinuous (lsc for short) and g is locally Lipschitz near v;*

• *if* $v = (v_1, v_2) \in E = E_1 \times E_2$, *and* $f(v) = f_1(v_1) + f_2(v_2)$ *with* f_1 *locally Lipschitz and* f_2 *lsc near* v, *then* $\partial f(v) \subset \partial f_1(v_1) \times \partial f_2(v_2)$.

These properties are satisfied for a wide variety of subdifferentials [9], [22]. Our interest is here mainly with the particular case of *distance functions.* The distance from $e \in E$ to a nonempty set $C \subset E$ is given by

$$d(e, C) := \inf \|e - C\| = \inf_{c \in C} \|e - c\|.$$

Define the *normal cone to* C at $c \in clC$ to be $N(c, C) := \mathbb{R}_+ \partial d(c, C)$. We suppose that all subdifferentials considered subsequently are such that the correspondence $c \longrightarrow \partial d(c, C)$ will be norm×weak* closed at any $c \in clC$. We say that C is *locally* ∂*-compact* at $c \in clC$ if there exists a locally compact cone $\mathcal{K} \subset E^*$ (in the $\sigma(E^*, E)$ topology) and a vicinity V of c such that $\partial d(c', C) \subset \mathcal{K}$ for all $c' \in C \cap V$.[12] Examples include the following important case [26]:

Let $A, C \subset E$ *be such that* A *contains* 0 *and is closed convex;* C *is compact; and* $0 \in int(A + C)$. *Then* A *is locally* ∂*-compact at each of its points.*

It is time to focus on a Pareto optimum z - be it weak or strong. The feasibility of z tells that $0 \in L[\Pi_{k \in K} clP_k(z)]$. But, by weak optimality $0 \notin L[\Pi_{k \in K} P_k(z)]$. So, after slight translation the set $L[\Pi_{k \in K} clP_k(z)]$ no longer seems to comprise 0. We shall describe such a situation by saying that 0 and $L[\Pi_{k \in K} clP_k(z)]$ are *locally extremal.* That notion is crucial and properly defined next.

Definition 3. (Extreme systems, Kruger and Mordukhovich [32]) *A finite family* $\{C_k\}$ *of sets in a Banach space is declared* ***locally extremal*** *at a common vector* $v \in \cap C_k$ *if there exists a vicinity* V *of* v *and sequences* $v_k^n \to 0$ *in the ambient space such that* $\cap_k (C_k - v_k^n) \cap V = \varnothing$ *for all* n. *If* V *equals the entire space, we shall simply speak about extremal points of the family* $\{C_k\}$. *We call* $\{C_k\}$ *a (locally)* ***extreme system*** *if there exists at least one point at which the sets are (locally) extremal.*

Remarks: If $\cap_k intC_k \neq \emptyset$, then $\{C_k\}$ cannot be an extreme system. Two smooth manifolds C_1, C_2 cannot be extreme if they intersect transversally.

In Definition 3 one may take at least one sequence $n \mapsto v_k^n \equiv 0$. One may also arrange the situation so as to comprise only two sets. In fact, $\{C_k\}$ is locally extremal at v iff $C := \Pi_k C_k$ and the diagonal D in the product space are locally extremal at $\vec{v} := (v, v, ...)$.

Suppose $\{C_k\}$ is locally extremal at v and that the vicinity in question is V. Then $\{C_k \cap V\}$ is extremal there, and the translated family $\{C_k - v\}$ will be locally extremal at 0. More generally, if $\{C_k\}$ is locally extremal at v and mapped continuously by an open affine L into another Banach space, then $\{LC_k\}$ becomes locally extremal at Lv. In the other direction, suppose $\{C_k \subset$ Banach space $\mathcal{V}\}$ is locally extremal at

[12] For more about ∂-compactness see [27].

some $v \in \operatorname{Im}\mathcal{L} \subseteq \mathcal{V}$. Here $\mathcal{L}$ is linear and maps another Banach space continuously into $\mathcal{V}$. If $v^n \to 0 \Rightarrow d(\mathcal{L}^{-1}v^n, 0) \to 0$, then $\{\mathcal{L}^{-1}C_k\}$ is locally extremal at any point in $\mathcal{L}^{-1}v$.

A family $\{C_k\}$ of cones will be locally extremal at 0 precisely when those cones can be *pulled apart* in the sense that $\cap_k(C_k - e_k) = \emptyset$ for suitable displacements e_k.

Any pair $\{0\}$, C will be extremal at $0 \in C$ iff there exists a sequence $v^n \to 0$ in the ambient space such that $v^n \notin C$ for all n.

At locally extremal points the intervening sets are somehow normal to each other. Lemma 1, adapted from [26], provides a precise statement of this phenomenon, explaining how an extremal system can be separated by a normal vector. Such vectors will shortly provide price descriptions of Pareto optima. For any given Banach space let $\mathbb{B}$ denote its closed unit ball centered at the origin. $\mathbb{B}^*$ is the corresponding object in the dual space.

Lemma 1. (Almost or full separation, nonconvex or convex sets)
(i) *Let two closed subsets A and B of some Banach space be locally extremal at a common vector v. Then there exist $a^n \to v$ in A, $b^n \to v$ in B, unit length vectors p^n in the the dual space, and positive numbers $\alpha_n, \beta_n \to 0$ such that*

$$(p^n, -p^n) \in \alpha_n\mathbb{B}^* \times \beta_n\mathbb{B}^* + (2 + \frac{1}{n})\partial d(a^n, A) \times \partial d(b^n, B). \tag{4}$$

If moreover, at least one set A or B is locally ∂-compact at v, then from (4) we obtain in the limit a nonzero

$$p \in \partial d(v, A) \cap -\partial d(v, B). \tag{5}$$

(ii) *More specially, for a continuous linear $L : E^K \to \mathcal{E}$ let the two closed sets $A = \ker L$ and $B = \Pi_{k\in K}B_k$, $B_k \subset E$, be locally extremal at a common vector v. Then there exist $a^n \to v$ in $\ker L$, $b^n \to v$ in B, and unit length vectors $e^{*n} \in \mathcal{E}^*$ such that*

$$L^*(e^{*n}, -e^{*n}) \in \alpha_n\mathbb{B}^* \times \beta_n\mathbb{B}^* + (2 + \frac{1}{n})\partial d(a^n, A) \times \partial d(b^n, B) \tag{6}$$

with $\alpha_n := 2\|a^n - v\| + \frac{1}{n}, \beta_n := 2\|b^n - v\| + \frac{1}{n}$.

We write $L^*e^* = (L_k^*e^*)_{k\in K}$ with the undertanding that each $L_k^*e^* \in E^*$. In this setting, if some $k \in K$ has B_k locally ∂-compact at v_k, and

$$inf\,\{\|L_k^*e^*\| : \|e^*\| = 1\} > 0, \tag{7}$$

then from (6) we obtain in the limit a nonzero $L^*e^* \in \partial d(v, A)$ such that

$$-L_k^*e^* \in \partial d(v_k, B_k) \textit{for all } k \tag{8}$$

(iii) *If* A, B *are nonempty convex disjoint, with* $cl(A-B)$ *a proper subset of the space, and* $0 \in int(A-B+C)$ *for some norm compact* C, *then*

$$\sup \langle p, A\rangle \leq \inf \langle p, B\rangle \quad \textit{for some nonzero } p. \tag{9}$$

Proof. It is expedient to consider (ii) first. Without loss of generality suppose $\{A, B\}$ is extremal at v. Then there exist a sequence $v^n \to 0$ such that $A \cap (B - v^n) = \emptyset$ for all n. To simplify notations we write $Z := E^K$ and $\mathbf{Z} := Z \times Z$. When $\mathbf{z} = (a, b) \in A \times B$ let

$$f_n(\mathbf{z}) := \|L(-a + b - v^n)\| + \|a - v\|^2 + \|b - v\|^2 .$$

Elsewhere on $\mathbf{Z}$ let $f_n = +\infty$. Pick $\mathbf{z}^n \in \mathbf{Z}$ such that $f_n(\mathbf{z}^n) \leq \inf f_n + 1/n^2$. Since f_n is lower semicontinuous, Ekeland's variational principle provides a point $\bar{\mathbf{z}}^n = (a^n, b^n) \in A \times B$ with distance $\leq 1/n$ from $\mathbf{z}^n$ such that $f_n(\bar{\mathbf{z}}^n) \leq f_n(\cdot) + \|\cdot - \bar{\mathbf{z}}^n\|/n$. It follows from $f_n(\mathbf{z}^n) \leq \|Lv^n\| + 1/n^2$ and $\|v^n\| \to 0$ that $\mathbf{z}^n \to (v, v)$. Since $\|\bar{\mathbf{z}}^n - \mathbf{z}^n\| \leq 1/n$ we also get $\bar{\mathbf{z}}^n = (a^n, b^n) \to (v, v)$. (In fact, $\|a^n - v\|^2 + \|b^n - v\|^2 \leq \|Lv^n\| + 2/n^2 - \|L(-a^n + b^n - v^n)\|$.)

Fix any $r \in (0, \frac{1}{2})$. For n large enough $\bar{\mathbf{z}}^n$ belongs to the closed ball centered at (v, v) with radius r. Moreover, $f_n(\cdot) + \|\cdot - \bar{\mathbf{z}}^n\|/n$ is then Lipschitz there with modulus ≤ 2. Thus, by Clarke's exact penalty technique the function $f_n(\cdot) + \|\cdot - \bar{\mathbf{z}}^n\|/n + (2 + \frac{1}{n})d(\cdot, A \times B)$ attains a global minimum at $\bar{\mathbf{z}}^n$. Consequently,

$$0 \in \partial \left[f_n(\cdot) + \|\cdot - \bar{\mathbf{z}}^n\|/n + (2 + \frac{1}{n})d(\cdot, A \times B) \right] (\bar{\mathbf{z}}^n). \tag{10}$$

The sets A, $B - v^n$ being disjoint, we must have $L(-a^n + b^n - v^n) \neq 0$ so that $\partial \|L(-\cdot + \cdot - v^n)\|(a^n, b^n) = L^* \{(-e^*, e^*) : e^* \in \mathcal{E}^*, \ \ \|e^*\| = 1\}$. Since $\partial \|\cdot - v)\|^2 (a^n) \subseteq 2\|a^n - v\| \mathbb{B}^*$ and $\partial \|\cdot - (a^n, b^n)\|(a^n, b^n) \subseteq \mathbb{B}^* \times \mathbb{B}^*$, inclusion (6) now follows from (10).

Pick a $k \in K$ for which the particular properties hold. Observe via (6) that for appropriate $e_k^{*n} \in E^*, \|e_k^{*n}\| \leq 1, b_k^{*n} \in \partial d(b_k^n, B_k)$ we have

$$-L_k^* e^{*n} = \beta_n e_k^{*n} + (2 + \frac{1}{n}) b_k^{*n}.$$

The sequence of pairs $(e^{*n}, b_k^{n*}), n = 1, 2, ...$ is bounded hence weak-star compact. Extract a weak-star convergent subnet having limit (e^*, b_k^*). We take it that such extraction has already been done. Suppose $b_k^* = 0$. Then, from the ∂-compactness of B_k at v_k we derive that $b_k^{n*} \to b_k^* = 0$ in norm (utilizing here an argument in [34]). This is a contradiction however, because $\lim \|b_k^{n*}\| = \lim \|L_k^* e^{n*}\|/2 > 0$ by (7). Thus $b_k^* = L_k^* e^*$ must be nonzero, and therefore so must e^* as well. Now (8) follows from the fact that $\partial d(v, \Pi_{k \in K} B_k) \subseteq \Pi_{k \in K} \partial d(v_k, B_k)$. This comletes the proof of (ii).

To demonstrate (i) simply take K to be a singleton, $L = Id$, and repeat the preceding arguments verbatim.

The final assertion (iii) is proven in [26]. For completeness we outline the verification. Set $\mathcal{A} := cl(A - B)$. If $0 \in bd\mathcal{A}$, then $\mathcal{A}$ is locally ∂-compact at 0, and $\{\mathcal{A}, \mathcal{B} := \{0\}\}$ becomes locally extremal at 0. Consequently, (i) applies, and (5) yields (9). If, on the other hand $0 \notin bd\mathcal{A}$, we can use the ordinary Hahn-Banach theorem to get separation, in fact, strict separation. □

Remarks: When $A = \Pi_{k\in K} A_k, A_k \subset E, v \in \cap A_k$, and $B = \left\{\vec{e} := (e, e, ..) \in E^K : e \in E\right\}$, we get $\partial d(\vec{v}, B) = \{(e_k^*) : \sum_k e_k^* = 0\}$. Inclusion (8) then yields a nontrivial sum $\sum_k e_k^* = 0$ such that $-e_k^* \in \partial d(v, A_k), \forall k$. This is what Mordukhovich calls *the extremal principle.* That principle is a most efficient tool for deriving necessary optimality conditions in programming and control; see [38] and references therein.

Lemma 1 stands apart from classical separation results - and from theorems of alternatives - in that convexity plays no dominant role. It entails at least *two* important advantages. *First*, no conical approximation or directional derivative was required in the primal (commodity) space. Instead, objects and elements in the dual space immediately come to the fore. This feature fits economic analysis focused on prices. *Second*, as brought out by Mordukhovich [37], by concentrating on small normal cones sharper estimates are obtained. In fact, the normal cones need here not be convex; they are not polars of primal sets.

The hypothesis behind (8) - that at least one set be locally ∂-compact at v - points to the condition in the Hahn-Banach separation theorem that one set should have nonempty interior. A closed subset C of a Banach space is declared *epi-Lipschitzian* at a member point iff its Clarke tangent cone there has nonempty interior [41]. Each closed convex set with nonempty interior is epi-Lipschitzian at each of its points. Borwein and Strojwas [7] extended this concept to what they named the *compactly epi-Lipschitz property,* now known to be equivalent to Loewen's local compactness property [34]. Mordukhovich [38] considers the still more general concept of *sequential normal compactness.*

The classical Dubovitskii-Milyutin lemma [13] follows from (8). That lemma says that if A, B are convex cones, one being nonempty open, and $A \cap B = \emptyset$, then (8) holds with $v = 0$. For relevant extensions and economic applications of that lemma in finite dimensions see [46].

The instance $A = \left\{(e_k) \in E^K : \sum_k e_k = e\right\}$ and $B = \Pi_{k\in K} B_k, B_k \subset E$, is of particular interest for a pure exchange economy having total resource endowment e. Then $J = \emptyset$, $Lx = \sum_i x_i = e$, and $\partial d(v, A) = \{(e^*, e^*, ...) : e^* \in E^*\}$. So, under the hypotheses leading up to (8), there exists a nonzero $-p$ belonging to all $\partial d(v_i, B_i)$. Letting now $v = z$ and $B_i = clP_i(z)$ we get the inclusions $-p \in \partial d(z_i, clP_i(z)), \forall i$. These we shall meet again below. If preferences are convex, the assumption preceding (9), that $cl(A - B)$ be a proper subset, seems most natural. Indeed, it makes no economic sense to be able to fabricate all sorts of consumption profiles by mere redistribution of aggregate resources. Also observe that $A - coB = E^K$ iff $\sum_k coB_k = E$, and then, in case E is Euclidean, $\{A, coB\}$ cannot be extreme

(8) implies that $p \in N(v, A) \cap -N(v, B)$. This is equivalent to (9) if both sets A

and B are convex. The two polar cones $N(v, A) = (A - v)^o$ and $N(v, B) = (B - v)^o$ are then positively dependent. More generally, A and B can be separated by some nonzero continuous linear functional p as in (9) if for some v the polar cones $(A-v)^o$, $(B - v)^o$ are positively dependent in the sense that $(A - v)^o, -(B - v)^o$ intersect out of origin.

Of particular interest to us is the instance when one of the two sets in Lemma 1 equals the kernel of a bounded linear operator $L : Z \to \mathcal{E}$. For any Banach space, such as $\mathcal{E}$, we let $\mathcal{E}^*$ denote its dual, consisting of all continuous linear $e^* : \mathcal{E} \to \mathbb{R}$. Associated to L is its transpose $L^* : \mathcal{E}^* \to Z^*$ defined by $\langle L^*e^*, z\rangle = \langle e^*, Lz\rangle$ for all $e^* \in \mathcal{E}^*$ and $z \in Z$.

Lemma 2. (Separation with respect to a kernel) *Given a linear continuous* $L : Z \to \mathcal{E}$ *with closed range, and let* $A := \ker L$.
(i) *If* $0 \in bdL(B)$ *for some* $B \subset Z$, *then* $\{A, B\}$ *is locally extremal at any common point.*
(ii) *Suppose* B *is closed and* $\{A, B\}$ *is locally extremal at* z *with at least one set locally* ∂*-compact there. Then there exists a nonzero* $-z^* \in \partial d(z, B)$ *such that* $z^* = L^*e^*$ *for some* $e^* \in \mathcal{E}^*$ *(unique for any given* z^* *if* L *is surjective). If moreover,* $Z = E^K$ *and* $B = \Pi_{k\in K}B_k, B_k \subseteq E$, *then*

$$-z_k^* \in \partial d(z_k, B_k) \subset N(z_k, B_k) \quad \text{for all} \quad k. \tag{11}$$

In that setting it suffices for (11) *that one* B_k *be locally compact at* z_k *while* (7) *holds.*

Proof. (i) There exists a sequence $e^n \notin L(B)$ in $\mathcal{E}$ which tends to 0. Since L has closed range, there is a positive number r such that $d(z, KerL) \leq \frac{r}{2} \|Lz\|$ for all $z \in Z$. Pick any $w^n \in L^{-1}e^n$ and find $a^n \in A$ satisfying $\|w^n - a^n\| \leq r \|Lw^n\| = r \|e^n\|$. Define $z^n := w^n - a^n$. Then $z^n \to 0$ and $Lz^n = e^n$. Thus $(A + z^n) \cap B = A \cap (B - z^n) = \varnothing$.

(ii) By (5) there exists $z^* \in \partial d(z, A) \cap -\partial d(z, B)$. Since A is a linear subspace of Z, its normal vector $z^* \in \partial d(z, A)$ must be an *annihilato*r, i.e., $\langle z^*, a\rangle = 0$ for all $a \in A$. Therefore, if L is surjective, we get $z^* \in (KerL)^\perp = \operatorname{Im} L^*$ and $z^* = L^*e^*$ for a unique $e^* \in \mathcal{E}^*$. Finally, when L merely has closed range (strictly contained in $\mathcal{E}$) there exists $r > 0$ such that $\partial d(z, KerL) \subseteq rL^*\mathbb{B}^*$ for all $z \in KerL$, see [23]. Thus, also in this case $z^* \in L^*e^*$ for some $e^* \in \mathcal{E}^*$. The final assertions obtain from Lemma 1 (ii). $\square$

The vector e^*, figuring in Lemma 1 and 2, can naturally be seen as a Lagrange multiplier associated with the balance constraint (1). Note though, that it emerges here *not* in terms of constrained optimization but instead as a feature of separation.

After these preparations, all dealing with separation of extremal sets, we are finally ready to address our main concern: that of describing a Pareto optimum z in terms of prices. Recall that $0 \in L\left[\Pi_{k\in K} clP_k(z)\right] \subseteq clL(\Pi_{k\in K}P_k(z))$. We say that *indifference levels are thin* (in the aggregate) at a Pareto optimum z iff there exists a sequence $(e^n) \subset imL$, converging to 0, such that

$$e^n \notin clL\left[\Pi_{k\in K}P_k(z)\right]. \tag{12}$$

Then clearly, $0 \notin intclL\left[\Pi_{k\in K}P_k(z)\right]$. Alternatively, we say that the *Cornet-Morduckhovich constraint qualification* holds at a Pareto optimum z iff there exists $\varepsilon > 0$, a sequence $(a^n) \subset imL$, converging to 0, and an agent $k_o \in K$ such that

$$L\left[\Pi_{k\in K}clP_k(z) \cap \mathbb{B}(z,\varepsilon)\right] + a^n \subset L\left[\Pi_{k\neq k_o}clP_k(z) \times P_{k_o}(z)\right]. \tag{13}$$

Theorem 1. (Prices and Pareto optimum) *Let z be a Pareto optimum, weak or strong.*
(i) *Suppose $\Pi_{k\in K}clP_k(z)$ is ∂-compact, that L is surjective, and that indifference levels are thin at z.*
(ii) *Alternatively, suppose at least one component set $clP_k(z)$ is ∂-compact with (7) in vigor, that L has closed range, and that qualification (13) holds.*

Then, in either case there exists a nonzero price regime $e^* \in \mathcal{E}^*$ and $z^* = L^*e^*$ such that $-z_k^* \in \partial d(z_k, clP_k(z)) \subset N(z_k, clP_k(z))$ for all k.

When the commodity space is finite-dimensional - or if $KerL$ has finite codimension - the assumptions about ∂-compactness are superfluous.

Proof. (i) By assumption, there exists a sequence (e^n) converging to 0 such that (12) holds. From the surjectivity of L we get another sequence (z^n) in Z, also converging to 0, such that

$$-z^n + KerL = L^{-1}(e^n).$$

Since $L\left[\Pi_{k\in K}clP_k(z)\right] \subseteq clL\left[\Pi_{k\in K}P_k(z)\right]$, we see that $(-z^n + KerL) \cap \Pi_{k\in K}clP_k(z) = \emptyset$. This shows that the system $\{KerL, \Pi_{k\in K}clP_k(z)\}$ is locally extremal at z so we may apply Lemma 2.

(ii) It is not difficult to check that $\{KerL, \Pi_{k\in K}clP_k(z)\}$ again is locally extremal at z and we conclude as above. $\square$

Remarks : Theorem 1 uses the concept of $\partial-$compcatness to obtain conditions at the point z_k, not just near it. Further, in accommodating a general linear operator Theorem 1 generalizes, for example, the results in [19], [24], [25], [38].

Examples of preferences satisfying the $\partial-$compactness condition (in infinite dimentional spaces) are given by Bellaassali and Jourani [3].

Condition (12) says that 0 and $clL\left[\Pi_{k\in K}B_k\right]$ are extremal with $B_k = P_k(z)$. In other words, $0 \in bd\ clL\left[\Pi_{k\in K}B_k\right]$. Thus $0 \in bdL\left[\Pi_{k\in K}clB_k\right]$ so that Lemma 2 (i) also applies. These observations lead us to explore conditions like (12) somewhat further:

Definition 4. (Open mappings) *We declare a linear $L : E^K \to \mathcal{E}$* ***open with respect to components*** *if for every collection of nonempty sets $B_k \subseteq E, k \in K$, at least one being open, the image $L(\Pi_k B_k)$ will also be open in $\mathcal{E}$.*

Any such operator L is of course surjective. (When $\mathcal{E} = E$, examples include

$L(z) = \sum_{k\in K} {}_k\alpha_k z_k$ with all coefficients $\alpha_k \neq 0$.) It is straightforward to prove that for any such L and $k_o \in K$ we have

$$L(intB_{k_o} \times \Pi_{k\neq k_o} B_k) = L(intB_{k_o} \times \Pi_{k\neq k_o} clB_k).$$

One will see that for some results below it suffices to have L be open merely with respect to a specific "coordinate" $k_o \in K$.

Lemma 3. (Extremality of $clL(\Pi_{k\in K} clB_k)$ at 0) *Consider a product $B = \Pi_{k\in K} B_k$ of sets $B_k \subseteq E$ and suppose $0 \in clL(clB)$. Then $clL(clB)$ is extremal at 0 provided one of the following properties is satisfied:*
(1) *$\exists\, k_o \in K$ such that $0 \notin L(clB_{k_o} \times \Pi_{k\neq k_o} B_k)$ and the latter image set has nonempty interior.*
(2) *$0 \notin L(\Pi_k B_k)$ and the image set has nonempty interior.*
(3) *L is open with respect to components, $\exists\, k_o \in K$ such that $intB_{k_o} \neq \varnothing$, and $0 \notin L(\Pi_k B_k)$.*
(4) *L is open with respect to components, $\exists k_o \in K$ such that $intclB_{k_o} \neq \varnothing$, and $0 \notin L(clB_{k_o} \times \Pi_{k\neq k_o} B_k)$.*

Proof. For (1) introduce the open set $U := L(clB_{k_o} \times \Pi_{k\neq k_o} B_k)$. Then either $0 \notin clU$ or $0 \in clU \setminus U$. In either case there is a sequence $e^n \to 0$ in $\mathcal{E}$ such that $e^n \notin clU \supseteq clL(\Pi_k clB_k)$ for all n, and now the conclusion follows without further ado. (2) is verified in the same manner. For (3) note that $0 \notin L(intB_{k_o} \times \Pi_{k\neq k_o} B_k) = L(intB_{k_o} \times \Pi_{k\neq k_o} clB_k) =: U$. From here on proceed as before. For (4) argue likewise, in fact verbatim. $\square$

Using Lemma 3 the next result follows forthwith:

Theorem 2. (Pareto optimum under assumptions about interiors) *Suppose z is a Pareto optimum (weak or strong) at which $\Pi_{k\in K} clP_k(z)$ is ∂-compact. Then the conclusion of Theorem 1 holds under each of the following conditions:*
(1) *$\exists\, k_o \in K$ such that $L\left[clP_{k_o}(z) \times \Pi_{k\neq k_o} P_k(z)\right]$ has nonempty interior but does not contain 0.*
(2) *$L\left[\Pi_{k\in K} P_k(z)\right]$ has nonempty interior but does not contain 0.*
(3) *L is open with respect to components, $\exists\, k_o \in K$ such that $intP_{k_o}(z) \neq \varnothing$, and $0 \notin L(\Pi_{k\in K} B_k)$.*
(4) *L is open with respect to components, $\exists k_o \in K$ such that $intclP_{k_o}(z) \neq \varnothing$, and $0 \notin L\left[clP_{k_o}(z) \times \Pi_{k\neq k_o} P_k(z)\right]$.*
Examples include:
I) ADDITIVE CONSTRAINTS, NO PUBLIC GOODS: *Suppose $Lz = \sum_{k\in K} L_k z_k$ for continuous surjective $L_k : E \to E$. Then $\mathcal{E} = E$ and there exists a nonzero price regime $e^* \in E^*$ such that $L_k^* e^* \in \partial d(z_k, clP_k(z))$ for all $k \in K$.*
II) PUBLIC GOODS: *Let here $E = E^{priv} \times E^{pub}$, $\mathcal{E} = E^{priv} \times (E^{pub})^I$, and define L as in (3). Then there exists a nonzero price regime $e^* = (e^{priv*}, e^{pub*}) \in E^*$ and*

$e_i^{pub*} \in E^{pub*}, i \in I$, *with* $\sum_{i\in I} e_i^{pub*} = e^{pub*}$ *such that*

$$-(e^{priv*}, e_i^{pub*}) \in \partial d(x_i, clP_i(z)) \text{ for all } i \in I$$

and $e^* \in \partial d(y_j, clP_j(z))$ *for all* $j \in J$. □

It is fitting to conclude with a brief reconsideration of the convex scenario. In doing so we also take the opportunity to generalize somewhat. We declare the economy *convex in the aggregate* at z if

$$L\left[\Pi_{k\in K} clP_k(z)\right] \quad \text{is convex} \tag{14}$$

and not dense. Recall that the "geometrical" result (8) has nice functional counterparts if one or both sets are convex - as exemplified by (9). This observation immediately yields:

Theorem 3. (Pareto optima in economies with convex aggregate)
Let z be a Pareto optimum (weak or strong) in an economy which is convex in the aggregate there. Suppose the commodity space E is finite-dimensional, or $KerL$ has finite co-dimension, or $0 \in int\{L\left[\Pi_{k\in K} clP_k(z)\right] + C\}$ for some norm compact set $C \subset \mathcal{E}$, or that one $clP_k(z)$ is ∂-compact at z_k with (7) in vigor.
Then there exists a nonzero continuous linear functional p on $\mathcal{E}$ such that $0 = pz = \min \langle p, L\left[\Pi_{k\in K} clP_k(z)\right]\rangle$.
If $\mathcal{E} = E$ and $Lz = L(x,y) = L_{-j}z_{-j} - y_j$ for some $j \in J$ and linear continuous L_{-j}, then job-shop j maximizes its profit: $\langle p, y_j\rangle = \max \langle p, clP_j(z)\rangle$.
If still $\mathcal{E} = E$ and $Lz = L(x,y) = x_i + L_{-i}z_{-i}$ for some $i \in I$ and continuous linear L_{-i}, then consumer i minimizes his expenditure: $\langle p, x_i\rangle = \min \langle p, clP_i(z)\rangle$.
If, however, $\mathcal{E} = E^{priv} \times (E^{pub})^I$ and L is defined as in (3), then there exists $p_i = (p^{priv}, p_i^{pub}) \in E^{priv*} \times E^{pub*}$ such that

$$\langle p_i, x_i\rangle = \min \langle p_i, clP_i(z)\rangle \quad \text{for all} \quad i, \text{ and}$$
$$\langle p\ , y_j\rangle = \max \langle p, clP_j(z)\rangle \quad \text{for all} \quad j \text{where } p := (p^{priv}, \sum_{i\in I} p_i^{pub}). \quad \Box$$

The convexity condition (14) is, of course, satisfied when all tastes/ technologies $P_k(z)$ are convex.

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