New Theory of Elasto-Plastic Stability with a Discussion of the Contradictions, Inherent in Present Inelastic Column Theories

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NEW THEORY OF ELASTO-PLASTIC STABILITY WITH A DISCUSSION OF THE CONTRADICTIONS, INHERENT IN PRESENT INELASTIC COLUMN THEORIES

NEW THEORY OF ELASTO-PLASTIC STABILITY WITH A DISCUSSION OF THE CONTRADICTIONS INHERENT IN PRESENT INELASTIC COLUMN THEORIES

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FOREWORD

I owe much of the credit for this work to Professor Techn.D. Ove Pettersson, whose rich ideas and scientific approach to complex problems of technology have always been a source of inspiration. Despite his personal preoccupation with problems of great importance in the field of Fire Technology and Structural Engineering, as well as his numerous academic and administrative responsibilities, he always had the time to express his views on the development of this paper. Professor Pettersson can well be described as the busiest who yet always has time. For his profound interest in the development of science and technology and as a token of gratitude for his scholarly supervision of the accomplishing of this dissertation, I dedicate this work to him.

I would like to express my appreciation to Techn.D. Åke Holmberg from whom I got much encouragement to pursue research at Lund Institute of Technology. For a period of four years in the Consulting Civil Engineering Firm, Centerlöf & Holmberg AB, I acquired much enthusiasm for the development of engineering sciences and the pursuit of advanced research as a construction engineer.

Perhaps it may be of interest to include a short review of the starting point when I got interested in the ideas presented in this dissertation. Thus, it would be appropriate to mention that as soon as I moved to the Division of Structural Mechanics and Concrete Construction of Lund Institute of Technology I began a theoretical investigation of the problem of stability of steel columns at elevated temperatures. It soon occured to me that the traditional assumption that the unloading-modulus for any stress-strain curve and at any position on a given stress-strain diagram is constant and equal to the initial modulus of elasticity was quite arbitrary. After a thorough examination of the problem the hidden contradiction in Shanley's theory was uncovered and it became obvious that no conclusion on the general problem of stability could be validly drawn on the basis of the old theories of elasto-plastic stability. Thus, my preoccupation with specific problems turned to a general investigation of the fundamental concepts of elasto-plastic stability.

I would like to thank all the following persons for their generous help in the preparation of this paper: Mrs. Birgitta Hellström for handwriting the equations and drawing the diagrams; Mrs. Marianne Dahlqvist, Mrs. Ingrid Nilsson and Miss Kerstin Krahner for typing the manuscript; Mrs. Urte Dougan for translating the summary of the dissertation to German; Mr. J.A. Schweitzer for translating the summary of the dissertation to French; Mr. Harold M. Koch for reading the manuscript and reviewing the linguistic style and my brother Mahmood Mazlumolhosseini for reading the manuscript and checking the numerical calculations.

A.A. Mazlumolhosseini

CONTENTS

NOTATIONS

A	nominal cross sectional area
A_o	"initial total effective area"
$A_{e\varepsilon}$	"elastic or effective area" corresponding to strain ε
a_i, a_{oi}	nominal cross sectional area of a single crystal element, i, and its corresponding effective area
$a_{e\varepsilon}$	"elastic part of unit area"
$a_{p\varepsilon}$	"plastic part of unit area"
b	width of rectangular cross section; subscript "buckling"
C	resultant of compressive forces on the cross section (in bending)
D	alternative symbol for "modulus of elasto-plasticity"
E	modulus of elasticity
E_o	initial modulus of elasticity
$\bar{E}$	Considère-Engesser's reduced modulus
E_r	alternative symbol for reduced modulus
E_t	tangent modulus
E_{ep}	modulus of elasto-plasticity
$E_{a\varepsilon}$	"unloading-modulus" at strain ε
E_{an}	unloading-modulus at point n on the stress-strain diagram
e	strain at point of unloading; subscript "elastic"
h	height of rectangular cross section
h_1, h_2	maximi distances from the neutral axis (in bending)
I, J	moment of inertia
i, j	symbols indicating a single crystal element or a point on the stress-strain diagram; numerical factors
k	a numerical factor; a point on the stress-strain diagram

L	length of the structural model
M	ratio of effective area to nominal cross sectional area
M_x	bending moment
m	a numerical factor
n	symbol for point of unloading on the stress-strain curve; numerical factor
o	subscript "initial"
P	axial load
R_{ij}	ratio of effective areas between two points i and j on the stress- strain diagram
$S_{e\varepsilon}$	elastic reversible energy stored in the structural element of unit cross sectional area and unit length corresponding to strain ε
$S_{f\varepsilon}$	frictional energy losses for the structural element of unit cross sectional area and unit length corresponding to strain ε
$S_{t\varepsilon}$	total area under the stress-strain diagram corresponding to strain ε
T	resultant of tensile forces on the cross section (in bending)
W_e	elastic energy stored in the structural model
W_f	frictional energy lost in the structural model
x, y	rectangular coordinate axes
α	angle between the simplified physical model's vertical and inclined elements
β_i	angle between the strain axis and the tangent to the stress-strain curve at point i
Δ	symbol for "increment of ..."
δ	lateral deflection; interval length on the stress-strain diagram; symbol for "increment of ..."

ε	normal strain
θ_i	angle between the strain axis and the unloading-line at point i on the stress-strain curve
ρ	radius of curvature; coefficient of transformation of effective area
Σ	operator "summation"
σ	normal stress
$\sigma_{e\varepsilon}$	"elastic stress" corresponding to strain ε
$\sigma_{p\varepsilon}$	"plastic stress" corresponding to strain ε
σ_y	yield-point stress

1. General Introduction

11. Definition of various column theories with a historical review [1]

A) Euler's theory. The critical load for an *ideal* elastic column, viz., for an initially straight centrally loaded elastic column is given by the general Euler formula:

$$P_{cr} = \frac{\pi^2 EI}{(\beta L)^2} \qquad (a)$$

In the above formula P_{cr} represents the critical axial load; E, the modulus of elasticity; EI, the flexural rigidity of the column in the plane of bending; L, the length of the column; and β , a coefficient which depends on the boundary conditions of the column. Historically, the problem of elastic stability was first formulated mathematically by Leonhard Euler, who studied the stability of an ideal perfectly elastic column subjected to a compressive axial load P and with the boundary conditions fixed vertically at the base and free at the upper end. Euler published the solution of this specific problem in the appendix, "De curvis elasticis," of his book "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes," Lausanne and Geneva, 1744. The critical load for this specific Euler problem is found in formula (a) by replacing β by the number 2.

The validity of Euler's formulation of the problem of stability for a perfectly elastic medium has never been questioned and on the basis of the original idea tremendous progress has been made in the field of elastic stability as a result of his theory[2]. The more general problem of stability for an elasto-plastic medium, on the other hand, had not been treated before late in the nineteenth century and the solution of the problem has been the subject of much controversy ever since that time. The various theories of elasto-plastic stability will be defined below in their historical order.

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1. The emphasis here is on the definition of various column formulas with a brief historical glimpse. For a more detailed historical discussion see Timoshenko, "History of Strength of Materials", McGraw Hill Book Company, Inc., New York, 1953.

2. Cf., S.P. Timoshenko and G.M. Gere, "Theory of Elastic Stability", McGraw Hill Book Company, Inc., New York, 1961.

B) The tangent modulus theory. The critical load for an ideal elasto-plastic column is obtained by replacing the modulus of elasticity, E, in Euler's formula by the derivative of the stress-strain diagram, $d\sigma_\varepsilon/d\varepsilon$, corresponding to the column load. Designating the derivate, $d\sigma_\varepsilon/d\varepsilon$, by the tangent modulus, E_t, the critical column load would be

$$P_{cr} = \frac{\pi^2 E_t \cdot I}{(\beta L)^2} \qquad (b)$$

The critical column load, given by Eq. (b), is sometimes referred to as the tangent modulus load. The tangent modulus theory was proposed by Fr. Engesser in 1889[1].

C) The reduced modulus theory. The critical load for an ideal elasto-plastic column is obtained by replacing the modulus of elasticity, E, in Euler's formula by the reduced modulus, $\bar{E}$, which is a function of the initial modulus of elasticity, E, the tangent modulus, E_t, and the shape of the cross section. The reduced modulus theory has been called also the double modulus theory, Considère-Engesser theory and Von-Kármán's theory. The concept of the reduced modulus was first initiated by A. Considère and presented in a report to the Congrès International des Procédés de Construction, held in Paris on September 9-14, 1889, a few months after Engesser had proposed his tangent modulus theory. Considère pointed out in his report that as the bending of an ideal column begins at a stress beyond the proportional limit the stress on the concave side of the column increases along the stress--strain diagram and the stress on the convex side of the column decreases according to Hooke's law and thus the critical column load would be given by

$$P_{cr} = \frac{\pi^2 \cdot \bar{E} I}{(\beta L)^2} \qquad (c)$$

Considère predicted that $\bar{E}$ is a function of the average stress in the column the value of which lies between the initial modulus of elasticity, E, and the tangent modulus, E_t. However, he did not specify his reduced modulus, $\bar{E}$, more closely. Considère's report was published in 1891 as an appendix to the Proceedings of the above mentioned congress where he had originally presented it.

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1. Z. Architek. u. Ing. Ver. Hannover, Vol. 35, P. 455, 1889.

The contradiction of the tangent modulus theory was demonstrated for the first time by Félix Jasinski in 1895[1]. The contradiction in the tangent modulus theory is attributed to the fact that the theory does not take into account the extra rigidity possessed by the column in the bent position. Jasinski pointed out this contradiction and referred to the Considère's concept of the reduced modulus as the correct alternative. He stated, however, that at that time it was impossible to determine precisely a functional relationship which would describe the reduced modulus, $\bar{E}$. A month later, Engesser replied to Jasinski by admitting the incorrectness of his tangent modulus theory. However, he did not agree with Jasinski that at that time it was impossible to determine the reduced modulus, $\bar{E}$, theoretically. Engesser even put down an expression for $\bar{E}$ in the general form[2].

The next step in the development of elasto-plastic stability was taken by T. Von-Kármán, who once again in 1910 stated the reduced modulus theory and determined the reduced modulus, $\bar{E}$, for a rectangular and an idealized I-section[3]. (An idealized I-section is assumed to have negligible web area and infinitely thin flanges).

After the completion of the reduced modulus theory by Von-Kármán the experiments carried out on the elasto-plastic columns by different investigators revealed that the critical load for such columns is always smaller than the value predicted by the reduced modulus theory. Theoretically, however, the theory was considered to be valid up to 1947, when Shanley proposed his theory[4], which will be explained below.

D) Shanley's theory. This theory does not give any unique critical column load which would correspond to Euler's formula. The theory is based upon three assumptions: in the first assumption the column begins to bend as soon as the column load reaches the tangent modulus load; in the second, the axial load increases simultaneously with bending; in the third, some stress reversal takes place as soon as the bending begins. On the basis

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1. Félix Jasinski: "Noch ein Wort zu den Knickfragen," Schweizerische Bauzeitung, Vol. XXV, No. 25, P. 172, June 22, 1895.

2. Fr. Engesser, "Ueber Knickfragen", Schweizerische Bauzeitung, Vol. XXVI, No. 4, P. 24, July 27, 1895.

3. T Von-Kármán, "Untersuchungen über Knickfestigkeit", Forschungsarbeiten Nr. 81, Berlin 1910.

4. F.R. Shanley: "Inelastic Column Theory," Journal of the Aeronautical Sciences, Vol. 14, No. 5, P. 261, May, 1947.

of these assumptions Shanley concludes that the tangent modulus load is the lowest critical load for an ideal column and that between the tangent modulus load and the reduced modulus critical load the column can have different equilibrium positions with different lateral deflections. Thus, each equilibrium position beyond the tangent modulus load corresponds to a certain increase of axial load beyond that level. Shanley's theory gives a column load which for a certain finite value of lateral deflection would lie somewhere between the tangent modulus load and the value obtained by reduced modulus theory. In his inelastic column theory Shanley has given a mathematical analysis which determines the column load for a simplified two--flange column with infinitely rigid legs joined by an elasto-plastic hinge. After the appearance of Shanley's theory Von-Kármán acknowledged its correctness and affirmed that Shanley's procedure can be considered as a generalization of the reduced modulus theory[1]. Eversince 1947 Shanley's theory has been universally considered to be valid.

In this dissertation the author presents a new theory of elasto-plastic stability and demonstrates that Shanley's theory as well as all the possible elasto-plastic column theories that would assume the tangent modulus load as a critical load would lead to unavoidable contradictions.

12. Outline of the dissertation

The discussions, mathematical analyses and theoretical investigations presented in this dissertation treat the various aspects of the stability of an initially straight, centrally loaded elasto-plastic column. The paper is divided into four chapters and each chapter in turn is divided into a number of sections.

Chapter 1 is devoted to a general explanation of the theory together with a short history of various column theories including Euler's theory of elastic stability.

Chapter 2 will discuss the contradictions in the inelastic column theories that assume the tangent modulus load as a critical load. The discussion begins with a review of Shanley's theory including an examination of his

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1. T.V. Kármán: "Discussion," Journal of the Aeronautical sciences, Vol. 14, No. 5, P. 267, May, 1947

remarks, assumptions, test data, conclusions, and his mathematical analysis. Shanley's theory is based principally upon three assumptions: in the first assumption the column starts to bend at the tangent modulus load; in the second assumption the axial loading continues to increase simultaneously with bending; in the third assumption some stress reversal takes place in the column as soon as bending begins. It will be demonstrated that the second and third assumptions are mutually contradictory. Thus, Shanley's theory as well as all other investigations and conclusions based upon that theory lose their validity.

In Section 23 the interpretations of Shanley's theory by Timoshenko and Von-Kármán will be discussed. Timoshenko's incompatible explanation of Shanley's theory will be mentioned because his book, Theory of Elastic Stability, is available at almost every technical library throughout the world even in places where there is no direct access to Shanley's original literature. Von-Kármán's interpretation of Shanley's theory will be mentioned because it was just the shortcomings in Von-Kármán's reduced modulus theory which led Shanley to the development of his theory.

The paradoxical idea of buckling or bending at the tangent modulus load appears to have created a great deal of scientific confusion ever since Engesser proposed his tangent modulus theory in 1889. Therefore, it is worth while to investigate all the alternative possibilities of the elasto--plastic buckling including all the alternatives which assume the tangent modulus load as a critical load. The latter alternatives include: a) the tangent modulus theory, b) Shanley's theory, and c) a third alternative proposed by the author. However, it will be demonstrated that all these three alternatives, including the new alternative, lead to unavoidable contradictions and therefore are no longer valid. In Chapter 2 we conclude that an elasto-plastic column has only one critical load which exceeds the tangent modulus load. The fact that the critical column load is always less than the value predicted by the reduced modulus theory will be proved after discovery of load deformation properties of deformable elasto-plastic media during the process of unloading and after a thorough discussion of various aspects of elasto-plastic stability in the chapters which follow.

Chapter 3 will be devoted to the development of a mathematical theory of plastic flow which is equivalent to the physical reality. In Section 32 the essential concepts of elasto-plasticity will be introduced and defined in connection with a detailed discussion of load-deformation properties of a structural, mathematical model built up in cross section by three distinct layers of homogeneous ideal elasto-plastic media with the same modulus of elasticity but with varying yield-point stresses. It will be demonstrated that the unloading curve coincides with a straight line. Thus, the tangent of the angle between the unloading-line and the strain axis will be called the "unloading modulus of elasticity" or simply the "unloading modulus". In Section 33 the theory will be generalized by demonstrating that we could construct a mathematical model which would be equivalent to any given stress-strain diagram and vice versa. The load deformation equivalence between the mathematical model and the physical or experimental model would hold as long as creep deformations or thermal effects do not begin to change the form of the experimental stress-strain diagram.

In Section 34 two types of problems of elasto-plasticity will be treated in connection with a discussion of the transformation of a non-homogeneous physical model to its homogeneous mathematical equivalent. The first type of problem is encountered when one wishes to determine the load-deformation characteristic of a given physical model with well-defined components (including both loading and unloading processes). The second type of problem arises when the stress-strain diagram of a physical model during the loading process is given and when one wishes to determine its stress-strain relationship during the process of unloading.

Section 35 will be devoted to a detailed study of a simplified problem of the first type, whereas Sections 36 and 37 will be devoted to the general solution of the problems of the second type. In Section 36 a general equation for determination of the unloading-modulus as a function of the shape of the stress-strain curve and the position of the point of unloading will be derived. In Section 37 a general mathematical investigation for determining the variational possibilities of the unloading-modulus along any stress-strain curve will be carried out and a final general analytic expression for the unloading-modulus will be set up. The investigations of Section 37 can be considered as the most decisive, generalized mathematical development as far as the problem of elasto-plastic stability is concerned. The final result of the investigations in this section will provide us with the means of drawing the far-reaching conclusion that the unloading-modulus is in general always less than the initial modulus of elasticity. The cases in which the unloading-modulus can be equal to the initial modulus of elasticity, viz., the cases of the ideal elastic and the ideal elasto-plastic media, are insignificant for the general problem of elasto-plastic stability. The unloading-modulus can never be greater than the initial modulus of elasticity. Section 38 will be devoted to the development of a numerical method for calculating the unloading-modulus and in Section 39 the results of the previous section will be used to calculate the unloading-moduli at various points on an actual experimental stress-strain curve.

From the outset it should be emphasized that the assumption of ideal elasto-plastic model is convenient and does not in any way affect or restrict the load-deformation properties of the individual crystal grains inside the physical model. Obviously, if we were to solve a problem of the first type, for example the model studied in Section 35, then we would have to begin by assuming a certain stress-strain relationship for each component of the model. However the solution of the problem of the second type does not require any assumption regarding the load-deformation properties of the inside structure of the physical model because, regardless of the causes of plastic flow and no matter how the individual crystal grains deform, one can always construct a mathematical model whose stress-strain properties would be equivalent to the experimental stress-strain diagram of the physical model. Thus, the general theory proposed in Chapter 3 is neither restricted by the causes of plastic flow nor limited by the stress-strain relationship of individual crystal grains inside the crystal structure of the physical experimental model.

Chapter 4 will be devoted to the development of the new theory of elasto-plastic stability based on the new fundamental theorem of stability; this new theorem, in turn, will be deduced from the generalization of the statements and conclusions drawn in Section 24. This fundamental theorem asserts that the stability of any column must be examined under the influence of constant loading since the simultaneous increase of axial load cannot affect the phenomenon of buckling. Realizing that Von-Kármán's original inelastic buckling load was derived by an examination of the column stability under the influence of constant loading and bearing in mind that he subsequently accepted Shanley's theory as a generalization of his own procedure, we conclude that historically the confidence in the validity of the tangent modulus load, the consequences of which have remained as a paradox for more than three quarters of a century, has been stronger than the confidence in the procedure which is demonstrated in this paper to be the fundamental theorem of stability. Thus, in the author's opinion the first step towards approaching any problem of stability scientifically would be to dissociate the incompatible idea of simultaneous increase of axial loading from the concept of stability of the column.

In the next part of Chapter 4 the various aspects of elasto-plastic instability will be discussed on the basis of the new fundamental theorem of stability. A general formula for the buckling load of an elasto-plastic column will be derived and finally after using the result of the general mathematical investigation of Section 37 we will conclude that the new theory of elasto-plastic stability would result in a critical column load which would lie somewhere between the tangent modulus load and the value predicted by the reduced modulus theory. The last section of Chapter 4 is devoted to a numerical comparison of the various column theories. The final conclusions appear at the end of the dissertation and immediately afterwards there follows a summary of the dissertation in English, German and French.

As a continuation of the work presented in this dissertation the author has worked out a fundamentally new procedure for investigation of deflections of eccentrically loaded elasto-plastic columns. This new development which is now under preparation will be covered by a second paper to be published soon by the Division of Structural Mechanics and Concrete Construction of Lund Institute of Technology.

2. Discussion of Contradictions in the Inelastic Column Theories Assuming the Tangent Modulus Load as a Critical Load

21. Short Introduction

The following discussion begins with a review of Shanley's Theory, including an examination of his remarks, assumptions, test data, conclusions and mathematical analysis. The discussion continues with the intepretations of the theory by Timoshenko and Von-Kármán and then turns to a general investigation of all the alternative possibilities of elasto-plastic buckling. The investigation includes all the alternatives that assume the tangent modulus as a critical load. It is proved, in this chapter, that all the latter alternatives, including Shanley's theory, lead to contradictions, and for this reason they lose their validity.

It is further proved that an inelastic column has only one critical load that always exceeds the tangent modulus load. It is also pointed out that the critical load is always smaller than the value predicted by the reduced modulus theory. This last statement is proved by investigations in the chapters which follow.

22. Discussion of Shanley's theory

Shanley's remarks, under the title, "The Column Paradox", appeared for the first time in Journal of the Aeronautical Sciences, vol. 13, no. 12, december, 1946. His inelastic column theory appeared in the same journal, vol. 14, no. 5, may, 1947. These two articles will be referred to here as Reference 1, and Reference 2. Shanley's remarks, assumptions, test data, and mathematical analysis will be discussed here in some detail.

A. Shanley's remarks and assumptions

Quotation from Ref. 1:

> «But there is an implied assumption in the derivation of the reduced modulus theory that is open to question. It is in effect, assumed that something keeps the column straight while the strain increases from that predicted by the tangent modulus theory to the higher value derived from the reduced modulus theory. Actually, there is nothing (except the column's bending stiffness) to prevent the column from bending simultaneously with increasing axial loading. ... Evidently, the maximum

column load will be reached somewhere between the loads predicted by the two theories. The entire problem should be reviewed on the basis that axial loading and bending can occur simultaneously. The use of the principle of superposition is not valid in this case ...»

B. Shanley's test data and his conclusions

It should be observed that all the figures and equations reproduced from Ref. 2, are given new numbers in this paper.

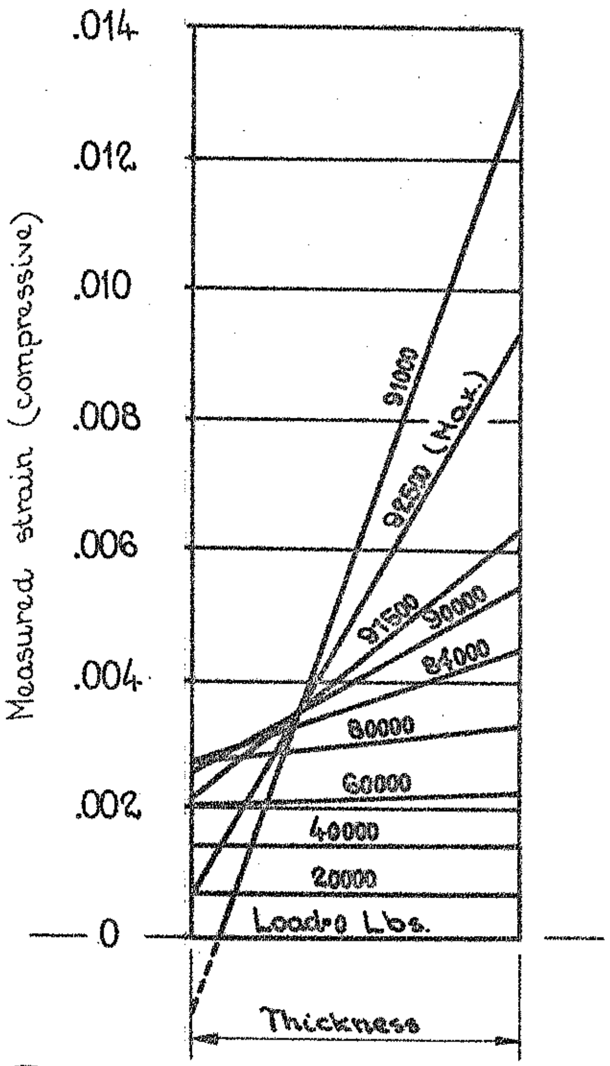

Fig. 1 Strain distribution as determined in a column test

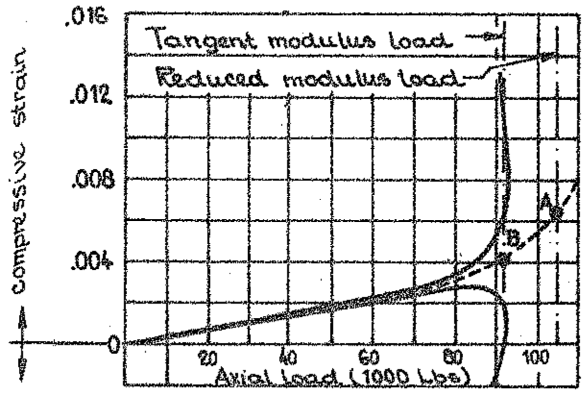

Fig. 2 Strain on opposite face of column from test data

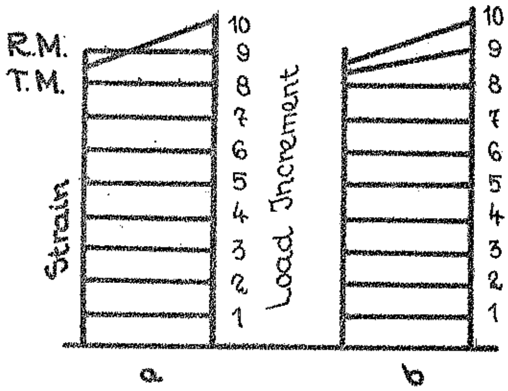

Fig 3 Alternative types of strain distribution across column cross section

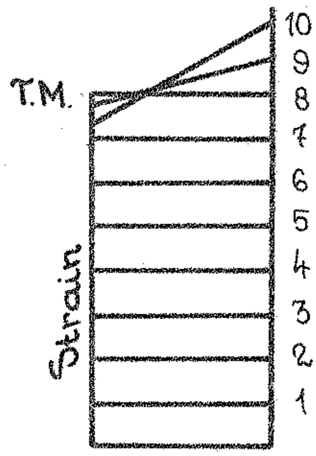

Fig. 4 Type of strain distribution needed to permit loading beyond tangent-modulus load.

The preceding figures and diagrams are reproduced from Reference 2.

Quotation from Ref. 2:

«Fig. 2 shows that if the column were to remain straight up to the reduced modulus load there could be no strain reversal below that load. What, then, can supply the extra effective value of E needed to prevent buckling beyond the tangent modulus load?[1] The obvious answer is that the column cannot remain straight beyond the tangent modulus load; there must be a definite amount of strain reversal as soon as the load is further increased. This should cause the curves to separate at point B, one starting downward and the other upward. It can now be seen that in the derivation of the reduced modulus theory a questionable assumption was made. It was assumed, by implication at least, that the column remains straight while the axial load is increased to the predicted critical value, after which the column bends, or tries to bend. Actually the column is free to bend at any time. There is nothing to prevent it from bending simultaneously with increasing axial load. Under such a condition it is possible to obtain a nonuniform strain distribution without any stress reversal taking place.[2] The difference between the two assumptions is shown diagramatically in Fig. 3. Fig. 3(b), however, still represents a paradox. There is no strain reversal indicated; hence the value of E_t must apply over the entire cross-section; therefore, the column load cannot exceed the tangent-modulus value. If the load is to go any higher, some strain reversal must begin at the tangent-modulus load.[3] The picture might then look something like Fig. 4, in which each succeeding increment of loading beyond the tangent-modulus load causes some additional strain reversal. The fact that this picture resembles the actual distribution shown in Fig. 1 is significant.[4] ... on the basis of the foregoing reasoning the authoer predicted, in reference 1, that (a) bending will begin as soon as the tangent-modulus load is exceeded; (b) the maximum column load will be reached somewhere between the loads predicted by the two theories.»

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1. The underlining does not occur in the original article. The question by Shanley is underlined here, to show that his approach to the problem is principally based on the assumption that the tangent modulus load is a critical load.

2. This possibility is rejected by Shanley. However, the author proves here that *if* the column is going to bend, before the actual buckling load, then the possibility, rejected by Shanley, is the only possibility at the beginning of bending.

3. This statement is underlined, to emphasize what Shanley considers as a necessity that if the load is to go any higher, some strain reversal must begin at the tangent modulus load.

4. An explanation will be made why Shanley's test data can give the results, mentioned by him, in spite of the fact that the author does not agree with his assumptions.

One can deduce from these remarks and conclusions that Shanley bases his theory upon three assumptions: in the first assumption the column starts bending at the tangent modulus load; in the second assumption the axial load continues to increase as soon as the bending begins; in the third assumption the stress reversal takes place as soon as the bending starts. Before discussing Shanley's first assumption, the author will prove that Shanley's second and third assumptions are mutually contradictory; more specifically, the type of stress distribution given in Fig. 4, which Shanley considers as necessary to permit loading beyond tangent modulus load, cannot be true. Consider Fig. 5 below:

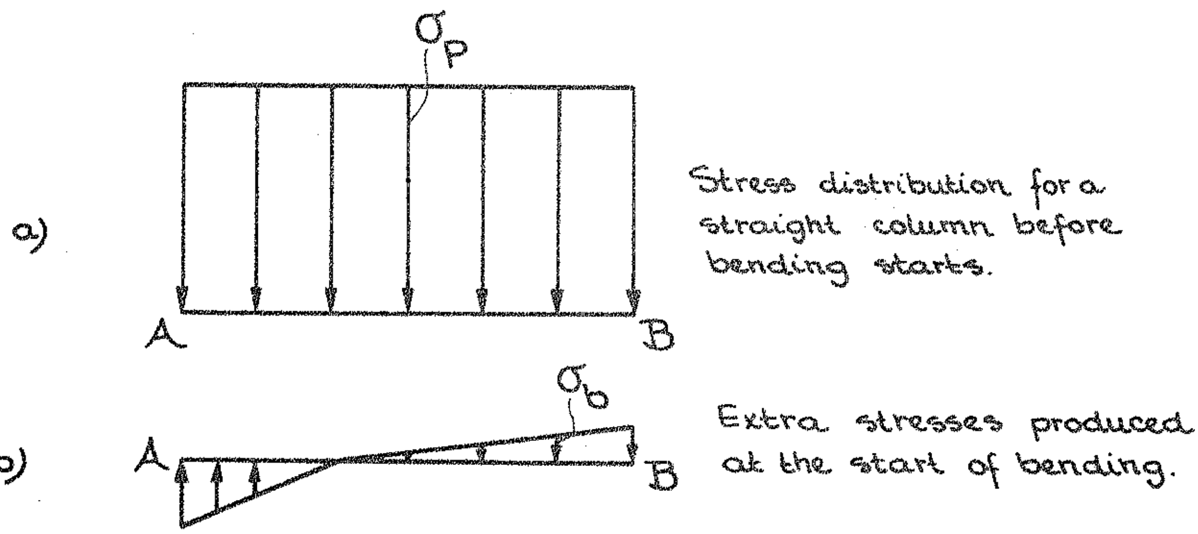

Fig. 5

Fig. 5(a) shows the uniform stresses for a straight column under the centric load P before bending starts. Fig. 5(b) shows the extra stresses, produced at the elasto-plastic column's cross-section, at the start of bending, provided the load P is constant. The stresses, σ_b , are infinitesimal in comparison with σ_p. Therefore, even if the neutral axis of the column has shifted considerably for the infinitesimal stresses σ_b , the displacement of the neutral axis for the total stresses $\sigma_p + \sigma_b \approx \sigma_p$ can be neglected provided that the values of bending stresses remain infinitesimal with respect to the uniform stresses due to the tangent modulus load. Thus while the resultant of the stresses are still concentric over any cross-section of the column, and while the bending stresses are still infinitesimal, the increase of axial load can accompany stress-reversal if and only if the stresses due to increase of axial load, $\sigma_{\Delta P}$, are smaller than the stresses due to bending.

This condition expressed mathematically would be $\sigma_{\Delta p} < \sigma_b$. If one were to accept Shanley's second and third assumptions simultaneously, the following inequality would result:

$$\sigma_{\Delta p} < \sigma_b < \text{any infinitesimal stress}$$

The logical conclusion from the above inequality is that $\sigma_{\Delta p} = 0$. Therefore, if the lateral deformations of the column at the start of bending are not large enough, it would be impossible to get stress reversal, simultaneously with an increase of axial load. According to Shanley's theory, on the other hand, the bending starts at the tangent modulus load, when the column is still straight and the lateral deflections are zero. Thus Shanley's assumption that the increase of axial load beyond the tangent modulus load takes place simultaneously with stress-reversal is a fundamental error, which is further reflected in his mathematical analysis.

Thus, one may conclude that if an initially straight centrally loaded column begins bending, at any stage of loading, then the tangent modulus applies to the whole cross-section and no stress reversal can take place at the start of bending.

The reason that Shanley has observed stress reversal at the tangent modulus load in his experiments can be attributed only to the unavoidable eccentricity and curvature of his test column. In fact, one can observe in Fig. 2 that the two curves corresponding to strain on opposite faces of column from test data have measurably separated at a level much lower than the tangent modulus load. The formulation of a theory can never be based on experimental results, if the underlying assumptions are mutually contradictory.

C. Shanley's mathematical analysis

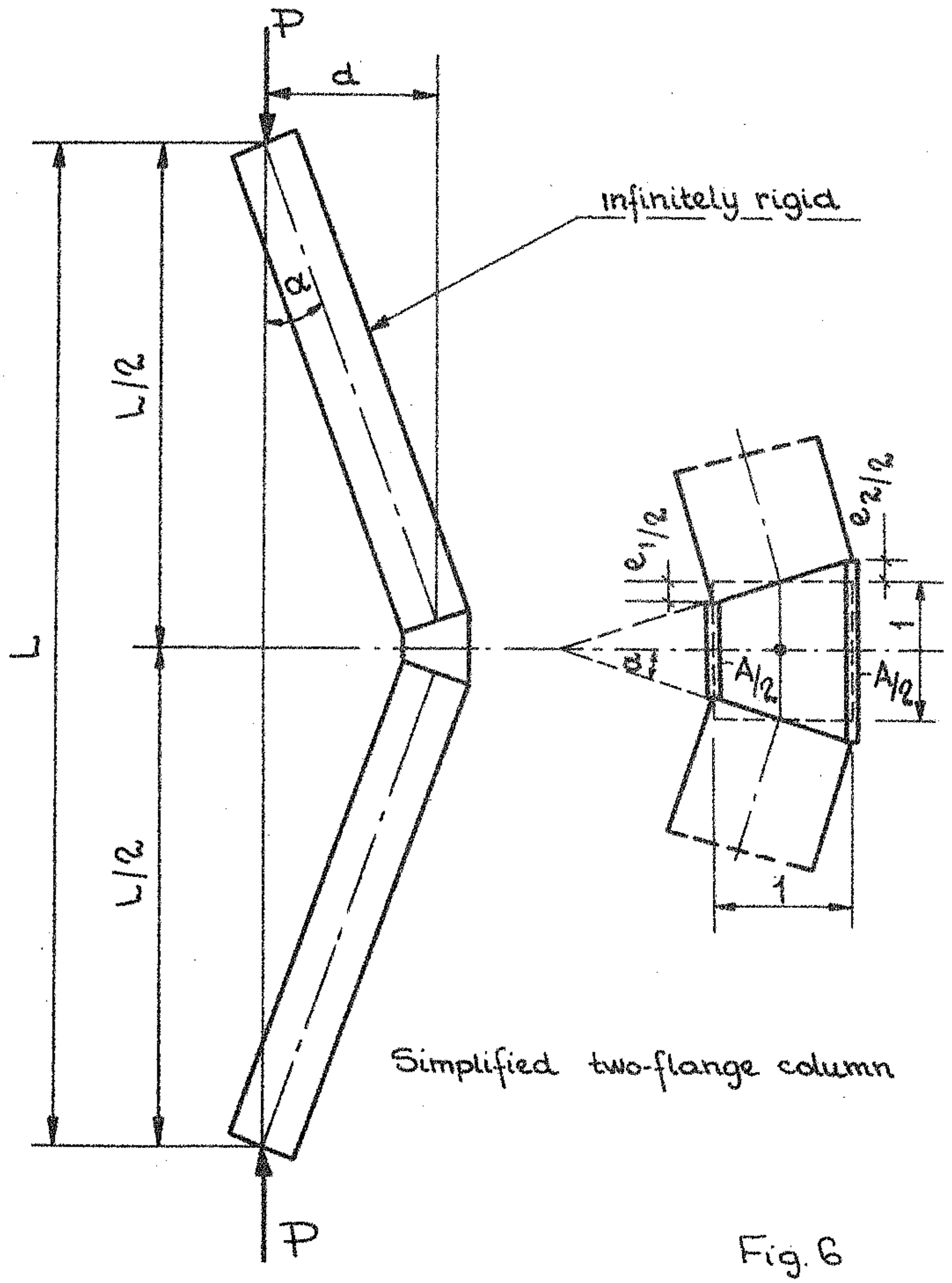

Fig. 6

In order to formulate his theory, Shanley gives a mathematical analysis based on a simplified two-flange column. The description of the hypothetical column is quoted here from Reference 2:

«This[1] consists in working with a two-legged hinged column in which the hinge consists of a unit cell formed from two small axial elements.... The two legs of the column are assumed to be infinitely rigid. If the dimensions of the unit cell are sufficiently small with respect to the column length L, it can be assumed that there is a simple hinge action about the center of the cell. This device reduces the problem to elementary form by eliminating the mathematical work involved in integrating over the cross-section and over the length of the column... The two elements of the column cell are assumed to have deflected in opposite directions through the distances e_1 and e_2 which may be regarded as strains that occur after the column starts to bend ... e_1 and e_2 may have different values indicating combined bending and variation in axial load.»

Shanley continues his calculation of this model with the assumption that while bending the two axial elements 1 and 2 possess the moduli of elasticity E_t and E respectively. By equating internal and external moments for the column and setting $\frac{E}{E_t} = k$, he arrives at the following column formula:

$$P = \frac{AE_t}{L}\left[1 + \frac{L}{4d}(k-1)e_2\right] \qquad (1)$$

In order to derive Eq. (1) Shanley has assumed that the two elements, 1 and 2, of the middle hinge of the two-flange column have equal areas but different moduli of elasticity. Thus, Eq. (1) gives the column load for an eccentrically applied compressive load P. In the next step Shanley relates the column-load according to Eq. (1) with the buckling load for an elasto plastic column. Quotation from Reference 2:

«... Another expression for P may be obtained by assuming that, after the tangent modulus is reached the column load continues to increase. This increase is given by the difference between the element loads P_1 and P_2, which can be expressed as $\Delta P = P_1 - P_2 = e_1 E_1 (A/2) - e_2 E_2 (A/2)$. Substituting for

E_1 and E_2, $\Delta P = (A/2) E_t (e_1 - k e_2)$

$$\Delta P = (A/2) E_t \left(\frac{4d}{L} - e_2 - k e_2\right) \quad \text{substitutning for } e_1,$$

$$\Delta P = A/2\, E_t \left[\frac{4d}{L} - (1+k) e_2\right]$$

1. This refers to the two-flange column.

This value should be added to the tangent modulus load to obtain the total value for P

$$P = P_t + \Delta P = \frac{AE_t}{L} + \frac{A}{2} E \left[\frac{4d}{L} - (1+k) e_2 \right]$$

$$P = \frac{AE_t}{L} \left\{ 1 + \left[2d - L/2(1+k) e_2 \right] \right\} \qquad (2) \quad »$$

Shanley equates Eq. (1), already calcalated, with Eq. (2) and arrives at the following relation for the column load:

$$P = \frac{AE_t}{L} \left(1 + \left\{ 1 \Big/ \left[\frac{1}{2d} + \left(\frac{k+1}{k-1} \right) \right] \right\} \right) \qquad (3)$$

Eq. (3) in Shanley's opinion, represents the complete theory of column action. Shanley then derives an equation for the reduced modulus column load.

Quotation from Reference 2:

> «... The equation for the critical column load will now be derived on the basis of the assumption originally used in the reduced-modulus theory. It will be assumed that the column remains straight up to the critical load P_r, after which it bends. The derivation proceeds as before, up to and including Eq. (1). Now, instead of assuming that there can be an increase in load, ΔP, it will be assumed that $\Delta P = 0$. Then $P_1 = P_2$ and $E_t e_1 = E e_2$ ...»

Using these relations Shanley arrives at the following formula for the reduced mudulus load:

$$P_r = (AE_t/L) \left\{ 1 + \left[(k-1)/(k+1) \right] \right\} \qquad (4)$$

Quotation from Reference 2:

> «Eq. (4) obviously represents the limiting value of Eq. (3) as <u>d</u> approaches infinity.»

It is clear from this mathematical analysis that all of Shanley's three assumptions appear in the derivation of his equations. Eq. (1) gives the column-load, in the bent position, while it is assumed that the stress reversal has already taken place. The fundamental error in the development of Shanley's theory is reflected mainly in the derivation of Eq. (2), where in the transitional stage from the straight to the bent position, it is assumed that stress reversal has taken place right from the start of bending. Eq. (3), which according to Shanley represents the complete theory of column action, is derived by equating Eqs. (1) and (2) and is thus affected by the same fundamental error.

The fact that Eq. (3), derived by Shanley, represents the limiting value of Eq (4), as d approaches infinity, does not at all mean that the assumption of bending at the tangent modulus load is not arbitrary. To prove this it will be assumed, now, that the column starts bending not necessarily at the tangent modulus load P_t, but at any arbitrary load $P_m = m \cdot P_t$, where m is any constant. Following the steps taken by Shanley, we proceed to find out the final value of the column load: $P = P_m + \Delta P = m \cdot P_t + \Delta P$, leading to the following equation for P,

$$P = \frac{AE_t}{L}\left\{m + \left[2d - L/2(1+k)e_2\right]\right\} \qquad (5)$$

Eq. (1) can be written in the following form:

$$P = \frac{AE_t}{L}\left[m + 1 - m + \frac{L}{4d}(k-1)e_2\right] \qquad (6)$$

Comparing Eqs. (5) and (6) we get:

$$2d - L/2(1+k)e_2 = \frac{L}{4d}(k-1)e_2 + 1 - m$$

calculating e_2 from the above equation gives

$$e_2 = \frac{8d^2 + 4d(m-1)}{L\left[(k-1) + 2d(1+k)\right]}$$

substituting e_2 in Eq. (1), will give the following equation:

$$P = \frac{AE_t}{L}\left[1 + \frac{2d(k-1)}{(k-1)+2d(1+k)} + \frac{(k-1)(m-1)}{(k-1)+2d(1+k)}\right] \qquad (7)$$

This may be reduced to

$$P = \frac{AE_t}{L}\left(1 + \left\{ 1 \Big/ \left[\frac{1}{2d} + \frac{(k+1)}{(k-1)}\right]\right\} + \left[\frac{(m-1)}{1 + \frac{2d(1+k)}{(k-1)}}\right]\right) \qquad (8)$$

Eq. (8) shows that, for the same value of lateral deflection as in Eq. (3), the value of P increases as m increases, and the value of P decreases as m decreases. The upper limit for Eq. (8), however, is the same as that for Eq. (3). As the lateral deflection d approaches infinity the load P reaches the limiting value given by Eq. (4).

It should be pointed out, however, that neither Eq. (3) nor Eq. (8) is correct. In spite of this, Eq. (8) was derived to prove that by following the steps taken by Shanley we arrive at an incorrect column load whose limiting value, as d approaches infinity, is independent of the stage of loading, where bending is assumed to start. This argument leads to the conclusion that Shanley's mathematical analysis does not in any way prove that bending must start at the tangent modulus load. In other words such a mathematical treatment does not introduce any new element to be discussed; furthermore, the equations derived do not provide any support for any of the assumptions on which they are based.

In his theory Shanley attempted to find a column formula which would agree with the experimental results of many investigators, including Shanley himself, that the true value of the inelastic buckling load must lie somewhere between the tangent modulus load and the reduced modulus buckling load. Because of the contradiction in Shanley's second and third assumptions, his first assumption that the column starts to bend at the tangent modulus load loses its significance because of the fact that the theory fails to achieve its aim, even if the first assumption would be accepted.

It is interesting to observe Shanley's recent explanation of the justification for his first assumption that the column starts to bend at the tangent modulus load:

> «In order to exceed the Engesser load, it is necessary that the effective modulus be greater than E_t. Such a situation can occur only if a portion of the cross section is subjected to a decreasing stress. But this means that the column would begin to bend before reaching the double-modulus load ... »[1]

We finally conclude from the above argument that Shanley justifies the assumption of bending at the tangent modulus load on the basis of stress reversal, unaware of the hidden paradox that the increase of axial loading, which takes place simultaneously with stress reversal, leads to a mutual contradiction.

23. Interpretation of Shanley's theory by Timoshenko and Von-Kármán

The idea of bending at the tangent modulus load is not, in fact, an accidental assumption. There are theoretical considerations which seemingly support this assumption. The start of bending at the tangent modulus load has been advocated by very prominent engineers and well-known investigators in the field of structural engineering, the world over. In the following quotations from the original published literature, Timoshenko and Von-Kármán express their ideas on this subject. Concerning the discussion of Shanley's theory Timoshenko writes the following:

> «... During the testing of an actual column, the axial force increases simultaneously with lateral deflection. In such a case, the decrease of stress on the convex side of the column during the initial stages of bending may be compensated by the increase of direct compressive stress due to the continually increasing axial force. Thus the actual deformation may proceed without any release of stress in the fibers on the convex side, as was assumed in Fig. 3 - 16[2], and the stress-strain relation for the entire column is defined by the tangent-modulus E_t. The differential equation of the deflection curve then becomes, $P_y = E_t I \frac{d^2y}{dx^2}$, and for a column with hinged ends the critical load is, $(P_t)_{cr} = \frac{\pi^2 E_t I}{\ell^2}$ (3-15) and the critical stress is $(\sigma_t)_{cr} = \frac{\pi^2 E_t}{(\ell/r)^2}$ (3-16)

1. F.R. Shanley, "Mechanics of Materials", Mc Graw-Hill Book Company, 1967.

2. Fig. 3 - 16 refers to Timoshenko's discussion of Von-Kármán's theory.

> These latter expressions for critical load and critical stress differ from Eqs. (3-12) and (3-13)[1], since they contain the tangent-modulus E_t, which is somewhat smaller than the reduced modulus E_r and independent of the shape of the cross-section. From this discussion it follows that under a continuously increasing load the column begins to buckle as soon as the load reaches the value (3-15)[2]»

One should note that Timoshenko, in a footnote to the above quotation attributes the theory, called the tangent modulus theory, to F.R. Shanley. Obviously this is not Shanley's theory because his theory is based on the three assumptions already discussed in this paper and was mainly developed to calculate a column load which exceeds the tangent modulus load. In his exposition of the inelastic column theory, Timoshenko somehow avoids the contradiction in Shanley's theory by stating that the deformations may proceed without the release of stress in the fibers on the convex side. In spite of this, Timoshenko falls back to the contradiction of the tangent modulus theory. One may deduce that Timoshenko conceives the idea of simultaneous increase of axial load as the cause of buckling at the tangent modulus load . In the following discussions in this paper, the author proves that the increase of axial loading can never affect the stability of the column.

The concept of bending, which takes place simultaneously with the increase of axial load as expressed by Von-Kármán, is set forth as follows:

> «Both Engesser's and my own analyses of the problem were based on the assumption that the equilibrium of the straight column becomes unstable, when there are equilibrium positions infinitesimally near to the straight equilibrium position, under the same axial load. The correct answer to this question is given by replacing, in Euler's equation, young modulus by the so-called reduced modulus. Mr. Shanley's analysis represents a generalization of the question. His procedure can be formulated as follows: what is the smallest value of the axial load at which a bifurcation of the equilibrium positions can occur, regardless of whether or not the transition to the bent position requires an increase of the axial load? The answer to this question is that the first equilibrium bifurcation from the straight equilibrium

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1. These Eqs. refer to corresponding equations for reduced modulus theory.

2. S.P. Timoshenko and J.M. Gere, "Theory of Elastic Stability", Mc Graw-Hill Book Company, Inc. 1961.

configuration occurs at a load given by the Euler formula when the young modulus is replaces by the tangent modulus. In fact, one can construct sequences of equilibrium positions starting from any load between the two limiting values corresponding to the tangent and the reduced moduli.» [1]

Von-Kármán considers Shanley's procedure to be a generalization of his own reduced modulus theory. He stipulates that all the stages of loading between the tangent modulus and the reduced modulus loads are critical, where the equilibrium bifurcation from the straight position can occur at any stage between the two loads. However, if Von-Kármán were to accept tangent modulus load as the lowest critical load, then his own theory which he considers to be special case of Shanley's theory would lose all its practical significance.

It is interesting to note that both Timoshenko and Von-Kármán agree in principle with Shanley's theory, whereas their explanations of the meaning of the theory are quite different. What is common between Timoshenko's and Von-Kármán's explanation of the Shanley's theory is the idea of bifurcation of the equilibrium position at the tangent modulus load. As already pointed out in this paper, this concept loses its value because of the contradictions arising out of Shanley's second and third assumptions. It is interesting to observe that all the investigators have, so far, focused their attention on the Shanley's first and second assumptions: that bending starts at the tangent modulus load, and that the axial load increases simultaneously; they have, however, failed to notice Shanley's third assumption: that some stress reversal must take place at the column, as soon as the bending starts. It is this third assumption that plays such a central role in the development of Shanley's procedure and which, as pointed out in this paper, contradicts the second assumption, leading to the final rejection of the theory. Logically, then, all conclusions based on Shanley's theory lose their validity.[2]

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1. J. Aeronaut. Sci., Vol. 14, p. 267, May 1947.

2. Cf. U. Müllersdorf, "Zur Theorie der plastischen Knickung". Der Bauingenieur, vol. 27, p. 57, 1952.

24. Discussion of all the alternative possibilities of elasto-plastic buckling

Timoshenko and Von-Kármán, as was pointed out above, have different explanations of the meaning of Shanley's theory. However they are in agreement on one point: that the tangent modulus is the smallest critical load for a centrically loaded, initially straight elasto-plastic column. This makes it worth-while to investigate all the following four alternative possibilities of the elasto-plastic buckling, including all the alternatives considering the tangent modulus as a critical load.

1. The first alternative: the column buckles at the tangent modulus load without any increase of axial load.

2. The second alternative: the column starts bending at the tangent modulus load, while the increase of axial load is accompanied by some stress reversal in the column from the very start of bending.

3. The third alternative: the column starts bending at the tangent modulus load, simultaneously with the increase of axial load, while the tangent modulus applies to the whole cross-section, up to a certain lateral deflection, where the stresses due to bending become large enough to allow some stress reversal in the column to take place. This process gives the column extra rigidity after a certain amount of lateral deflection.

4. The fourth alternative: the column does not bend at the tangent modulus load, while the axial load increases, with the tangent modulus governing over the whole cross-section, up to a critical stage of loading where the column loses its stability.

Discussion

A. The first alternative coincides with the tangent modulus theory. This alternative, however, leads to the contradiction, as pointed out by earlier investigators, that it does not take into account the extra rigidity possessed by the column in the bent position.

B. The second alternative coincides with Shanley's theory, which in turn leads to the contradiction that stress reversal cannot take place simultaneously with the increase of axial load from the very start of bending. This contradiction in Shanley's theory has been pointed out, here for the first time, in this paper.

C. The third alternative has not been mentioned by previous investigators. The investigation of this alternative is important, because it is the last alternative which predicts the start of bending at the tangent modulus load. This alternative avoids the contradiction inherent in Shanley's theory and allows equilibrium sequences in the bent position. This third alternative also leads to a contradiction which will be explained below.

Before explaining the contradiction in the third alternative, some of the fundamental concepts of the elastic stability will be reviewed in brief. Consider a centrically loaded, initially straight, elastic column with hinged ends. See Fig. 7.

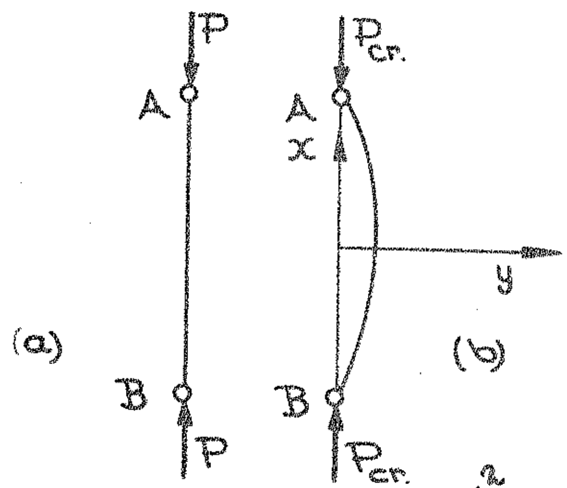


Fig. 7

The critical Euler load, $P_{cr} = \frac{\pi^2 EJ}{l^2}$, refers to a stage of loading where, due to an infinitesimal lateral deflection of the column, the work done by the load P because of a shortening of the distance AB just exceeds the elastic energy stored in the column due to bending. The limitation of the above formula is that, in its derivation, the curvature of the column is approximated by the term, $\frac{d^2y}{dx^2}$ which can be considered as valid for small lateral deflections. In general the increase of axial load P, from an infinitesimal value below the critical load to an infinitesimal value above the critical load, is accompanied by a change in the state of energy of the system which is composed of the load and the column. Once the load has reached the

the critical value, the change in the state of energy of the system is independent of time; this concept means that for the infinitesimal range of lateral deflections, where the formula for the critical load is theoretically correct, buckling takes place instantaneously.

Now, suppose that the column, shown in Fig. 7, is elasto-plastic. The tangent modulus load is the smallest critical load if and only if the tangent modulus applies to the whole column. This is true, if and only if the increase of axial load compensates for the decrease in the stresses on the convex side of the column in the bent position. This means in reality that the cause of lateral deflection is the increase of axial load. However, the increase of axial load beyond the tangent modulus load is dependent on time, whereas the assumed loss of stability at the tangent modulus load is independent of time. This argument leads to the conclusion, that no matter how fast the rate of loading beyond the tangent modulus load, the rate of the assumed lateral deflection must be faster, which idea results in the contradiction that the column in the bent position must be free from the effect of increasing axial load.

The following problem in mechanics could be considered to be analogous to the above argument.

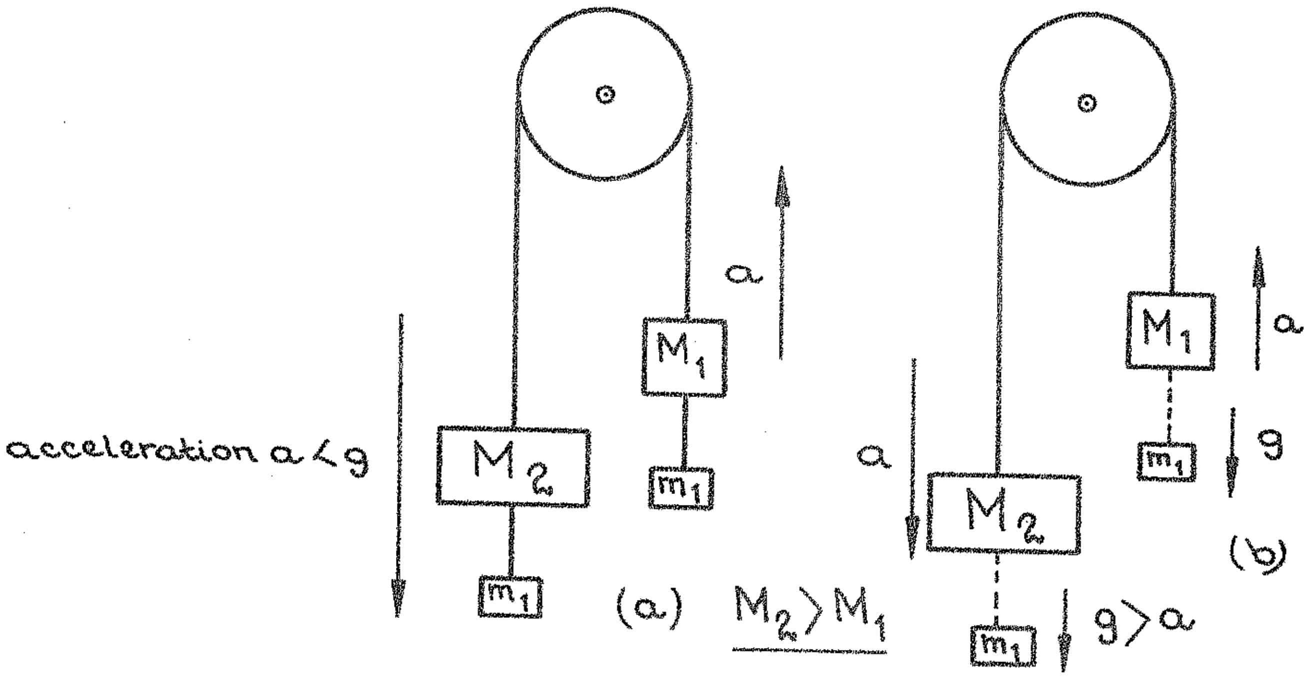


Fig. 8

The frictionless pulley, shown in Fig. 8 is fixed at the point O, the system of masses, m_1, M_1, and M_2 are attached by frictionless strings, as shown in Fig. 8. The masses M_2 and m_1, on the left side of the pulley, accelerate downward with a constant acceleration $a < g$, where g denotes the acceleration of the field in which the system is situated (Fig. 8 a). If now the two masses m_1, to the left and right of the system, were to be cut off simultaneously, in an identical way, they would fall freely downwards, with an acceleration $g > a$, whereas the downward acceleration of the mass M_2 would remain unchanged, that is to say equal to a (Fig. 8 b). If one observes the separation of the two masses m_1 and M_2, one cannot attribute the incident to the action of the mass M_2. In other words, the mass M_2, with a smaller rate of change of velocity than the gravitational acceleration, cannot cause the free fall of the mass m_1.

By an identical reasoning as above, we conclude that an elasto-plastic column can not buckle at the tangent modulus load, since the incident of buckling is a critical stage which corresponds to a change in the state of energy of the system. It is independent of time, taking place instantaneously. Thus, the time-dependent increase of axial load never finds the opportunity to prevent the column from getting the extra rigidity at the bent position. The above argument can be summarized by the statement that

An effect can not be initiated by a cause that does not find the opportunity to create that effect.

25. Conclusion

We conclude that there is no alternative possibility, where the tangent modulus can be considered as the smallest critical load. For a centrically loaded, initially straight column, there exists only one critical load, greater than the tangent modulus load. The paradoxical idea of bending or buckling, at the tangent modulus load is not correct.

Thus, the fourth alternative that the column possesses only one critical load, greater than the tangent modulus load, is theoretically the only valid alternative.

On the basis of the above conclusions, we realize that the procedure originally followed by Von-Kármán is not a special case of Shanley's theory, as admitted by Von-Kármán himself later. On the contrary, Von-Kármán's original method of calculating column rigidity in the bent position under the action of a constant load is theoretically correct. However, Von-Kármán, in his approach to the problem, failed to realize a very important property of elasto-plasticity, viz, that the unloading-line is not parallell with the initial tangent to the stress-strain curve. The unloading modulus, which Von-Kármán took as a constant, is in reality a variable which depends on the position of the point of unloading and the shape of the stress-strain curve.

After having proved the contradictions in all the existing inelastic column theories, and having realized that the column has only one critical load, the author can now proceed to develop a theory of elasto-plasticity of his own. The primary aim of this theory is to determine the nature of the unloading process and to develope equations by which the unloading modulus can be calculated at any point on the stress-strain diagram. The results of the investigations in the following sections of this paper are applied to the author's new theory of elasto-plastic stability with the final conclusion that the critical load for an elasto-plastic column always lies between the tangent modulus load and the value predicted by the reduced modulus theory.

3. New Theory of Elasto-Plasticity for the Determination of the Unloading-Modulus of Elasticity

31. Short introduction

The following chapter is devoted to the development of a theory of elasto-plasticity for determination of the equation of the unloading-line. Section 32 begins with a discussion of fundamental concepts and ideas leading to the formulation of theory proposed here. The essential ideas and concepts of elasto-plasticity are introduced and defined, in connection with a detailed discussion of load-deformation properties of a structural mathematical model, built up in cross section by three distinct layers of homogeneous ideal elasto-plastic media, with the same modulus of elasticity, but with different yield-point stresses. By discussing the unloading process, it is demonstrated that the unloading curve coincides with a straight line. The slope of the unloading-line with respect to the strain-axis is called the "unloading modulus of elasticity", or simply, the "unloading-modulus".

Section 33 is devoted to the development of a generalized mathematical model. It is proved that, for any given well defined mathematical model, one can always determine its load-deformation characteristic, and conversely, for any given stress-strain diagram one can determine a mathematical model which can produce an identical stress-strain diagram. Furthermore, the condition of equivalence of stress-strain diagram between the mathematical and physical models is established.

Section 34 is devoted to the various aspects of the transformation of a non-homogeneous physical model to its homogeneous mathematical counterpart. Two types of problems of elasto-plasticity are discussed, in general. The first type of problems is encountered when a physically well defined model is given, and when one wishes to determine its load-deformation characteristic during both loading and unloading processes. The second type of problems is encountered when an experimental stress-strain diagram for the loading of a physical model is given and when one desires to determine its stress-strain relationship during the unloading-process. The general solution of these two types of problems is thoroughly discussed.

In Section 35 a simple space-truss is presented as the model for a simplified crystal unit. A detailed calculation is carried out to demonstrate the various steps involved in transforming a physical model to its mathematical equivalent. The calculations further illustrate the general concepts introduced in Section 33 and 34 and review the fundamental concepts presented in Section 32.

Section 36 is devoted to the derivation of the unloading-modulus for the most general case of an elasto-plastic physical model which can have any stress-strain diagram. In Section 37 the variational possibilities of the unloading modulus, along any stress-strain curve, is investigated. It is proved, in general, that the unloading modulus is always less than the initial modulus of elasticity. In Section 38 a numerical method for calculation of the unloading modulus is introduced, and finally in Section 39 a numerical example based on an actual experimental stress-strain curve is worked out. Sections 36 to 39 all treat the problems of the second type, discussed in Section 34.

32. A discussion of fundamental concepts, with a detailed study of the load-deformation properties of a structural mathematical model

We begin the discussion by posing the following questions:

Is the unloading-curve a straight line?

If it is a straight line, is it parallel with the initial tangent to the stress-strain diagram?

If not parallel with the initial tangent, has it always the same slope regardless of at which point on the stress-strain diagram the unloading begins?

What is the equation of the unloading-line?

To answer the above questions we have to establish a mathematical theory of plastic flow, which would be equivalent to the physical reality. To make the model for such a theory we choose the concept of a continuous, homogeneous and deformable medium which does not necessarily have to be isotropic, because it is understood that we study the stress-strain diagram of the medium in the direction of the applied force, for which no condition of isotropy is required. It is assumed that the medium deforms elastically up to a certain loading level, after which either the deformations continue to increase under constant loading (Fig. 9 a), or the load continues to decrease with constant deformation (Fig. 9). Again it is assumed that all the deformations take place

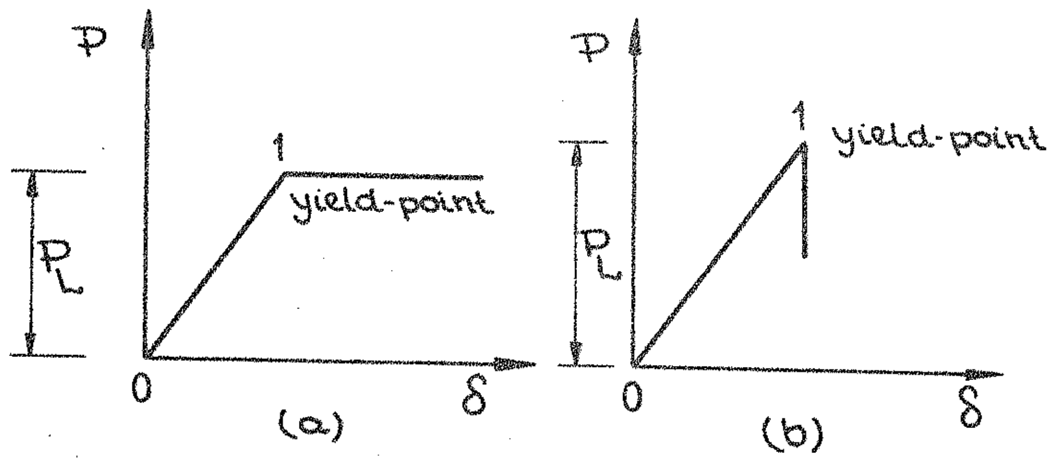


Fig.9

under isothermal conditions in order to avoid thermal effects. In Fig. 9 (a) complete plasticization takes place in the medium at Point 1; after plasticization the medium preserves the load P_L and continues to deform without any further increase of loading. We define such a medium as an ideal elasto-plastic medium. In Fig. 9 (b), the medium plasticizes completely at Point 1; after plasticization the load decreases at a constant deformation. The decrease of the load at Point 1 cannot take place instantaneously for that would mean an abrupt decay or separation of parts of the medium; this, however, is not the case. Thus the decrease of loading, with constant deformation, must be a function of time. In reality, all elasto-plastic materials demonstrate this phenomenon which is known as creep. For the present study, however, we assume the loading to take place in such a time span that the effects of creep in the material are negligible for all practical purposes. Thus, we construct our model on the basis of a homogeneous, deformable, ideal-elasto plastic material with the load-deformation diagram, shown in Fig. 9 (a).

From the outset it should be pointed out that the condition of isothermality requires slow loading; whereas a situation in which creep deformation is negligible demands that the loading be done as quickly as possible. These two conditions seem to contradict each other; however, both conditions can be satisfied if one considers the fact that the time required for the loading to assure a reasonable isothermality for a structure with ordinary dimensions is much shorter than the time required for the creep effects to become appreciable. Therefore we construct the following model based on homogeneous ideal elasto-plastic medium under isothermal conditions.

Consider the following three structural models, with cross sectional areas shown in Fig. 10.

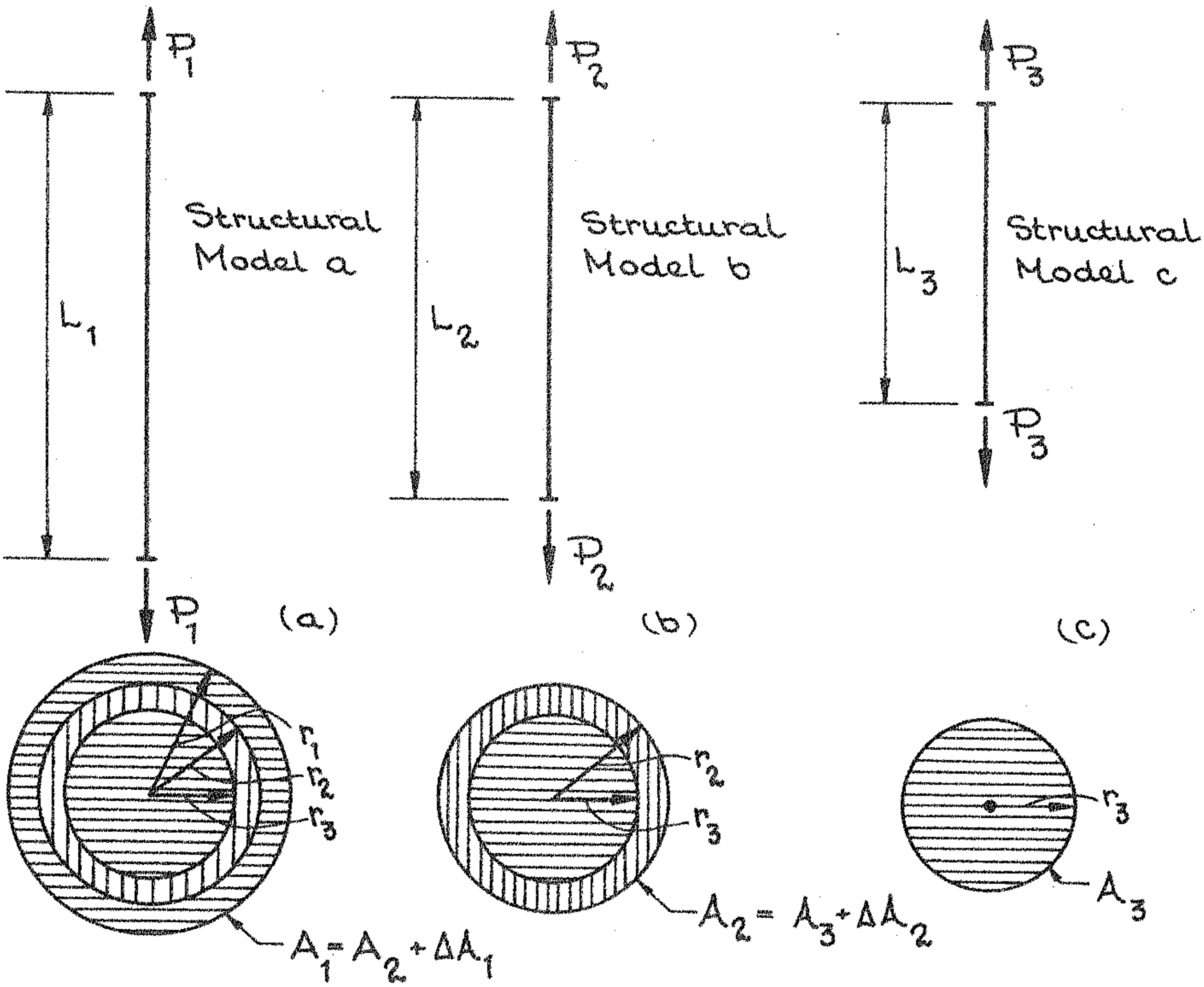


Fig. 10

Model c has a solid circular cross sectional area, A_3, with radius r_3, shown in Fig. 10 (c). The cross section of Model b is built up from the cross section of Model c with the addition of an outer ring having the inner radius r_3 and the outer radius r_2. The cross sectional area of this outer ring is designated by ΔA_2, while the total built-up section forms an area equal to A_2, shown in Fig. 10 (b). The cross sektion of Model a is built up of the cross section of Model b with the addition of an outer ring having the inner radius r_2 and the outer radius r_1. The cross sectional area of this outer ring is designated by ΔA_1, while the total built-up section forms an area equal to A_1, shown in Fig. 10 (a). All the joints between various built-up sections are supposed to be so strong as to prevent relative sliding between joined surfaces during loading. We assume that the solid part with cross section A_3 and the hollow parts with cross sectional areas ΔA_2 and ΔA_1 are built of homogeneous, ideal elasto-plastic media with the same modulus of elasticity but with varying yield-point stresses. We assume, further, that A_3 has a higher yield-point stress than ΔA_2, while ΔA_2 has a higher yield-point stress than ΔA_1. This means that by denoting yield-point stress by σ_y, the following inequality holds:

$$\sigma_{y(A_3)} > \sigma_{y(\Delta A_2)} > \sigma_{y(\Delta A_1)} \qquad (9)$$

Designating the strain by ε the stress by σ, and the centrally applied load by P, the following three equations hold for the three models, within the range of their elastic deformation:

$$\varepsilon_a = \frac{\sigma_a}{E} = \frac{P_a}{A_1} \cdot \frac{1}{E} \qquad (10)$$ Load-strain relationship for model a

$$\varepsilon_b = \frac{\sigma_b}{E} = \frac{P_b}{A_2} \cdot \frac{1}{E} \qquad (11)$$ — " — " — " — " — " b

$$\varepsilon_c = \frac{\sigma_c}{E} = \frac{P_c}{A_3} \cdot \frac{1}{E} \qquad (12)$$ — " — " — " — " — " c

A graphical representation of the above load-strain relationships within the range of elastic deformations of the models are given in Fig. 11.

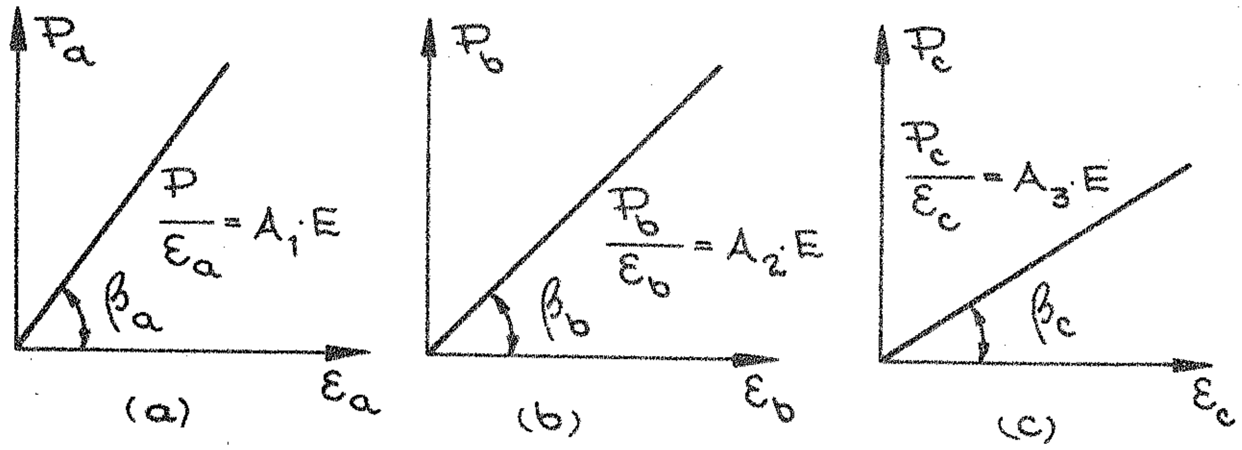


Fig. 11

If, in the case of Model a, we were to continue loading beyond the range of its elastic deformations, shown in Fig. 12, the inequality (9) would show that the first breaking point would occur at a stress equal to $\sigma_{y \Delta A_1}$, at which point the outer ring, ΔA_1 plasticizes completely. Point 1 in Fig. 12 correspons to this stage of loading. At infinitesimal distance to the left of Point 1, all the initial area, A_1, is effective in carrying the load, while an infinitesimal distance to the right of Point 1, a part of the initial area, equal to ΔA_1, is plasticized. Any further increase in loading beyond Point 1 has to be carried by the non-plasticized effective area, equal to

$$A_1 - \Delta A_1 = A2$$

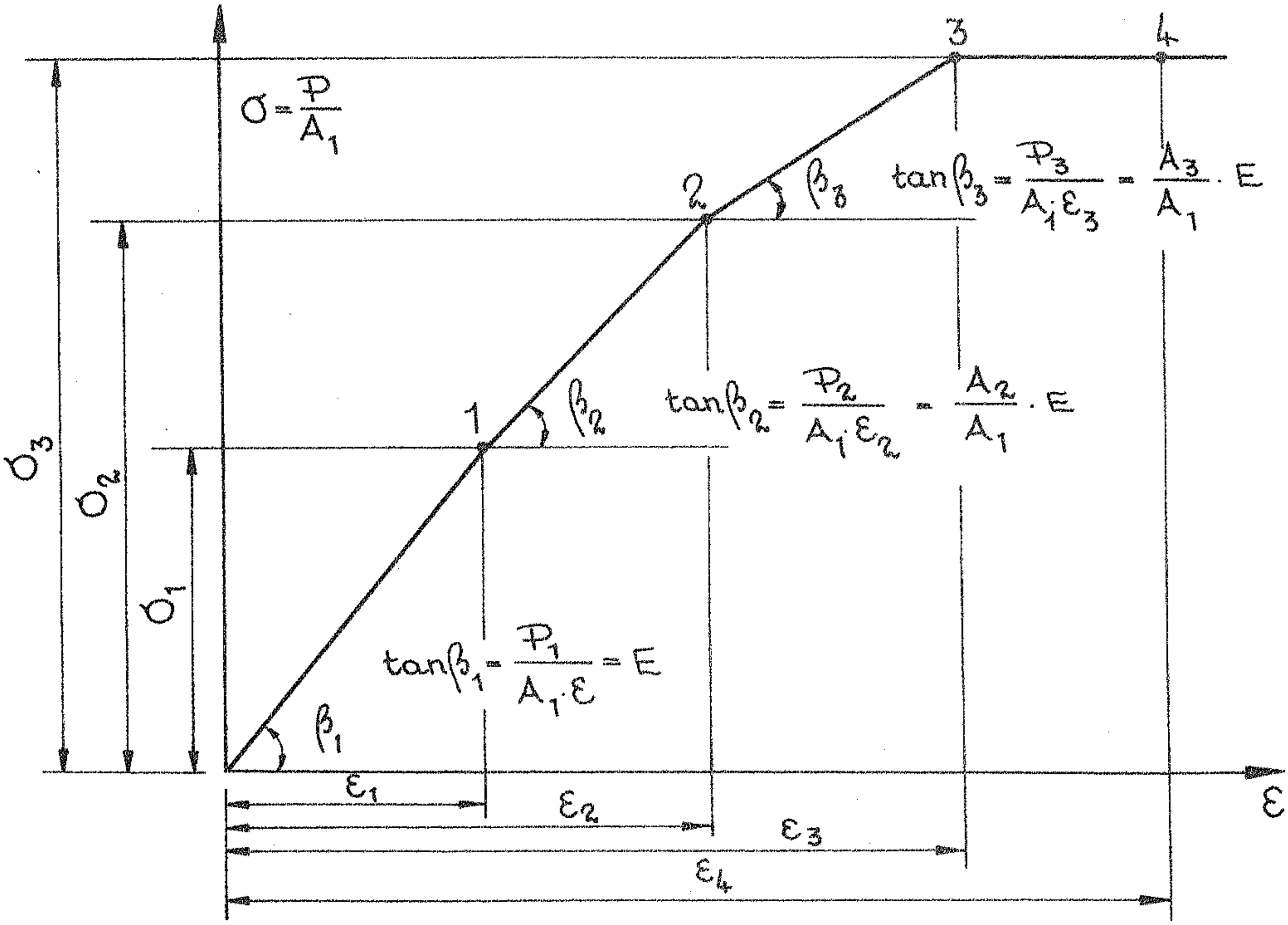


Fig. 12

Stress-strain diagram for Structural Model a

As the load continues to increase, the second breaking point occurs at Point 2, where the middle ring, ΔA_2, becomes completely plasticized. Any further increase in loading has to be carried by the remaining non-plasticized effective area, equal to $A_2 - \Delta A_2 = A_3$. The third breaking point occurs at Point 3, where the last effective area, A_3, becomes plasticized. After Point 3, the column is unable to bear any more load.

In connection with the model, just studied, the following new terms and concepts are introduced in technical literature.

Part of the original area which becomes plasticized, just to the right of Point 1, is called the "plastic area" at Point 1, A_{P1}; while the force carried by this area is called the "plastic force" at Point 1, P_{P1}; the corresponding stress is then called the "plastic stress" at Point 1, σ_{P1}. The corresponding terms for the non-plasticized part of the area are called the "elastic area", "elastic force" and "elastic stress" at Point 1, denoted by A_{e1}, P_{e1} and σ_{e1} respectively. For the other points, 2 and 3, we use the same terms and notations mutatus mutandis. Thus, the elastic area at a point is equivalent to the "effective area" at that point.

With the above definitions, we proceed to calculate the condition of elasto-plasticity at Points 1, 2 and 3 in Fig. 12.

1. Condition at Point 1

a) An infinitesimal distance to the left of the point:

Elastic area = Effective area = A_1

Plastic area = 0

Elastic stress = σ_1

Elastic force = $A_1 \cdot \sigma_1$

Plastic stress = 0

Plastic force = 0.

b) An infinitesimal distance to the right of the point:

Elastic area = Effective area = $A_1 - \Delta A_1 = A_2$

Plastic area = ΔA_1

Elastic stress = σ_1

Elastic force = $A_2 \cdot \sigma_1$

Plastic stress = σ_1

Plastic force = $A_1 \cdot \sigma_1$.

2. Condition at point 2

a) An infinitesimal distance to the left of the point:

Elastic area = Effective area = A_2

Plastic area = ΔA_1

Elastic stress = σ_2

Elastic force = $A_2 \cdot \sigma_2$

Plastic stress = σ_1

Plastic force = $A_1 \cdot \sigma_1$

b) An infinitesimal distance to the right of the point:

Elastic area = Effective area = $A_2 - \Delta A_2 = A_3$

Plastic area = $\Delta A_1 + \Delta A_2$

Elastic stress = σ_2

Elastic force = $A_3 \cdot \sigma_2$

Plastic stress = σ_1 on ΔA_1, and σ_2 on ΔA_2

Plastic force = $\sigma_1 \cdot \Delta A_1 + \sigma_2 \cdot \Delta A_2$

3. Condition at Point 3

a) An infinitesimal distance to the left of the point:

Elastic area = Effective area = A_3

Plastic area = $\Delta A_1 + \Delta A_2$

Elastic stress = σ_3

Elastic force = $A_3 \cdot \sigma_3$

Plastic stress = σ_1 on ΔA_1 and σ_2 on ΔA_2

Plastic force = $\Delta A_1 \cdot \sigma_1 + \Delta A_2 \cdot \sigma_2$.

b) An infinitesimal distance to the right of the point:

Elastic area = Effective area = $A_3 - A_3 = 0$

Plastic area = $\Delta A_1 + \Delta A_2 + A_3 = A_1$

Elastic stress = 0

Elastic force = 0

Plastic stress = σ_1 on ΔA_1, σ_2 on ΔA_2 and σ_3 on A_3

Plastic force = $\Delta A_1 \cdot \sigma_1 + \Delta A_2 \cdot \sigma_2 + A_3 \cdot \sigma_3$.

The ratio of effective areas between the two Points, i, and j, on the stress-strain diagram will be denoted by R_{ij}. Using the notations, A_{ei} for the elastic area at Point, i, and A_{ej} for the elastic stress at Point, j, we get

$$R_{ij} = \frac{A_{ei}}{A_{ej}} \tag{13}$$

Applying Eq. (13) to Fig. 12, we get the following ratios between the effective areas, to the right of the breaking points, and the initial effective area:

$$R_{10} = \frac{A_{ei}}{A_{eo}} = \frac{A_2}{A_1} = \frac{\tan\beta_2}{\tan\beta_1}; \tag{14}$$

$$R_{20} = \frac{A_{e2}}{A_{eo}} = \frac{A_3}{A_1} = \frac{\tan\beta_3}{\tan\beta_1}; \tag{15}$$

$$R_{30} = \frac{A_{e3}}{A_{eo}} = \frac{0}{A_1} = \frac{\tan 0}{\tan\beta_1} = 0. \tag{16}$$

In order to complete investigations of the stress-strain diagram, shown in Fig. 12, we proceed with our calculations for determining the unloading-modulus at Points 1, 2, 3, and 4. However we must first discuss the concept of loss of energy due to plasticization.

As soon as a part of the effective area becomes plasticized during loading, the cross section loses part of its resistance against deformation. This is because the plastic area, while retaining its yield-point load, can not take up any extra loading. All the work done on the plasticized area, beyond its yield-point deformation, will be converted into heat energy, which can not be recovered during the process of unloading. However, the energy stored in the structure, in the form of elastic energy, will be recovered during unloading. Thus, the energy developed due to the action of plastic forces, W_f, is irreversible, while the energy stored in the structure, due to the action of elastic forces, W_e, is reversible. The total work done on the structure, W_t, is equal to the sum of the reversible and irreversible energies:

$$W_t = W_e + W_f \tag{17}$$

During the process of unloading, all the potential energy, stored in the structural model, in the form of elastic energy, will be recovered. No further plasticization will take place during this period. The path of unloading will be a straight line, because the energy to be recovered is the sum of all the elastic energy, due to the action of the elastic forces during the loading process. Since there will be no plasticization as far as unloading is concerned, there will be no difference, whatsoever, in the structural behaviour of an elasto-plastic, or an ideal elastic material. Because of the elastic behaviour of the unloading process, the tangent of the unloading-line is called the "unloading-modulus of elasticity" or simply, the "unloading-modulus".

For example, we consider the unloading of an ideal elasto-plastic material which is elastic under the external load up to a certain limit P_L; beyond this limit the material begins to plasticize. The deformations continue under the constant load P_L. If the unloading begins before P_L is reached, the unloading-line coincides with the loading-line. If the unloading begins after the load P_L is reached, all the energy put in the material, thereafter, is converted to heat energy and consequently the unloading-line is parallel with the loading-line, Fig. 13 (b).

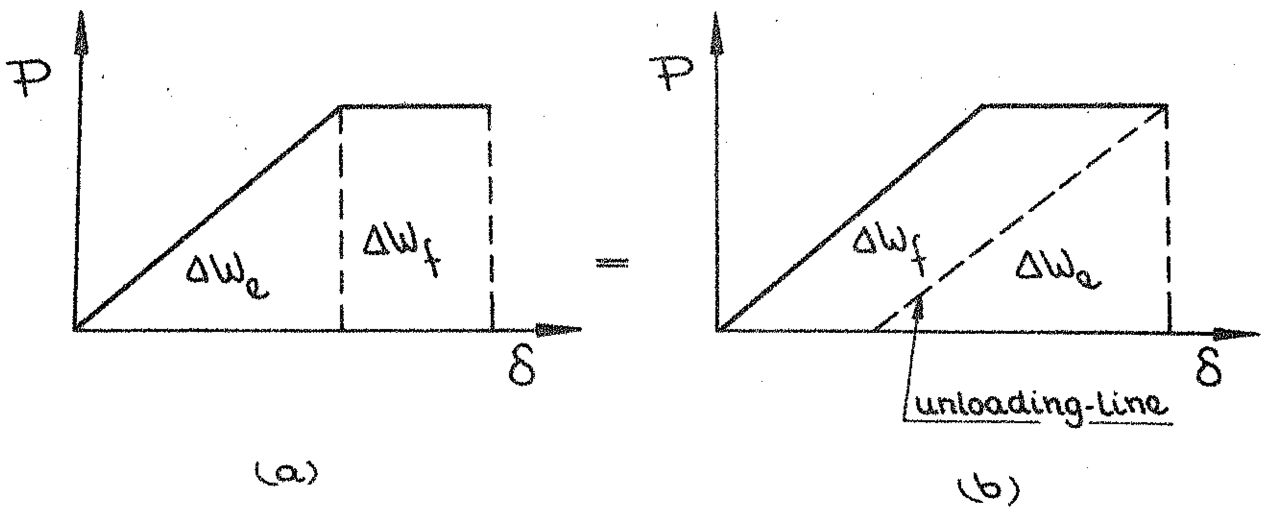


Fig. 13

From the above fundamental energy concepts, we proceed to calculate the unloading modulus, for the model stress-strain diagram, shown in Fig. 12, at Points 1, 2, 3, and 4.

1. Unloading-modulus at Point 1

No plasticization takes place between the origin of coordinate axes, 0, and the Point 1, hence $W_f = 0$, and $W_t = W_e$; consequently the unloading-line coincides with the line 01, and the unloading modulus, E_a, is equal to the initial modulus of elasticity, E.

2. Unloading-modulus at Point 2

As calculated previously, the plastic force, an infinitesimal distance to the right of Point 1, is equal to $\Delta A_1 \cdot \sigma_1$; frictional energy losses, W_{f2}, are then developed between Points 1 and 2.

$$W_{f2} = \Delta A_1 \cdot \sigma \left(\varepsilon_2 - \varepsilon_1\right) \tag{18}$$

The irreversible energy per unit area of the initial cross section will be denoted by S_f, the reversible energy for the same unit area by S_e, and the total energy for the same unit area, by S_t. This last term, S_t, is measured by the area under the stress-strain diagram.

$$S_{f2} = \frac{W_f}{A_1} = \frac{\Delta A_1}{A_1} \cdot \sigma_1 \left(\varepsilon_2 - \varepsilon_1\right) \tag{19}$$

$$S_{t2} = \sigma_1 \cdot \varepsilon_1 /_2 + \sigma_1 \cdot \left(\varepsilon_2 - \varepsilon_1\right) + \left(\sigma_2 - \sigma_1\right) \cdot \left(\varepsilon_2 - \varepsilon_1\right) /_2 \tag{20}$$

see Fig. 12.

$$\varepsilon_2 - \varepsilon_1 = \frac{\sigma_2 - \sigma_1}{\tan\beta_2} = \frac{A_1(\sigma_2 - \sigma_1)}{A_2 \cdot E} \tag{21}$$

see Fig. 12.

$$\varepsilon_1 = \frac{\sigma_1}{\tan\beta_1} = \frac{\sigma_1}{E} \tag{22}$$

substitution in Eqs. (19) and (20) gives:

$$S_{f2} = \frac{\Delta A_1}{A_2 E} \cdot \sigma_1 \left(\sigma_2 - \sigma_1\right) \tag{23}$$

$$S_{t2} = \frac{\sigma_1^2}{2E} + \frac{A_1}{A_2 \cdot E} \cdot \sigma_1 (\sigma_2 - \sigma_1) + \frac{A_1 (\sigma_2 - \sigma_1)^2}{2 A_2 E} \tag{24}$$

The elastic energy per unit area of the cross-section, at Point 2, can be written as

$$S_{e2} = S_{t_2} - S_{f_2} \tag{25}$$

$$S_{e2} = \frac{A_2\sigma_1^2 + A_1\cdot(\sigma_2-\sigma_1)(\sigma_2+\sigma_1) - 2\Delta A\sigma_1(\sigma_2-\sigma_1)}{2A_2\cdot E} \tag{26}$$

For determination of the unloading-modulus, at any Point, i, on any stress-diagram, the following calculations can be carried out, provided that the value of S_{ei} is known:

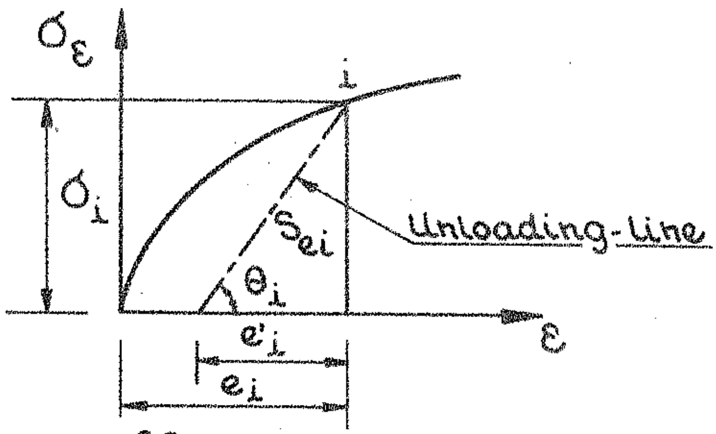


Fig. 14

$$e'_i = \frac{2S_{ei}}{\sigma_i} \tag{27}$$

$$E_{ai} = \tan\theta_i = \frac{\sigma_i}{e'_i} = \frac{\sigma_i^2}{2S_{ei}} \tag{28}$$

For a numerical evaluation we suppose that $A_3 = \Delta A_2 = \Delta A_1$, and that the yield-point stress for ΔA_2 is 1.7 times the yield-point stress for ΔA_1, while the yield-point stress for A_3 is 2.2 times that of ΔA_1,

$\therefore \sigma_2 = 1,7\,\sigma_1$, and $\sigma_3 = 2,2\,\sigma_1$ see Fig. 10

substitution in Eq. (26) gives,

$$S_{e2} = \frac{2A_3\sigma_1^2 + 3A_3\cdot 0,7\sigma_1\cdot 2,7\sigma_1 - 2A_3\cdot\sigma_1\cdot 0,7\sigma_1}{4A_3\cdot E}$$

$$\therefore S_{e2} = \frac{1,57\,\sigma_1^2}{E} \tag{29}$$

using Eq. (28),

$$E_{a2} = \frac{\sigma_2^2\cdot E}{3,14\,\sigma_1^2} = 0,922\,E \tag{30}$$

3. Unloading-modulus at Point 3

As previously calculated, the plastic force has the value $\sigma_1 \cdot \Delta A_1$, an infinitesimal distance to the right of Point 1, and the value $\sigma_1 \cdot \Delta A_1 + \sigma_2 \cdot \Delta A_2$, just to right side of Point 2, developing frictional energy losses, W_{f3} between Points 1 and 3. Carrying out the calculation as for the proceeding point, we get

$$W_{f3} = \sigma_1 \cdot \Delta A_1 \cdot (\varepsilon_2 - \varepsilon_1) + \sigma_1 \cdot \Delta A_1 \cdot (\varepsilon_3 - \varepsilon_2) + \sigma_2 \cdot \Delta A_2 \cdot (\varepsilon_3 - \varepsilon_2). \quad (31)$$

The irreversible energy, S_{f3}, and the total energy, S_{t3} per unit area of the cross section at Point 3, are calculated to be

$$S_{f3} = \frac{\Delta A_1}{A_1} \sigma_1 (\varepsilon_2 - \varepsilon_1) + \frac{\Delta A_1}{A_1} \sigma_1 (\varepsilon_3 - \varepsilon_2) + \frac{\Delta A_2}{A_1} \cdot \sigma_2 (\varepsilon_3 - \varepsilon_2) \quad (32)$$

$$S_{t3} = S_{t2} + \sigma_2 \cdot (\varepsilon_3 - \varepsilon_2) + (\sigma_3 - \sigma_2) \cdot \frac{(\varepsilon_3 - \varepsilon_2)}{2} \quad (33)$$ See Fig. 12.

$$\varepsilon_3 - \varepsilon_2 = \frac{\sigma_3 - \sigma_2}{\tan\beta_3} = \frac{A_1(\sigma_3 - \sigma_2)}{A_3 \cdot E} \quad (34)$$ See Fig. 12.

$(\varepsilon_2 - \varepsilon_1)$ and ε_1 have already been calculated in Eqs. (21) and (22).

Substituting in Eqs. (32), and (33),

$$S_{f3} = \frac{\Delta A_1}{A_2 E} \cdot \sigma_1 (\sigma_2 - \sigma_1) + \frac{\Delta A_1}{A_3 \cdot E} \sigma_1 (\sigma_3 - \sigma_2) + \frac{\Delta A_2}{A_3 E} \cdot \sigma_2 (\sigma_3 - \sigma_2) \quad (35)$$

$$S_{t3} = \frac{A_2 \sigma_1^2 + A_1 (\sigma_2 - \sigma_1)(\sigma_2 + \sigma_1)}{2 A_2 \cdot E} + \frac{A_1 (\sigma_3 - \sigma_2)(\sigma_3 + \sigma_2)}{2 A_3 \cdot E} \quad (36)$$

using the same numerical data as before, viz., $\Delta A_1 = \Delta A_2 = A_3$, $\sigma_3 = 2{,}2\,\sigma_1$, and $\sigma_2 = 1{,}7\,\sigma_1$, we get

$$S_{f3} = \frac{\sigma_1 \cdot 0{,}7\sigma_1}{2E} + \frac{\sigma_1 \cdot 0{,}5\sigma_1}{E} + \frac{1{,}7\sigma_1 \cdot 0{,}5\sigma_1}{} = \frac{1{,}70\,\sigma_1^2}{E} \quad (37)$$

$$S_{t3} = \frac{2\sigma_1^2 + 3 \cdot 0{,}7\sigma_1 \cdot 2{,}7\sigma_1}{2E} + \frac{3 \cdot 0{,}5\sigma_1 \cdot 3{,}9\sigma_1}{2E} = \frac{4{,}835\,\sigma_1^2}{E} \quad (38)$$

The elastic energy per unit area of the cross section at Point 3, S_{e3}, is calculated to be

$$S_{e3} = S_{t3} - S_{f3} = \frac{3{,}135\,\sigma_1^2}{E} \quad (39)$$

Using Eq. (28),

$$E_{a3} = \frac{\sigma_3^2 \cdot E}{2 \cdot 3{,}135\,\sigma_1^2} = 0{,}722\ E \quad (40)$$

4. Unloading-modulus at Point 4

At an infinitesimal distance to the right of Point 3, all the initial effective area is plasticized; hence, for any deformation beyond Point 3, all the work done on the model structure will turn into frictional losses. Consequently the unloading-line, at any point beyond Point 3, will be parallel to the unloading-line at Point 3. See the unloading-line for the ideal elasto-plastic model, shown in Fig. 13.

Determination of coordinates of Points 1, 2, 3, and 4 on the stress-strain diagram of the structural model.

To draw the unloading-lines in relation to the stress-strain diagram, the exact coordinates of points 1, 2, 3 and 4 must be calculated. For this purpose, we use the same numerical data used for calculating the unloading-muduli, viz., $\Delta A_1 = \Delta A_2 = A_3$, $\sigma_2 = 1{,}7\sigma_1$, and $\sigma_3 = 2{,}2\sigma_1$

Referring to Fig. 12 and using the above numerical data,

$$\tan\beta_1 = E \tag{41}$$

$$\tan\beta_2 = \frac{A_2}{A_1} \cdot E = 2/3 \cdot E \tag{42}$$

$$\tan\beta_3 = \frac{A_3}{A_1} \cdot E = 1/3 \cdot E \tag{43}$$

The stress-strain diagram, together with the unloading-lines, calculated for the structural model is presented in the following diagram:

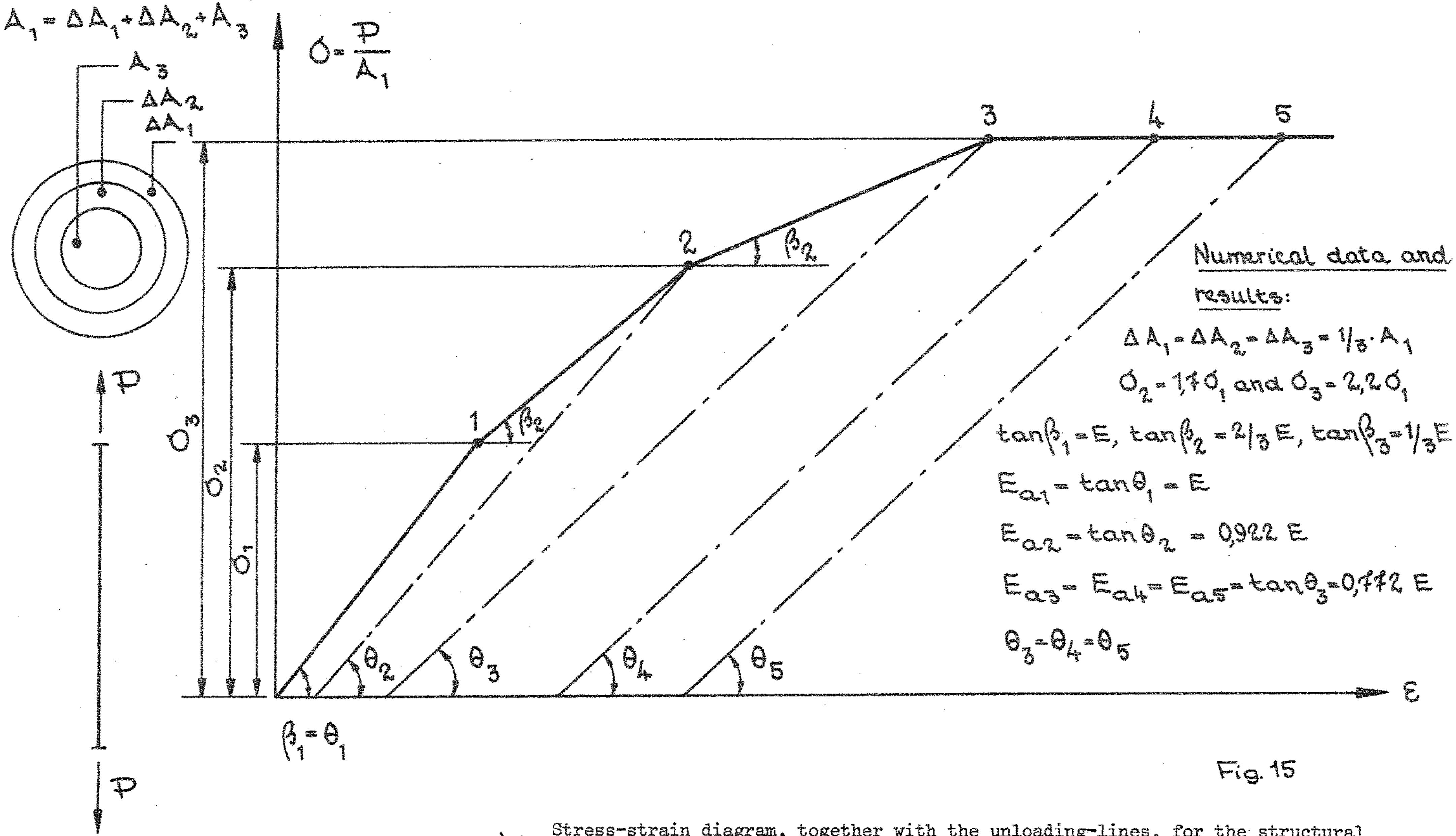


Fig. 15

a) Structural model with its cross section

b) Stress-strain diagram, together with the unloading-lines, for the structural model, built up in cross section with three layers of homogeneous, ideal elasto-plastic media, with the same modulus of elasticity, but with varying yield-point stresses.

33. Generalization of the structural mathematical model

The model studied in the previous section was used for introducing the fundamental concepts of elasto-plasticity. We can generalize the results of the preceeding discussion by the following argument. Let the cross sectional area of the structural model be denoted by A_0, consisting of n layers of incremental area, ΔA_i, built up by homogeneous ideal elasto-plastic media, with the same modulus of elasticity but with different yield-point stresses. Thus the area A_0 can be represented by the following relation:

$$A_0 = \sum_{i=1}^{n} \Delta A_i \tag{44}$$

In this case n is any positive integer.

In the previous section we assumed that the yield-point stress for A_3 is greater than the yield-point stress for ΔA_2, and the yield-point stress for ΔA_2 is greater than the yield-point stress for ΔA_1. This assumption was presented through the inequality, $\sigma_{yA3} > \sigma_{y\Delta A_2} > \sigma_{y\Delta A_1}$. However, we did not restrict the position of the incremental areas inside the cross section. In fact any of the incremental areas could be considered as constituting any of the three possible layers, the inner layer, the middle layer, or, the outer layer, without changing the load-deformation properties of the model. In this way, the incremental areas inside the cross section of the model, studied in the previous section, could be arranged in 3 ! = 6 different ways, without affecting the final results.

In an identical way, we define the yield-point stresses for the incremental areas, over the cross section of the generalized structural model, by the following inequality:

$$\sigma_{y\Delta A_n} > \sigma_{y\Delta A_{(n-1)}} > \sigma_{y\Delta A_{(n-2)}} > \cdots\cdots > \sigma_{y\Delta A_1} \tag{45}$$

The n differential elements of area, can be arranged in n! different ways, over the cross section. Choosing one arrangement out of the n! possible permutations, we realize that, by choosing the number n large enough, and by giving proper values to the yield-point stresses, we can reconstruct any arbitrary stress-strain diagram to any degree of accuracy. This means that, given any model, we can determine its load-deformation behaviour and conversely, given, any load-deformation behaviour, that is to say, any stress-strain diagram, we can determine and reconstruct the model which can produce an identical load-deformation behaviour. The converse procedure, that is to say the equivalence of the experimental and the model stress-strain diagrams, is true if and only if there exists a one to one correspondence between the two sets, consisting of the two stress-strain diagrams. This equivalence relation holds as long as creep deformations or thermal effects do not begin to change the form of the experimental stress-strain diagram.

The fact that the arrangement of the incremental areas over the cross section of the model is immaterial is significant, because it allows us to conceive a plasticization process which takes place continuously over the whole cross section simultaneously. To clarify this notion, consider the rectangular cross section A_0, shown in Fig. 16

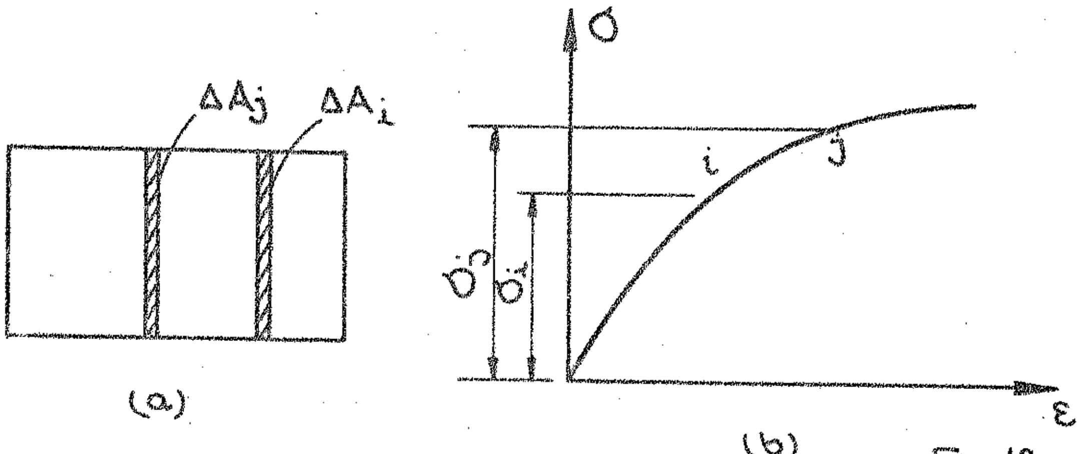


Fig. 16

suppose that the structural model with the cross section as shown in Fig. 16 (a) is loaded centrally and a stress-strain diagram, as shown in Fig. 16 (b), is obtained. We can construct a structural model, by assuming, for example, that the element of area ΔA_i gets plasticized at a stress equal to σ_i, while another element, such as ΔA_j, gets plasticized at a stress equal to σ_j. However, because the arrangement of the elements of area over the cross section is immaterial, we can conceive that the weaker and stronger elements are mingled and spread out over the whole cross section, in a uniform way. In this way, one can imagine that each one of the n incremental areas, such as ΔA_i, is stretched and uniformly spread out, covering the whole cross section as a membrane.

From the above argument, we can conceive of a cross section of the structural model, covered by n ideal elasto-plastic membranes, each of which reaches the yield-point stress at a certain stage of loading. We can proceed a step further, by conceiving the membranes, not as rigidly separate layers, but as a single entity, in which the properties of all membranes are uniformly mingled and spread out identically throughout the section. Such a hypothetical model is significant, since it accounts for continuous and uniform plasticization over the whole cross section. Such a model allows for introducing the new concept of rate of plasticization, identical to the rate of decrease of the effective area. Thus, we construct a structural model which does not only approximate the experimental stress-strain diagram to any degree of accuracy but rather theoretically coincides with it.

34. Transformation of a non-homogeneous physical model to its homogeneous mathematical equivalent

Based on the above purely mathematical concepts, we can construct a model which coincides with any given stress-strain curve without any postulation concerning the physical causes of plastic flow. For the sake of completeness, however, we may try to demonstrate how a physically non-homogeneous elasto-plastic medium can be transformed to a mathematically homogeneous elasto-plastic model. In the structural model studied in the previous section, we assumed that the incremental areas, composing the cross section, have the same modulus of elasticity with different yield-point stresses. By gradually developing that concept we have arrived at a generalized structural model with n ideal elasto-plastic membranes which are uniformly stretched out over the whole cross section. Despite the incremental areas of the model in the previous section, the hypothetical membranes of the generalized model do not necessarily have to possess different yield-point stresses. They are restricted, on the other hand, in a more generalized sense so that each membrane reaches the yield-point stress at a different stage of loading. This last statement is important, because, it accounts for two possible alternatives: in the first alternative it is assumed that different membranes have different yield-point stresses, as was the case for the incremental areas of the model in the previous section;

in the second alternative it is assumed that all the membranes have the same yield-point stress, assuming that they are not stressed identically. The second alternative is practically impossible for a mathematically homogeneous medium. For a physically non-homogeneous medium, on the other hand, the second alternative reflects a picture of the physical reality, for the following reasons:

An elasto-plastic material is thought to be made up of tiny molecular elements which constitute the building stones inside the crystals. The position and the inclination of the molecular elements define the geometric configuration of the crystal structure. All the molecular elements are supposed to have the same modulus of elasticity, which can be considered as the average value for the material. When a continuously increasing external load is centrally applied to a test sample of an elasto-plastic material with the nominal cross sectional area A, the stress on the cross section is arbitrarily measured by dividing the force by the nominal cross sectional area and is defined as the average stress. Nevertheless, the average stress is only a quantitative criterium, which does not give an actual picture of the stress-distribution over the cross section. In reality, all the molecular elements, crossing a certain section of the test sample, cooperate in carrying the load, even if they are not stressed identically, because, different elements have different inclinations with respect to the direction of the external force. The stress-distribution is a function of the geometric configuration of the crystals and their orientation with respect to the direction of the force.

Thus, with the above physical picture of the orientation of the crystal grains over the cross section, we can establish a correspondence between the ideal elasto-plastic membranes of the homogeneous mathematical model and the crystal grains of the non-homogeneous physical model by the following observation: During loading of an actual model each single crystal grain or group of crystal grains, which are stressed identically, corresponds to a membrane of the mathematical model. Those crystal grains which are under greater stresses are affected by plasticization at a lower stage of loading than those crystal grains which are under smaller stresses. In the following lines, we demonstrate how a non-homogeneous physical model can be transformed to a homogeneous mathematical model. The aim of the mathematical derivations which follow is to establish an equivalence relation between the two models, despite the fact that in the sections to follow the calculations will be based only on the mathematical model.

The concept of the effective area was defined, in the previous section, to be identical with the elastic area at any stage of loading. For the mathematical structural model the initial effective area is equal to the total area of the cross section at zero load. For the physical model, on the other hand, the monimal area, measured by the units of area covering the cross section, is proportional but not equal to the initial effective area. The proof of the last statement can be deduced from the following definition of effective area for the physical model: Among all the elements, crossing a certain section of the model, the first molecular element is singled out at random. All the other elements are replaced by imaginary ones, parallel to the direction of the first element, with fictitious cross sectional areas, chosen in such a way that the deformation properties of the model remain unchanged. The total cross sectional area presented in this way is called the total effective area. For the convenience of formulation the first molecular element, chosen at random, is taken, below, to be parallel with the direction of the external force.

The nominal cross sectional area, A, and the initial effective area, A_0, can be defined according to the following relationships:

$$A = \sum_{i=1}^{N} a_i \qquad (46)$$

$$A_0 = \sum_{i=1}^{N} a_{0i} \qquad (47)$$

N, in the above equations, is a very big number, which represents the total number of molecular elements passing through the cross sectional area of the test-sample. a_i, in Eq. (46) represents the cross sectional area of the element i, projected on the plane of the whole cross section.

Among the N elements, passing through the whole cross section, there are a certain number which are parallel with the direction of the applied external force. The element i is assumed to be one such element. Now the other N - 1 elements are replaced by N - 1 immaginary elements, with ficticious areas, a_{01}, a_{02}, a_{03}, ... $a_{0(i+1)}$, ... a_{0N}. The immaginary elements are taken to be parallel with the element i and the ficticious areas a_{01}, a_{02}, a_{03}, ... $a_{0(i+1)}$, ... a_{0N}, are chosen, in such a way as, to give the same deformation characteristic to the physical model at all stages of loading. According to the above discussion, the element i is replaced by itself, in other words, $a_i = a_{0i}$. The values of the ficticious areas of the other N - 1 elements depend solely on the internal geometric configuration of the crystal structure inside the material. For example the value of the ficticious area of the element i + 1 is a function of the internal angles of the crystal unit in which the element i + 1 is situated. The ficticious area, also, is a function of the angle which the element makes with the direction of the force. However the geometry and the orientation of crystal unit for a given material at a given equilibrium are definitely known in nature notwithstanding the fact that we do not know their numerical values. Thus, expressed in terms of a_i, the following relationships can be written, for the ficticious areas:

$$a_{01} = m_1 \cdot a_i, \quad a_{02} = m_2 \cdot a_i, \quad a_{03} = m_3 \cdot a_i,$$
$$\ldots a_{0i} = a_i, \quad \ldots \quad a_{0N} = m_N \cdot a_i$$

Thus the initial effective area, A_0, given by Eq. (47), can be rewritten as

$$A_0 = a_i \cdot \sum_{j=1}^{N} m_j \tag{48}$$

The nominal area, A, given in Eq. (46), can also be expressed in terms of a_i:

$$A = a_i \cdot \sum_{j=1}^{N} n_j \tag{49}$$

m_j, and n_j, in Eqs. (48) and (49), are constants.

From Eqs. (48), and (49) we get

$$A_0 = \frac{\sum_{j=1}^{N} m_j}{\sum_{j=1}^{N} n_j} \cdot A = M \cdot A \tag{50}$$

where

$$M = \frac{\sum_{j=1}^{N} m_j}{\sum_{j=1}^{N} n_j} \tag{51}$$

M is a constant, and is a function of the crystal structure inside the physical model, which in actuality can be very complicated.

Eqs. (46) to (51) indicate the steps which must be taken to transform a physically non-homogeneous model to a mathematically homogeneous model. Eq. (50) demonstrates that the initial effective area, A_0, for a physical model, is proportional to the nominal cross sectional area, A. Once the initial effective area, A_0, is determined, the model can be regarded as a mathematical one, and the necessary calculations can be carried out as in section 32. One may conclude that, in transforming a physical to a mathematical model, without making any specific assumptions, we have only one condition to be satisfied: that the corresponding physical and mathematical models must have identical load-deformation properties.

There are, in general, two types of physical problems of elasto-plasticity to be investigated. The first type of problems arises when a physical model with well-defined components is given and one wishes to determine its load-deformation characteristic (including both loading and unloading processes). By well-defined components it is understood that the dimensions, geometric configurations and elasto-plastic properties of all components of the model are precisely known. The solution of such a problem can be carried out in two steps. The first step would be to transform the physical model to the corresponding mathematical model, by employing the general procedure outlined in Eqs. (46) to (51) and by using the condition of equivalence of load-deformation properties for the two models. The determination of the mathematical model can, in general, be an extremely involved problem, because of the combined effect of the geometric configuration, the frame-action at the joints and the generally non-linear elasto-plastic deformation properties for various components of the model. Theoretically, however, a solution always exists; because, no matter how complicated a physical model, it always deforms in a certain manner under a continuously increasing external load. Thus, according to the discussion of the generalized mathematical model, in Section 33, it is always possible to find out a mathematical model which coincides with any stress-strain curve. Having found the mathematical model, the second step would be to determine load-deformation properties for the mathematical model, similar to the procedure followed in Section 32.

The second type of problems of elasto-plasticity arises, when the stress-strain diagram of a physical model, during the loading process, is given and when one wishes to determine its stress-strain relationship during the process of unloading. Such a problem can be solved by reversing the procedure followed for the first type; this can also be done in two steps. The first step would be to obtain the mathematical model from the given stress-strain diagram, and the second step would be to determine the stress-strain relationship during unloading process, for the mathematical model, similar to the procedure employed in Section 32. The difficulty encountered here is that the stress for a physical model is measured, in practice, with respect to the nominal area of the cross section, A, whereas the stress for a mathematical model is measured with respect to the initial effective area of the section, A_0. The ratio $\frac{A_0}{A} = M$ is given in Eq. (50), where M is a constant which is a function of the crystal structure inside the physical model. In the absence of precise knowledge of the numerical value of the constant M, the initial effective area, A_0 of the mathematical model cannot, be determined. This difficulty can be overcome by a transformation of the effective area, A_0, of the mathematical model. The key to the solution of the problem lies in the concept that, by keeping the number and geometrical configuration of the crystal elements unchanged, the calculated effective area, A_0, of the mathematical model, can be expanded or contracted, by a transformation, without changing the load-deformation properties of the model. Choosing the transformation coefficient, f, equal to $\frac{1}{M}$, the transformed initial effective area, $\frac{A_0}{M}$, would coincide with the nominal cross sectional area A. Thus, it is always correct to replace the total initial effective area, A_0, of the mathematical model by, any arbitrary area, especially, the nominal cross sectional area, A, for the physical model, provided that the transformation coefficient, f, applies for all the individual components, $a_i \cdot m_j$, of the initial effective area, defined by Eq. (48):

$$A_0 = a_i \cdot \sum_{j=1}^{N} m_j .$$

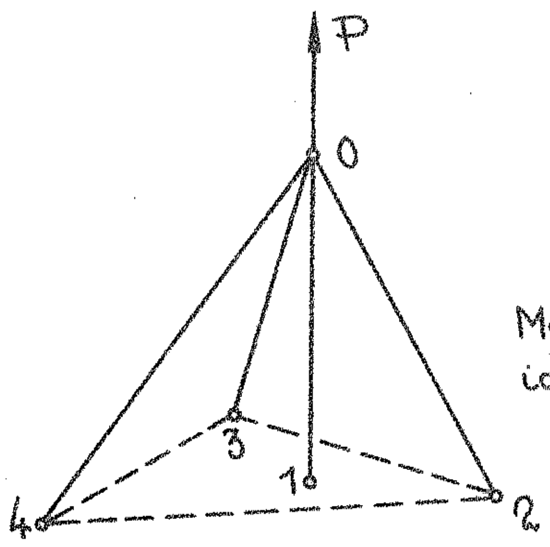
The problems of the first type may be met, in practice, only in connection with load-deformation study of simple physical models; hence, this type of problem is of limited practical significance. The second type of problem, on the other hand, is of great practical importance, because a reasonably accurate and continuous stress-strain diagram can be obtained experimentally for the loading of any type of physical elasto-plastic model. For the same test sample unloading can be obtained experimentally only at one point for each loading curve; hence a general and continuous evaluation of unloading cannot be achieved experimentally. Furthermore, a general theoretical approach to the problem of unloading is a matter of vital importance for the problem of elasto-plastic stability for which the range of unloading, to be accounted for, is infinitesimal. Thus, a theoretical evaluation of the infinitesimal unloading does not have to be affected by the experimental errors during unloading.

In the following section a problem of the first type is studied carefully to illustrate the general concepts introduced, in connection with the theory of transformation of a physical model to its mathematical counterpart. The remaining sections of this chapter are devoted to a general investigation of the problems of the second type.

35. Study of a space-truss as the physical model for a simplified crystal unit

The purpose of this section is to present a simple physical model, illustrating the various intermediate calculations for transformation of a given physical model to its mathematical equivalent. Once the mathematical model is determined, it is an easy matter to calculate its load-deformation characteristic, by following the procedure employed in Section 32. To simplify the calculations, we seek a model which is simple in geometric configuration; it has no moment resistance at the joints; and each of its individual components is governed by simple stress-strain relationship. The space-truss, shown in Fig. 17, fulfills the above simplifying assumptions.

The truss is constructed of an ideal elasto-plastic material, possessing four elements; a vertical element (01) of lenght, L, and three other inclined element, symmetrically placed with respect to the horizontal ring around Point 1, the inclined elements make an angle of α with the vertical element.



Model, constructed of an ideal elasto-plastic material

Fig. 17

It is assumed that all the four elements have the same cross sectional area A, the same modulus of elasticity E, and are all constructed of the same ideal elasto-plastic material. This last statement implies that all components of the model are assumed to have the same yield-point stresses. A tensile vertical force P is applied at the crown point 0. We proceed to calculate the load-deformation behaviour of the model under the increasing load P. The forces produced in the elements 01, 02, 03 and 04, are designated by S_1, S_2, S_3 and S_4 respectively.

$$S_2 = S_3 = S_4 = S \quad \text{(Symmetry requirement)} \tag{52}$$

$$S_1 + 3S\cos\alpha = P \quad \text{(Vertical projection)} \tag{53}$$

$$\frac{SL}{AE\cos^2\alpha} = \frac{S_1 \cdot L}{AE} \quad \text{(Compatibility of deformation)} \tag{54}$$

$$\therefore S_1 = \frac{P}{1+3\cos^3\alpha}; \tag{55}$$

$$S_2 = S_3 = S_4 = S = \frac{P\cos^2\alpha}{1+3\cos^3\alpha}. \tag{56}$$

Denoting the vertical deformation of the model by δ_p,

$$\delta_{(P)} = \frac{S_1 L}{EA} = \frac{PL}{AE(1+3\cos^3\alpha)} \qquad (57)$$

The vertical element is under greater stress than the other three elements (compare Eqs. 55 and 56). Thus, after the vertical element 01 has reached the limit of its load-carrying capacity under a limiting load P_L, any further increase of load, Δ_P, is taken up by the three remaining inclined elements:

$$\Delta S = \Delta S_2 = \Delta S_3 = \Delta S_4 = \frac{\Delta P}{3\cos\alpha}\,; \qquad (58)$$

and the corresponding vertical deformation is

$$\delta_{(\Delta P)} = \frac{\Delta S \cdot L}{AE\cos^2\alpha} = \frac{\Delta P \cdot L}{3AE\cos^3\alpha} \qquad (59)$$

Eq. (57) is a relation between P and δ where all the four elements deform elastically. Eq. (59) is a relation between P and δ where the vertical element deforms plastically.

The effect of the inclination angel α for the inclined elements can be studied in an alternative way. Each inclined element with the cross sectional area A is replaced by an immaginary vertical element with a ficticious cross sectional area, A_f, chosen in such a way that the load-deformation properties of the model remain unchanged. With such a definition, the equations (57) and (59) can be written in the alternative form:

$$\delta_{(P)} = \frac{P \cdot L}{E(A+3A_f)} \qquad (60)$$

$$\delta_{(\Delta P)} = \frac{\Delta P \cdot L}{3E\,A_f} \qquad (61)$$

A comparison of eqs. (57) and (59) with eqs. (60) and (61) gives

$$A_f = A \cdot \cos^3\alpha \qquad (62)$$

The preceding calculations show how a physical model can be transformed to its mathematical equivalent by following the general rule that the load-deformation properties of the model must remain unchanged during any equivalent transformation. Eq. (62) determines the initial effective area of each of the three inclined elements. The effective area of the vertical element, on the other hand, is taken to be equal to its nominal value, A. Thus, the total initial effective area, A_0, of the physical model's mathematical equivalent is calculated to be:

$$A_0 = A(1+3\cos^3\alpha) \tag{63}$$

Comparing Eq. (63) with Eq. (50), derived in general form in Section 34, we get the following specific relation for the constant M:

$$M = \frac{A_0}{\text{Nominal cross section}} = \frac{A(1+3\cos^3\alpha)}{A+\frac{3A}{\cos\alpha}} = \frac{1+3\cos^3\alpha}{1+\frac{3}{\cos\alpha}} \tag{64}$$

In Section 34, we asserted that M is a constant which is a function of the geometrical configuration of the crystal structure inside the physical model. Eq. (64) verifies the last statement and demonstrates that for the simplified model studied in this section the value of the constant M is a function of the geometry of the model, shown in Fig. 17.

Having determined the total initial effective area of the equivalent mathematical model, we proceed to investigate its load-deformation properties by a similar procedure which has been carried out in Section 32. The mathematical model resulting from the above transformation resembles the structural Model b, shown in Fig. 10 (b), where the outer ring, ΔA_2, corresponds to the initial effective area of the vertical element of the model, and the inner layer, A_3, corresponds to the initial effective areas of the inclined elements of the model altogether. Since the shape of the cross section of the mathematical model is immaterial, we choose the initial effective area of the resulting model to be rectangular in section. Thus, our physical model would be reduced to its mathematical counterpart, shown in Fig. 18

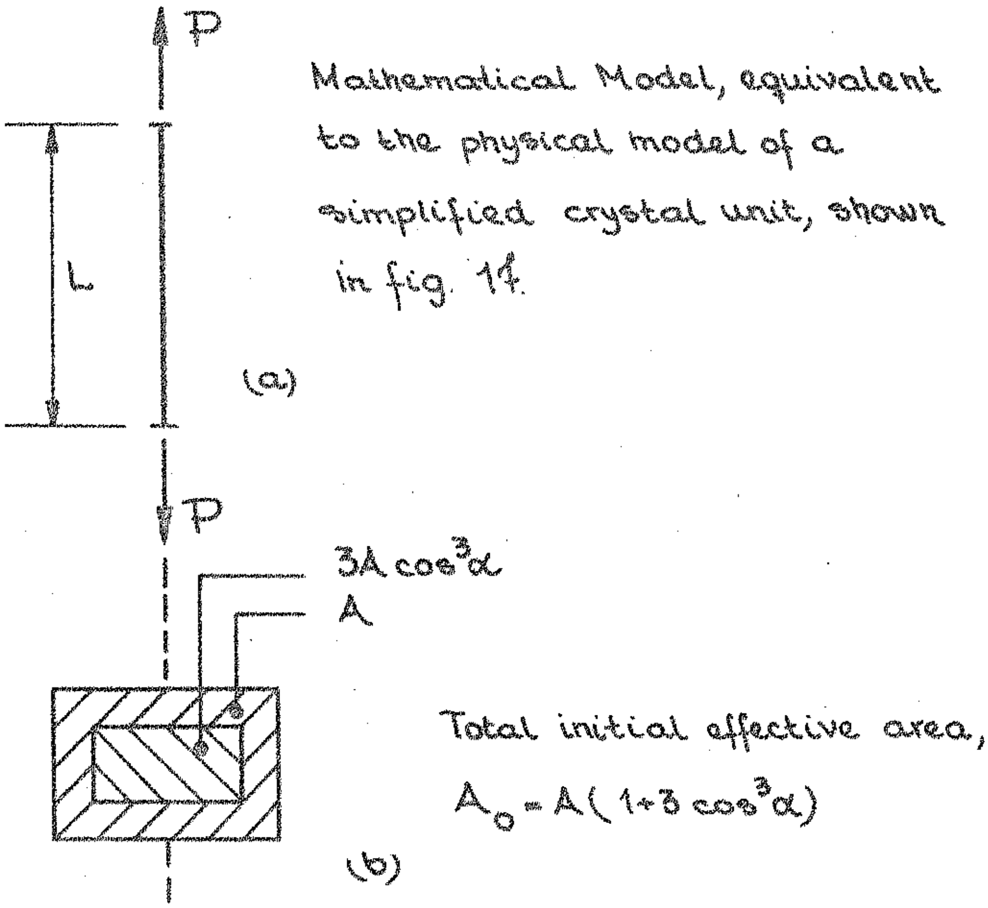


Fig. 18

In Fig. 18 (b) the inner layer represents the initial effective areas of the simplified crystal unit's inclined elements taken together, and the outer layer represents the effective initial area of its vertical element. The difference between the mathematical model studied in Section 32 and the model, shown in Fig. 18, is that in the former case, it was assumed that the three components of the model had the same modulus of elasticity but different yield-point stresses; whereas in the latter case it was assumed that the two components of the model, shown in Fig. 18 (b), have the same modulus of elasticity and the same yield-point stress but, because of geometrical configuration of its equivalent physical model, they are not stressed identically. Thus, the elements of area, shown in Fig. 18 (b), reach their yield-point stress at different stages of loading. This conclusion is in agreement with the results of the general discussions, set forth in Sections 33 and 34.

As the model in Fig. 18 is loaded the whole initial effective area, A_0, cooperates in bearing the load up to a limiting load, P_1, where the outer layer of the section, A, shown in Fig. 18 (b), plasticizes completely. Any further increase of loading, ΔP, beyond the value P_1, is carried by the non-plasticized effective area, $A_2 = A_1 - \Delta A_1$ shown in the inner layer of Fig. 18 (b). If we were to continue loading, we would reach another limiting load, P_2, at which the inner layer would also plasticize completely. The model could not bear any more load in excess of P_2; thus the deformations would continue under the constant limiting load, P_2. The load-deformation diagram for the model would, then, be represented by the following figure, (Fig. 19).

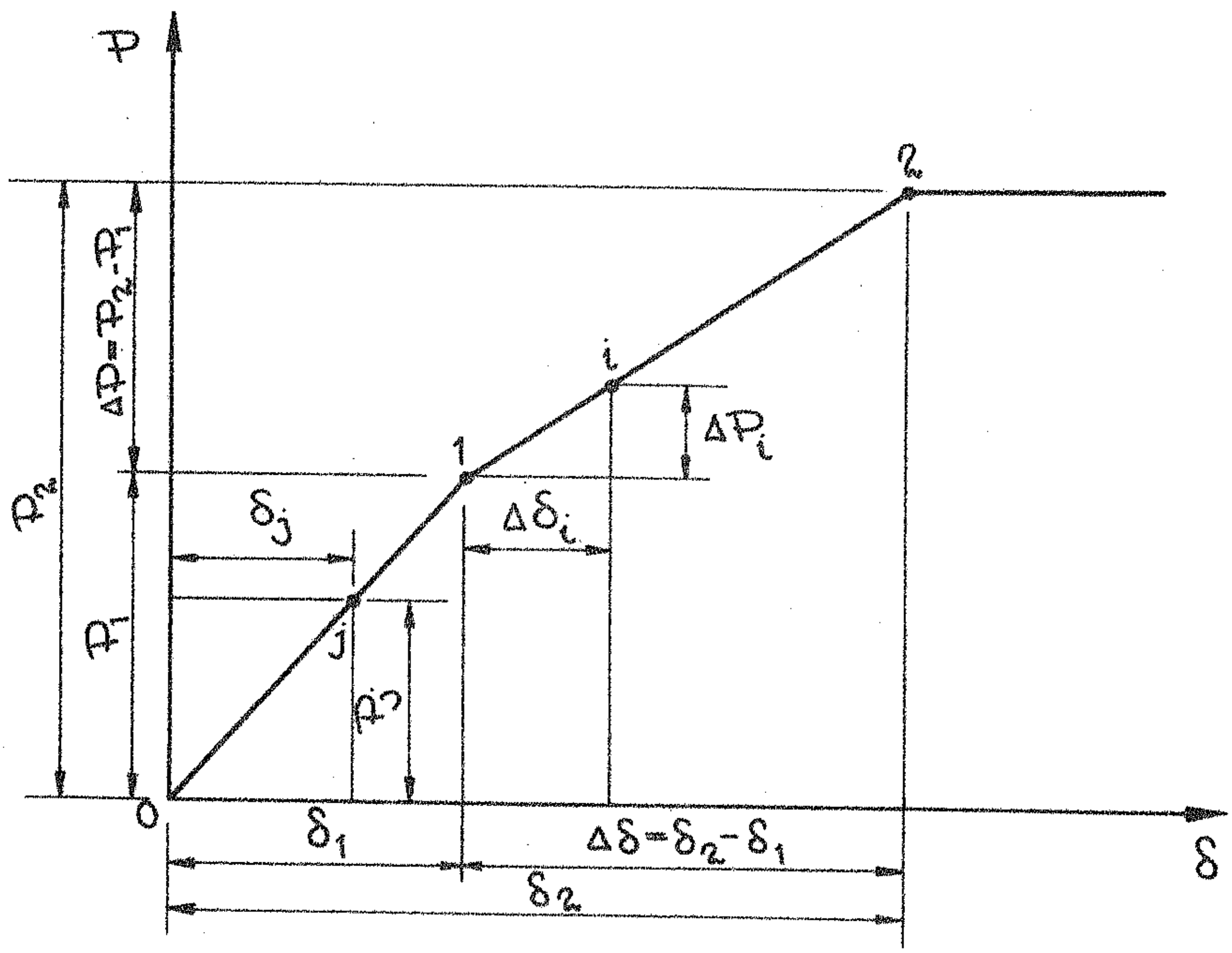


Fig. 19

The load-deformation relation between the origin of coordinate axes, 0, and the first breaking point, 1, is given by Eq. (67), and the corresponding relation between point 1 and the next breaking point, 2, is given by Eq. (59). Rewriting Eqs. (57) and (59), for any point j between points 0 and 1 and any point i between points 1 and 2, we get:

$$P_j = AE(1+3\cos^3\alpha)\cdot\frac{\delta_j}{L}; \quad (65) \quad \text{This relation holds from 0 to 1}$$

$$\Delta P_i = 3AE\cos^3\alpha\cdot\frac{\Delta\delta_i}{L}; \quad (66) \quad \text{This relation holds from 1 to 2}$$

The terms, $\frac{\delta_j}{L}$ and $\frac{\Delta\delta_i}{L}$ represent the strain for the two distinct stages of loading marked by their upper limits, 1 and 2, respectively. Measuring the stress with respect to the total initial effective area, A_0, and denoting the terms, $\frac{\delta_j}{L}$ and $\frac{\Delta\delta_i}{L}$ by ε_j and $\Delta\varepsilon_i$ respectively, we get the following two stress-strain relationships corresponding to Eqs. (65) and (66)

$$\sigma_j = \frac{P_j}{A_0} = E\cdot\varepsilon_j \quad (67)$$

$$\Delta\sigma_i = \frac{\Delta P_i}{A_0} = \frac{3\cos^3\alpha}{1+3\cos^3\alpha}\cdot E\cdot\Delta\varepsilon_i \quad (68)$$

The stress-strain diagram would then be given by the following figure, (Fig. 20):

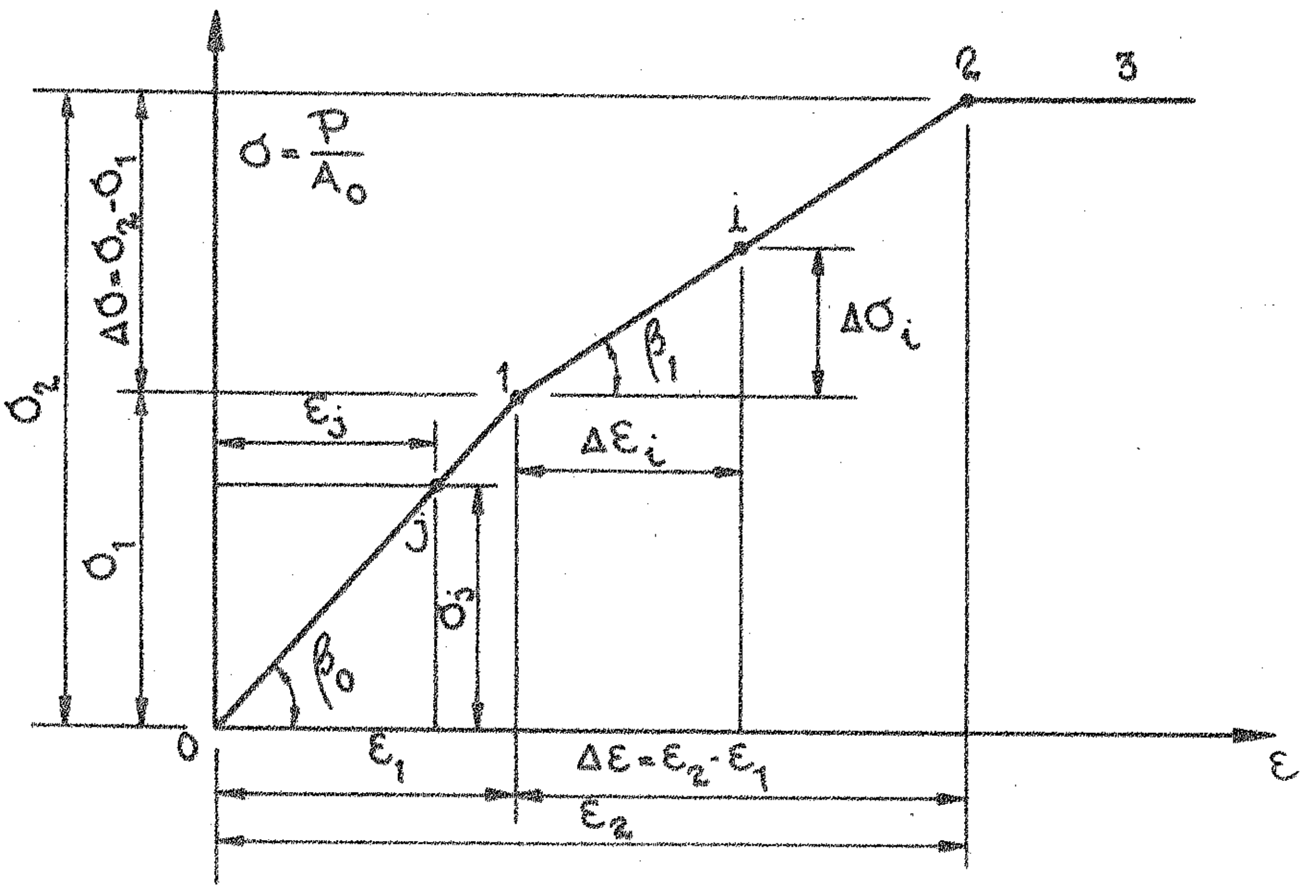


Fig. 20

By using Eqs. (67) and (68) and observing Fig. 20, we get the following relations:

$$\tan\beta_0 = \frac{\sigma_i}{\varepsilon_i} = E ; \qquad (69)$$

$$\tan\beta_1 = \frac{\Delta\sigma_i}{\Delta\varepsilon_i} = \frac{3\cos^3\alpha}{1+3\cos^3\alpha}\cdot E \qquad (70)$$

Eqs. (65) to (68) can be used to determine the correspending loads and stresses at points 1 and 2, as follows.

$$P_1 = AE(1+3\cos^3\alpha)\cdot\frac{\delta_1}{L} = AE(1+3\cos^3\alpha)\cdot\varepsilon_1 \qquad (71)$$

$$\Delta P = 3AE\cos^3\alpha\cdot\frac{\Delta\delta}{L} = 3AE\cos^3\alpha\cdot\Delta\varepsilon \qquad (72)$$

$$\sigma_1 = E\cdot\varepsilon_1 \qquad (73)$$

$$\Delta\sigma = \frac{3\cos^3\alpha}{1+3\cos^3\alpha}\cdot E\cdot\Delta\varepsilon \qquad (74)$$

The stage of loading at which the inner layer of the model in Fig. 18 (b) gets plasticized can be precisely determined from the condition that for the physical model, shown in Fig. 17, the stress on each inclined element must become equal to the maximum limiting stress on the vertical element. The above condition expressed mathematically would be

$$S + \Delta S = S_1 \qquad (75)$$

By using Eqs. (55), (56) and (58), Eq. (75) would be rewritten as following:

$$\frac{P_1\cos^2\alpha}{1+3\cos^3\alpha} + \frac{\Delta P}{3\cos\alpha} = \frac{P_1}{1+3\cos^3\alpha}$$

$$\therefore \; \Delta P = \frac{3P_1 \cos\alpha \cdot \sin^2\alpha}{1+3\cos^3\alpha} \qquad (76)$$

Substituting for P_1 from Eq. (71),

$$\Delta P = 3AE \cdot \cos\alpha \cdot \sin^2\alpha \cdot \varepsilon_1 \qquad (77)$$

Comparing Eqs. (77) and (72),

$$\cos^2\alpha \cdot \Delta\varepsilon = \sin^2\alpha \cdot \varepsilon_1$$

$$\therefore \; \frac{\Delta\varepsilon}{\varepsilon_1} = \tan^2\alpha \qquad (78)$$

Adding the value of ΔP from Eq. (77) to P_1 from Eq. (71),

$$P_2 = P_1 + \Delta P = AE(1+3\cos\alpha) \cdot \varepsilon_1 \qquad (79)$$

Similar to the procedure used for the model in Section 32, we continue to determine the condition of elasto-plasticity at the two breaking points, 1 and 2. We use the same fundamental concepts and terms which were introduced in Section 32.

1. <u>Condition at point 1</u>

a) An infinitesimal distance to the left of the point:

Elastic area = Effective area = $A(1+3\cos^3\alpha)$

Plastic area = 0

Elastic force = P_1 = AE $(1+3\cos^3\alpha) \cdot \varepsilon_1$

Plastic force = 0

Elastic stress = $E \cdot \varepsilon_1$

Plastic stress = 0

b) An infinitesimal distance to the right of the point:

Elastic area = Effective = $3A\cos^3\alpha$

Plastic area = A

Elastic force = $P_1 \cdot \frac{\text{Elastic area}}{A_0} = 3AE\cos^3\alpha \cdot \varepsilon_1$

Plastic force = $P_1 \cdot \frac{\text{Plastic area}}{A_0} = AE \cdot \varepsilon_1$

Elastic stress = $\frac{\text{Elastic force}}{\text{Elastic area}} = E \cdot \varepsilon_1$

Plastic stress = $\frac{\text{Plastic force}}{\text{Plastic area}} = E \cdot \varepsilon_1$

2. Condition at point 2

a) An infinitesimal distance to the left of the point:

Elastic area = Effective area = $3A\cos^3\alpha$

Plastic area = A

Elastic force = Elastic force to the right of point 1 + ΔP =

$$= 3AE\cos^3\alpha \cdot \varepsilon_1 + 3AE \cdot \cos\alpha \sin^2\alpha \cdot \varepsilon_1 = 3AE\cos\alpha \cdot \varepsilon_1$$

Plastic force = plastic force at point 1 = $AE \cdot \varepsilon_1$

Elastic stress = $\frac{3AE\cos\alpha \cdot \varepsilon_1}{3A\cos^3\alpha} = \frac{E \cdot \varepsilon_1}{\cos^2\alpha}$

Plastic stress = plastic stress at point 1 = $AE\ \varepsilon_1$

b) An infinitesimal distance to the right of the point

Elastic area = 0

Plastic area = $A(1 + 3\cos^3\alpha)$

Elastic force = 0

Plastic force = $P_2 = AE(1 + 3\cos\alpha) \cdot \varepsilon_1$

Elastic stress = 0

Plastic stress at point 2 is different for the two parts of area, shown in Fig. 18 (b). The plastic stress on the outer layer is equal to $E \cdot \varepsilon_1$ and the plastic stress on the inner layer is equal to the elastic stress just to the left of point 2, viz., $\frac{E \cdot \varepsilon_1}{\cos^2\alpha}$.

The calculation of elastic and plastic stresses, for the above model as well as for the model studied in section 32, is based on the concept of elastic or plastic force per unit of the corresponding elastic or plastic area. Such a definition gives a good physical picture of the intensity of various stresses at any loading level. However, when there is a continuous plasticization over the cross section of the model, it would be more convenient to define the elastic and plastic stresses as the elastic or plastic force per unit area of the initial total effective area. This latter definition of elastic and plastic stress will be employed in the next section when we are dealing with general problems of the second type with continuous plasticization over the whole cross-section. More explanation will be given in next section.

<u>Calculation of the unloading-modulus</u>

By observing Fig. (20), we conclude that there is no plasticization up to point 1; hence, the unloading-line anywhere between the two points 0 and 1 coincides with the loading-line 01. Immediately to the right of point 1, however, part of the area equal to A is completely plasticized. The force carried by this area, which is calculated above to be equal to $AE\varepsilon_1$ develops frictional energy losses, ω_{fi}, between point 1 and point i, which is assumed to lie between points 1 and 2. Denotning the strain between points 1 and i by $\Delta\varepsilon_i$, as shown in Fig. 20, we get the following relation for ω_{fi}:

$$\omega_{fi} = AE\varepsilon_1 \cdot \Delta\varepsilon_i = AE\varepsilon_1 \cdot \frac{\Delta\delta_i}{L} \tag{80}$$

$\Delta\delta_i$ is identical with the deformation, $\delta_{\Delta P}$, caused by an increase of load, ΔP, beyond the stage of loading at point 1 and up to any point i between points 1 and 2.

Substituting for $\Delta\delta_i = \delta_{\Delta P}$ from Eq. (59) in Eq. (80),

$$\omega_{fi} = \varepsilon_1 \cdot \frac{\Delta P_i}{3\cos^3\alpha} \tag{81}$$

The frictional energy losses per unit area of the initial total effective area $A_o = A(1+3\cos^3\alpha)$ will be denoted by S_{fi} and will be calculated as follows:

$$S_{fi} = \frac{\omega_{fi}}{A_o} = \varepsilon_1 \cdot \frac{\Delta\sigma_i}{3\cos^3\alpha} \tag{82}$$

Referring to Fig. 20 and using Eq. (69),

$$\varepsilon_1 = \frac{\sigma_1}{\tan\beta_o} = \frac{\sigma_1}{E} \tag{83}$$

Subustituting ε_1 in Eq. (82),

$$S_{fi} = \frac{\sigma_1 \cdot \Delta\sigma_i}{3E\cos^3\alpha} \tag{84}$$

The total energy per unit area of the total initial effective area, S_{ti}, is calculated by measuring the area under the stress-strain diagram up to point i, as shown in Fig. 20.

$$\therefore S_{ti} = \varepsilon_1 \cdot \sigma_{1/2} + \sigma_1 \cdot \Delta\varepsilon_i + \Delta\sigma_i \cdot \Delta\varepsilon_{i/2} \tag{85}$$

Calculating $\Delta\varepsilon_i$ from Eq. (68),

$$\Delta\varepsilon_i = \frac{\Delta\sigma_i(1+3\cos^3\alpha)}{3E\cos^3\alpha} \tag{86}$$

Substituting $\Delta\varepsilon_i$ from Eq. (86) and ε_1 from Eq. (83) in Eq. (85),

$$S_{ti} = \frac{\sigma_1^2}{2E} + \frac{\sigma_1 \cdot \Delta\sigma_i(1+3\cos^3\alpha)}{3E\cos^3\alpha} + \frac{\Delta\sigma_i^2(1+3\cos^3\alpha)}{6E\cos^3\alpha} \tag{87}$$

The corresponding elastic energy stored in the sample per unit area of the total effective area, S_{ei}, is calculated as the difference between S_{ti} and S_{ei}.

$$S_{ei} = S_{ti} - S_{fi} \tag{88}$$

Using Eqs. (87) and (84) and substituting in Eq. (88),

$$S_{ei} = \frac{3\cos^3\alpha\,(\sigma_1 + \Delta\sigma_i)^2 + \Delta\sigma_i^2}{6E\cos^3\alpha}$$

This may be reduced to

$$S_{ei} = \frac{(\sigma_1 + \Delta\sigma_i)^2 + \dfrac{\Delta\sigma_i^2}{3\cos^3\alpha}}{2E} \tag{89}$$

Using Eq. (28), derived in section 32,

$$E_{ai} = \frac{\sigma_i^2}{2S_{ei}} = \frac{E(\sigma_1 + \Delta\sigma_i)^2}{(\sigma_1 + \Delta\sigma_i)^2 + \dfrac{\Delta\sigma_1^2}{3\cos^3\alpha}}$$

This may be written in the following form:

$$E_{ai} = \frac{E}{1 + \dfrac{\Delta\sigma_i^2}{3\cos^3\alpha\,(\sigma_1 + \Delta\sigma_i)^2}} \tag{90}$$

Eq. (90) shows that the unloading-modulus for any point of unloading between points 1 and 2 in Fig. 20 is less than the initial modulus of elasticity E. The value E_{ai} is a function of the loading level and the geometry of the simplified crystal unit which is defined by the angle α .

Numerical evaluation of unloading-modulus

Eq. (90) is valid between the loading levels 1 and 2, as shown in Fig. 20. For any further deformation beyond the loading level 2, all the external work done on the model would turn into frictional energy losses; Thus, the unloading-line would be parallel with the unloading-line at point 2. This conclusion is based on the stress-strain diagram of the ideal elasto-plastic model discussed in section 32 and shown in Fig 13. In order to calculate the numerical variation of the unloading-modulus between points 1 and 2, we first determine the critical stress at point 2.

Assuming the angle α equal to $\pi/3$, Eq. (76) gives the following relation for the increment of loading, ΔP, beyond loading level 1:

$$\Delta P = \frac{3P_1 \cos\alpha \cdot \sin^2\alpha}{1+3\cos^3\alpha} = 0{,}818\, P_1 \qquad (91)$$

Accordingly, we get the following relationship between the stress at point 1 and the critical increment of stress between points 1 and 2:

$$\Delta\sigma = 0{,}818\,\sigma_1 \qquad (92)$$

Having determined the critical increment of stress, $\Delta\sigma$, we proceed to determine the unloading-modulus at four points, i_1, i_2, i_3 and i_4, defined by the following stress relationships:

$$\begin{aligned}
\Delta\sigma_{i1} &= 0 \\
\Delta\sigma_{i2} &= 0{,}3\,\sigma_1 \\
\Delta\sigma_{i3} &= 0{,}6\,\sigma_1 \\
\Delta\sigma_{i4} &= \Delta\sigma = 0{,}818\,\sigma_1
\end{aligned} \qquad (93)$$

Using Eq. (90), we get the following numerical values for the corresponding unloading-moduli:

$$E_{ai_1} = E_{ai} = E$$

$$E_{ai_2} = 0{,}875\ E \qquad (94)$$

$$E_{ai_3} = 0{,}728\ E \qquad \text{Given } \alpha = \pi/3$$

$$E_{ai_4} = E_{a2} = 0{,}650\ E$$

We observe that the value of the unloading-modulus decreases as we approach point 2. After passing point 2, the value of the unloading-modulus remains unchanged and equal to 0,650 E. The variation of the unloading-modulus depends also on the angle α. As the angle α decreases, the range of variation of the unloading-modulus decreases accordingly. By assuming the angle α equal to $\pi/6$, we get in similarity with Eqs. (91) and (92) the following increment of loading, ΔP, and the corresponding increment of stress, $\Delta\sigma$, beyond the loading level 1:

$$\Delta P = 0{,}22\,P_1 \qquad \text{and} \qquad \Delta\sigma = 0{,}22\,\sigma_1$$

For $\alpha = \pi/6$ we determine the unloading-moduli at three points i_1, i_2 and i_3 defined by the following stress-strain relationships:

$$\Delta\sigma_{i_1} = 0; \qquad \Delta\sigma_{i_2} = 0{,}1\,\sigma_1$$

$$\text{and } \Delta\sigma_{i_3} = \Delta\sigma = 0{,}22\,\sigma_1$$

Using Eq. (90) we arrive at the following numerical values for the corresponding unloading-moduli:

$$E_{\alpha i1} = E_{\alpha 1} = E$$

$$E_{\alpha i2} = 0{,}996\,E \tag{95}$$

$$E_{\alpha i3} = E_{\alpha 2} = 0{,}986\,E$$

Denoting the index 1 for the case $\alpha = \pi/3$ and the index 2 for the case $\alpha = \pi/6$ and using Eq. (55) we get the following relationship between the loads $P_{1\alpha 1}$ and $P_{1\alpha 2}$ corresponding to the loading level 1 on the stress-strain diagram where the vertical element, for both cases 1 and 2, gets plasticized:

$$S_{1\alpha 1} = S_{1\alpha 2} \quad \text{which may be redued to}$$

$$\frac{P_{1\alpha 2}}{P_{1\alpha 2}} = \frac{1 + 3\cos^3 \alpha_1}{1 + 3\cos^3 \alpha_2} = 0{,}467$$

The relation between the corresponding stresses at the loading level 1 would be

$$\frac{\sigma_{1\alpha 1}}{\sigma_{1\alpha 2}} = 0{,}467$$

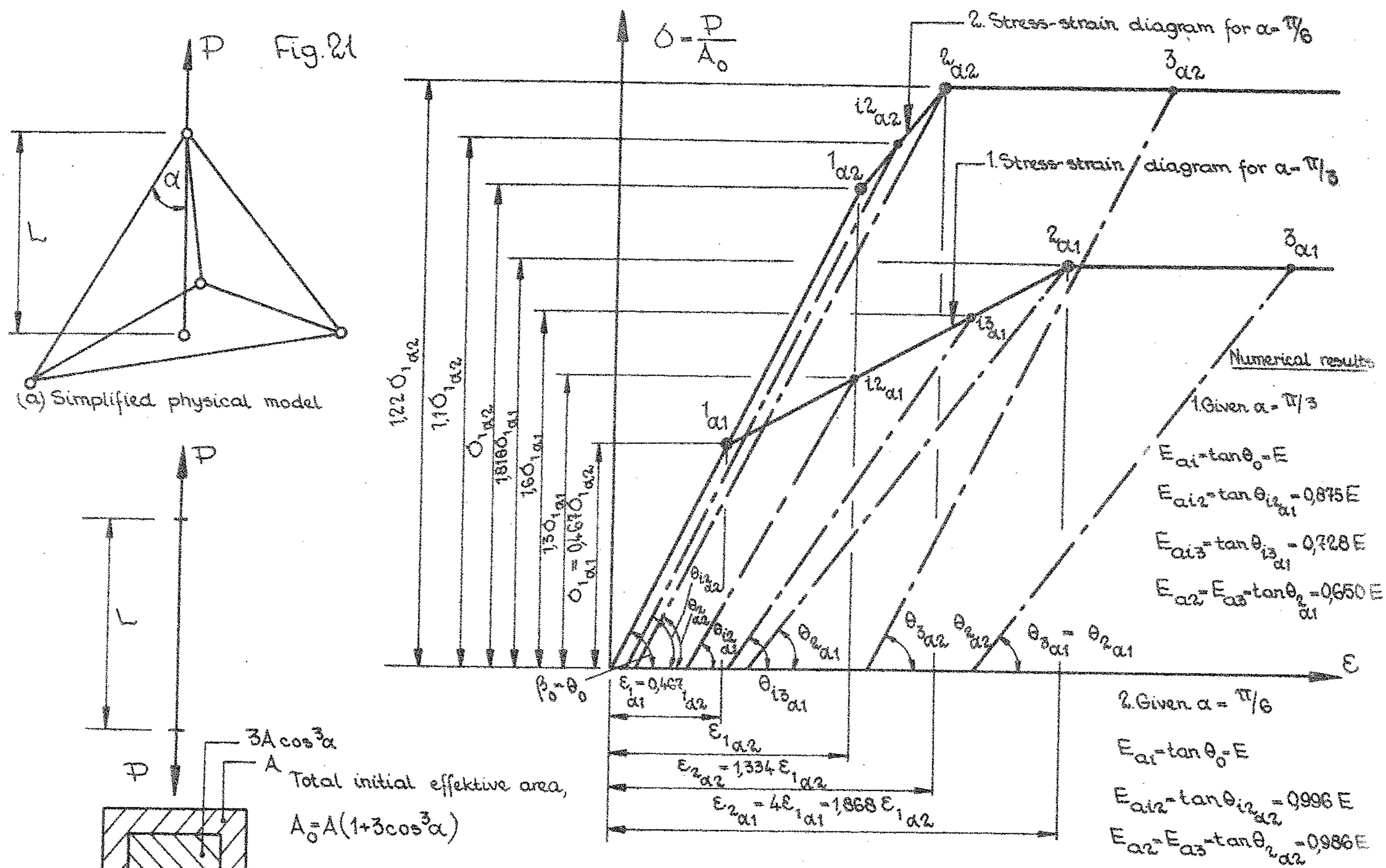
I order to determine the loading level 2 (Fig. 20) for both cases 1 and 2 we use Eq. (78)

$$\frac{\Delta\varepsilon_{\alpha 1}}{\varepsilon_{1\alpha 1}} = \tan^2 \alpha_1 = 3\,;$$

$$\text{and} \quad \frac{\Delta\varepsilon_{\alpha 2}}{\varepsilon_{1\alpha 2}} = \tan^2 \alpha_2 = 0{,}334$$

The stress-strain diagram with its corresponding unloading-lines, as calculated above, is shown in Fig. 21.

Fig. 21



(a) Simplified physical model

(b) Equivalent mathematical model

(c) The stress strain-diagram with the unloading lines for α = π/6 and α = π/3

Conclusion. The foregoing detailed study of the simplified model demonstrates that for the study of a given well-defined physical model the first step would be to find its mathematical equivalent. The equivalent mathematical model would be determined by its total initial effective area and the critical loading levels at which parts of the effective area get plasticized. For a simple model with simple stress-strain relationship governing each component of the model, the calculations can be carried out as shown in this section. For a more complicated model, on the other hand, the calculations can in general be extremely involved. Theoretically, however, one can always determine the mathematical equivalent of any physical model, no matter how complicated its crystal structure, and no matter what stress-strain relationship which governs its individul crystal grains. This general conclusion is true, because the construction of the equivalent mathematical model is based on the general concept of the equivalence of deformation properties of the physical model and its mathematical counterpart. Because of the fact that a mathematical model, based on the fundamental concepts introduced in Section 32, can be constructed for any predetermined stress-strain curve; and because of the fact that any physical model always deforms in a certain pattern with a certain stress-strain curve; we conclude that any physical model has always a mathematical equivalent. This conclusion is important for the investigation of the problems of the second type, when the stress-strain diagram is determined experimentally for any physical model, and when one wishes to determine the unloading-modulus at any point on the given stress-strain diagram. The following sections will be devoted to a general investigation of the problems of the second type. Finally, it should be emphasized that the elements of the space truss, shown in Fig. 17, were assumed to be ideal elasto-plastic only in order to simplify the calculations. The general theory proposed in this paper does not in any way restrict the stress-strain relationships of the individual crystal grains.

36. Determination of the unloading-modulus, based on the experimental stress-strain diagram of any arbitrary physical model.

We asserted in Section 34, in connection with second type of problems of elasto-plasticity, that when the stress-strain diagram of a centrally loaded physical model is given, and when one wishes to determine its stress-strain relationship during the process of unloading, the procedure can be carried out in two steps. The first step would be to determine the equivalent mathematical model from the given stress-strain curve; the second step then would be to determine the stress-strain relationship during the unloading process, for the mathematical model, similar to the procedure employed in Section 32. There is an apparent difficulty in determining the mathematical model from the stress-strain curve since the former is measured with respect to the total initial effective area, wheras the latter is measured with respect to the nominal cross sectional area of the physical model. The ratio between the total initial effective area, A_o, of the mathematical model and the nominal cross sectional area, A, of the physical model is denoted by the constant M and is given by Eq. (51). However, by keeping the number and crystal configuration of the crystal elements unchanged, the effective area can be expanded or contracted without changing the load-deformation properties of the physical model provided that all the various elements of the effective area are expanded or contracted identically. By all the elements of the effective area of the mathematical model (according to definition given in Section 33) we mean all the incremental areas of the effective cross section, each of which reaches the yield-point stress at a certain stage of loading. Thus, the effective area is a relative criterium and all the incremental areas of the effictive cross section can always be expanded or contracted in such a way that it coincides with the nominal cross section of the physical model.

One may conclude from the above argument that the total initial effective area, A_o, of the equivalent mathematical model can be considered to be equal to the nominal cross sectional area, A, of its physical counterpart. According to the discussion in Section 33, the generalized mathematical model is uniquely determined by simultaneously determining Eq. (44) and the inequality (45). Eq. (44) gives the total initial effective area which, according to the above reasoning, can be considered equal to the nominal cross sectional area, A, of the physical model. Inequality (45) can be determined from the stress-strain diagram of the given physical model, because the load-deformation properties of our mathematical model must coincide with that of its physical counterpart. Thus, the general problem of the second type is reduced to the problem of investigating the unloading properties of a mathematical model whose total initial effective area, A, and its stress-strain diagram are given.

In order to estabilish inequality (45) from the given stress-strain diagram, we use the important conclusions drawn from the discussion of the generalized mathematical model in Section 34, where we conceived of a cross section of the structural model, covered by n ideal elasta-plastic membranes, each of which reaches the yield-point stress at a certain stage of loading in Section 33, we proceeded to consider the membranes, not as rigidly separate layers, but as a single entity, in which the properties of all membranes are uniformly migled and spread out identically throughout the section. Such a generalized hypothetical model would allow us to introduce the new concept of the rate of plasticization, which is identical to the rate of decrease of the effective area, giving the impression of a continually decreasing modulus of elasticity; hence, the ratio between the tangent moduli at two points on the stress-strain diagram is equal to the ratio between the corresponding effective areas at those two points. This conclusion has already been verified, through Eqs. (14) to (16), for the mathematical model, discussed in section 32.

The concept of uniform plasticization over the whole section of the model implies that the ratio between elastic and plastic areas at any point on the stress-strain diagram applies to the whole section. Thus, the load deformation properties of the whole model can be studied by choosing a unit element of that model with a unit cross sectional area and a unit length. The load-deformation diagram for such element would coincide with the stress-strain curve for the whole model. Considering the unit element at any stage of loading corresponding to the strain, ε , the elastic part of the unit area would be denoted by $a_{e\varepsilon}$ and the plastic part of the unit area would be denoted by $a_{p\varepsilon}$. The force carried by $a_{e\varepsilon}$ would be called the "elastic stress" or the "elastic force per unit of the initial total effective area" and would be denoted by $\sigma_{e\varepsilon}$. The force carried by $a_{p\varepsilon}$ would be called the "plastic stress" or the "plastic force per unit of the total initial effective area" and would be denoted by $\sigma_{p\varepsilon}$. The total load carried by the unit element would be called the "the total stress" and would be denoted by σ_ε . Thus, according to definition we get the following relations:

$$a_{e\varepsilon} + a_{p\varepsilon} = 1 \tag{96}$$

and

$$\sigma_{e\varepsilon} + \sigma_{p\varepsilon} = \sigma_\varepsilon \tag{97}$$

One may conclude that according to the above definition the "elastic and plastic stresses" are measured not with respect to the unit of corresponding elastic or plastic area, as was the case for the study of the models in sections 32 and 35 but are measured rather with respect to the unit of the total initial effective area. Thus, in order to avoid confusion, the "elastic and plastic stresses", in the above definitions, are identified with equivalent phrases: the "elastic and plastic forces per unit of the total initial effective area". However, we shall henceforth refer to $\sigma_{e\varepsilon}$ and $\sigma_{p\varepsilon}$ merely as "elastic and plastic stresses" respectively.

Now, with the above introductory remarks, definitions and notations, we consider the following Figure:

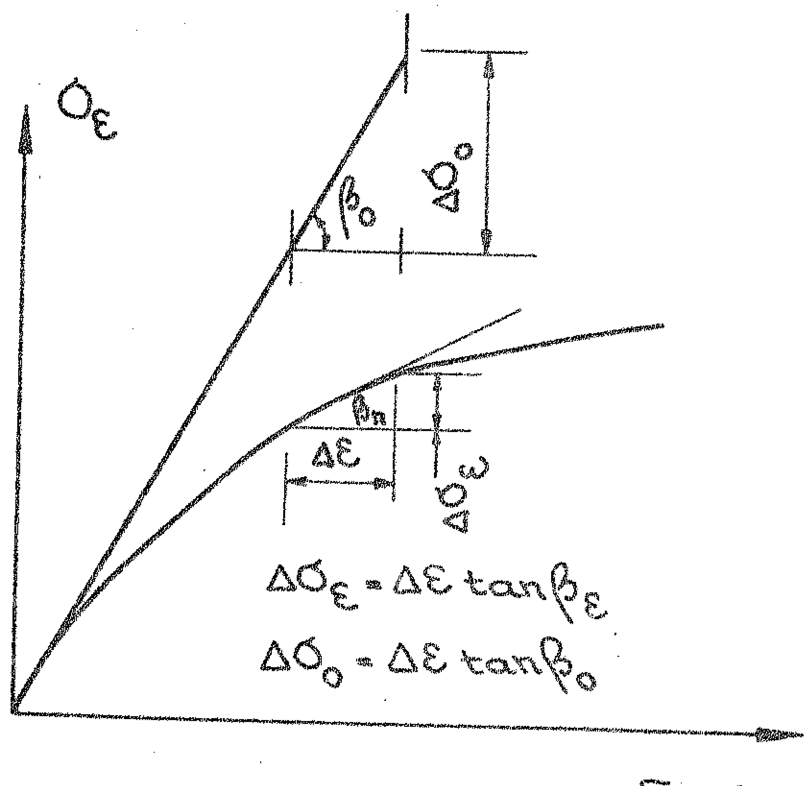


Fig. 22

If we were to assume that the material deformed elastically all the time, then the strain $\Delta\varepsilon$ would have corresponded to a stress increase of, $\Delta\sigma_o$, instead of the actual value, $\Delta\sigma_\varepsilon$. The ratio $\frac{\Delta\sigma_o}{\Delta\sigma_\varepsilon}$ is identical with the ratio between the initial effective area, 1, and the effective area, $a_{e\varepsilon}$, at the point $(\varepsilon, \sigma_\varepsilon)$:

$$\frac{\Delta\sigma_o}{\Delta\sigma_\varepsilon} = \frac{1}{a_{e\varepsilon}} = \frac{\tan\beta_o}{\tan\beta_\varepsilon} = \frac{E_o}{d\sigma_\varepsilon/d\varepsilon}$$

$$\therefore a_{e\varepsilon} = \frac{1}{E_o} \cdot d\sigma_\varepsilon/d\varepsilon \qquad (98)$$

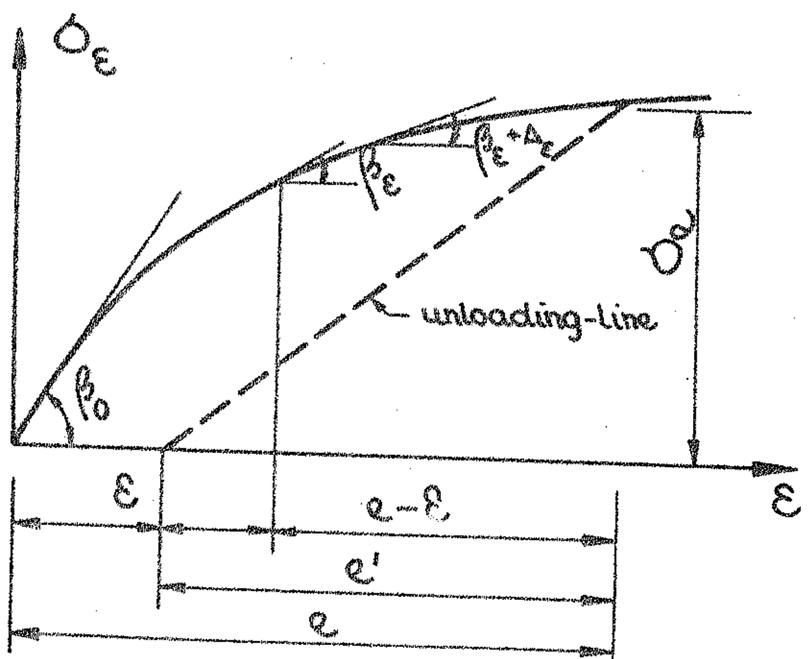


Fig. 23

Using Eqs. (96) and (98),

$$a_{p\varepsilon} = 1 - a_{e\varepsilon} = \frac{1}{E_o}\left(E_o - \frac{d\sigma_\varepsilon}{d\varepsilon}\right) \qquad (99)$$

$$a_{p(\varepsilon+\Delta\varepsilon)} = \frac{1}{E_o}\left[(E_o) - \frac{d}{d\varepsilon}(\sigma_\varepsilon + \Delta\sigma_\varepsilon)\right] \qquad (100)$$

$$\Delta a_{p\varepsilon} = a_{p(\varepsilon+\Delta\varepsilon)} - a_{p\varepsilon} = \frac{1}{E_o}\left[\tan\beta_{(\varepsilon+\Delta\varepsilon)} - \tan\beta_\varepsilon\right] \qquad (101)$$

$\Delta a_{p\varepsilon}$ denotes the infinitesimal portion of the effective area that gets plasticized between the point $(\varepsilon, \sigma_\varepsilon)$ and the point $(\varepsilon+\Delta\varepsilon, \sigma_\varepsilon+\Delta\sigma_\varepsilon)$. If

$\Delta\sigma_{p\varepsilon}$ represents the load carried by the plasticized infinitesimal area $\Delta a_{p\varepsilon}$, then the ratio between $\Delta\sigma_{p\varepsilon}$ and the "elastic stress", $\sigma_{e\varepsilon}$, is the same as the ratio between $\Delta a_{p\varepsilon}$ and $a_{e\varepsilon}$:

$$\frac{\Delta\sigma_{p\varepsilon}}{\sigma_{e\varepsilon}} = \frac{\Delta a_{p\varepsilon}}{a_{e\varepsilon}}$$ substituting from Eqs. (98)

and (101)

$$\frac{\Delta\sigma_{p\varepsilon}}{\sigma_{e\varepsilon}} = -\frac{\left[\tan\beta_{(\varepsilon+\Delta\varepsilon)} - \tan\beta_{\varepsilon}\right]}{\tan\beta_{\varepsilon}} \qquad (102)$$

$$\Delta\sigma_{p\varepsilon} = -\sigma_{e\varepsilon}\cdot\frac{\left[\frac{\tan\beta_{(\varepsilon+\Delta\varepsilon)} - \tan\beta_{\varepsilon}}{\Delta\varepsilon}\right]}{\tan\beta_{\varepsilon}}\cdot\Delta\varepsilon \qquad (103)$$

Denoting the infinitesimal frictional loss caused by the plasticised infintesimal area, $\Delta a_{p\varepsilon}$, by ΔS_{fe}, and denoting the strain at the point of unloading by e, we get the following relation:

$$\Delta S_{fe} = \Delta\sigma_{p\varepsilon}\cdot(e-\varepsilon)$$ (See Fig. 23)

Substituting for $\Delta\sigma_{p\varepsilon}$,

$$\Delta S_{fe} = -\sigma_{e\varepsilon}\cdot(e-\varepsilon)\cdot\frac{\left[\frac{\tan\beta_{(\varepsilon+\Delta\varepsilon)} - \tan\beta_{\varepsilon}}{\Delta\varepsilon}\right]}{\tan\beta_{\varepsilon}}\cdot\Delta\varepsilon \qquad (104)$$

$$\frac{\Delta S_{fe}}{\Delta\varepsilon} = -\sigma_{e\varepsilon}\cdot(e-\varepsilon)\cdot\frac{\left[\frac{\tan\beta_{(\varepsilon+\Delta\varepsilon)} - \tan\beta_{\varepsilon}}{\Delta\varepsilon}\right]}{\tan\beta_{\varepsilon}}$$

$$\frac{dS_{fe}}{d\varepsilon} = -\sigma_{e\varepsilon}\cdot(e-\varepsilon)\cdot\frac{d^2\sigma_{\varepsilon}/d\varepsilon^2}{d\sigma_{\varepsilon}/d\varepsilon}$$

$$\therefore\ S_{fe} = -\int_0^e \sigma_{e\varepsilon}\cdot(e-\varepsilon)\cdot\frac{d^2\sigma_{\varepsilon}/d\varepsilon^2}{d\sigma_{\varepsilon}/d\varepsilon}\cdot d\varepsilon \qquad (105)$$

Denoting the external work done on the unit element, up to the point of unloading e, by S_{te} and the energy stored in the unit element up to the same point of unloading by S_{ee},

$$S_{ee} = S_{te} - S_{fe} = \int_0^e \sigma_\varepsilon \cdot d\varepsilon - S_{fe}$$

$$S_{ee} = \int_0^e \left[\sigma_\varepsilon + \sigma_{e\varepsilon} \cdot (e-\varepsilon) \cdot \frac{d^2\sigma_\varepsilon / d\varepsilon^2}{d\sigma_\varepsilon / d\varepsilon}\right] \cdot d\varepsilon \qquad (106)$$

Finally, by substituting S_{ee} in Eq. (28), we arrive at the following general formula for the unloading-modulus, $E_{a\varepsilon}$:

$$E_{a\varepsilon} = \frac{\sigma_e^2}{2\int_0^e \left[\sigma_\varepsilon + \sigma_{e\varepsilon} \cdot (e-\varepsilon) \cdot \frac{d^2\sigma_\varepsilon / d\varepsilon^2}{d\sigma_\varepsilon / d\varepsilon}\right] \cdot d\varepsilon} \qquad (107)$$

The term, $\sigma_{e\varepsilon}$, appearing in Eq. (107) represents the stress carried by the non-plasticized effective area at the point ($\varepsilon, \sigma_{\varepsilon}$) on the stress-strain diagram. So far, no effort has been made to determine analytic expression for $\sigma_{e\varepsilon}$. The reason for this has been the extreme complexity in such an effort at this stage of development. What we know at this moment is that $\sigma_{e\varepsilon}$ is a continuously decreasing function of strain and is moreover always less than σ_{ε} . Eq. (107), so far, gives an expression for the unloading-modulus of elasticity in which the term $\sigma_{e\varepsilon}$, in turn, is a function of strain which has to be decided. For a deeper insight into the meaning of $\sigma_{e\varepsilon}$, a numerical procedure for its calculation will be shown in the following paragraph. We shall not however give up the search for a complete analytic solution; rather we shall come back to this question at a later stage when more important properties about the variational possibilities of the unloading-modulus of elasticity will have been discovered.

Numerical procedure for calculation of $\sigma_{e\varepsilon}$

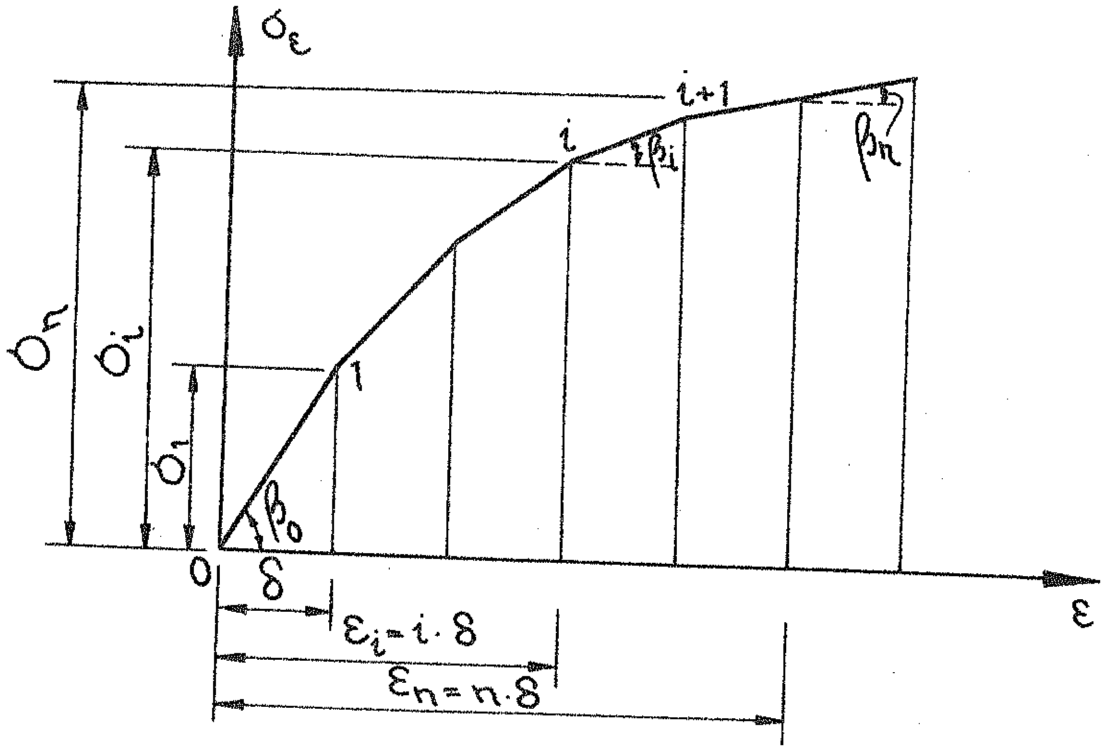


Fig. 24

The stress-strain diagram is divided into a series of straight lines, as shown in Fig. (24).

If one would denote the stress carried by the non-plasticized area at the point $(\varepsilon_i, \sigma_i)$ by σ_{ei}, then the stress carried by the non-plasticized area at the point $(\varepsilon_{i+1}, \sigma_{i+1})$, would be given by the following relation:

$$\sigma_{e(i+1)} = \sigma_{ei} + (\sigma_{i+1} - \sigma_i) - \Delta\sigma_p \tag{108}$$

$\Delta\sigma_p$ in Eq. (108) represents the load carried by the portion of effective area that gets plasticized between the points i and i + 1 on the stress strain diagram.

Eq. (102) can be rewritten in the form:

$$\Delta\sigma_{p\,(i,i+1)} = -\sigma_{ei} \cdot \frac{\tan\beta_i - \tan\beta_{(i-1)}}{\tan\beta_{(i-1)}} \tag{109}$$

Substituting in Eq. (108),

$$\sigma_{e(i+1)} = \sigma_{ei}\left(1-\frac{\tan\beta_{(i-1)}-\tan\beta_i}{\tan\beta_{i-1}}\right)+\left(\sigma_{i+1}-\sigma_i\right) \qquad (110)$$

Eq. (110) gives a general expression for the stress carried by the non-plasticized effective area at a certain point on the stress-strain diagram provided that the corresponding value for a neigbouring point is given.

For the total stress between two neighbouring points, i and i + 1, on the σ-ε curve, the following relation holds:

$$(\sigma_{i+1}-\sigma_i) = \tan\beta_i \cdot (\varepsilon_{i+1}-\varepsilon_i) = \tan\beta_i \cdot \delta \qquad (111)$$

Eq. (110) can be written in the following alternative form:

$$\sigma_{e(i+1)} = \sigma_{ei} \cdot \frac{\tan\beta_i}{\tan\beta_{(i-1)}} + (\sigma_{i+1}-\sigma_i)$$

$$\sigma_{e(i+1)} = \sigma_{ei} \cdot \frac{(\sigma_{i+1}-\sigma_i)}{(\sigma_i-\sigma_{i-1})} + (\sigma_{i+1}-\sigma_i) \qquad (112)$$

The recursive relation (112) can now be used to determine the successive elastic stresses by starting from point 1 on the stress-strain diagram where $\sigma_{ei} = \sigma_1$.

$$\sigma_{e2} = \sigma_1 \cdot \frac{(\sigma_2-\sigma_1)}{\sigma_1-\sigma_0} - (\sigma_2-\sigma_1) = 2(\sigma_2-\sigma_1)$$

$$\sigma_{e3} = 2(\sigma_2-\sigma_1) \cdot \frac{\sigma_3-\sigma_2}{\sigma_2-\sigma_1} + (\sigma_3-\sigma_2) = 3(\sigma_3-\sigma_2)$$

$$\sigma_{e4} = 3(\sigma_3-\sigma_2) \cdot \frac{\sigma_4-\sigma_3}{\sigma_3-\sigma_2} + (\sigma_4-\sigma_3) = 4(\sigma_4-\sigma_3)$$

$$\therefore \sigma_{ei} = (i-1)\left[\sigma_{i-1} - \sigma_{i-2}\right] \cdot \frac{\sigma_i - \sigma_{(i-1)}}{\sigma_{(i-1)} - \sigma_{(i-2)}} + \left[\sigma_i - \sigma_{(i-1)}\right]$$

$$\therefore \sigma_{ei} = i \cdot \left[\sigma_i - \sigma_{(i-1)}\right] \qquad (113)$$

This last result leads to the general simple formula for the elastic stress, σ_{en}, at the point $(\varepsilon_n, \sigma_n)$ on the stress-strain diagram:

$$\underline{\underline{\sigma_{en} = n \cdot \left[\sigma_n - \sigma_{(n-1)}\right]}} \qquad (114)$$

37. Investigation of variational possibilities of the unloading-modulus along any arbitrary stress-strain curve.

Now, we are faced with the decisive question of the possible variations of the unloading-modulus along any given arbitrary stress-strain curve. The general solution of the problem of elasto-plastic stability requires a precise knowledge of how the unloading modulus varies as a function of the two possible variables: The shape of the stress-strain curve and the position of the point of unloading on the σ-ε diagram. In order to begin the investigations we first pose the following questions.

1. Is the unloading modulus of elasticity always less than the initial modulus of elasticity?
2. Can the unloading-modulus under any circumstances be greater or equal to the initial modulus of elasticity?
3. What would happen if after some deformations the tangent modulus would remain constant?
4. What would happen if the tangent modulus would tend to zero?

One way to answer the above questions would be to choose various stress-strain diagrams, and apply Eq. (107) to the corresponding numerical investigations. This procedure, however, would be a never-ending task. For a profound answer to the above questions a general mathematical investigation is necessary. For this purpose we choose a general stress-strain diagram that is typical for all the visco-elastic materials. Such a general typical curve would have the following mathematical properties:

1. The stress is an increasing, continuous function of strain. Stress can also be constant.
2. The first derivative, $d\sigma/d\varepsilon$, is positive and a decreasing function of strain. $d\sigma/d\varepsilon$ can have a constant value, which includes the possibility of its having the value zero.
3. The second derivative is either negative or equal to zero.

Fig. (25), below, shows the various possibilities of such a typical stress-strain diagram:

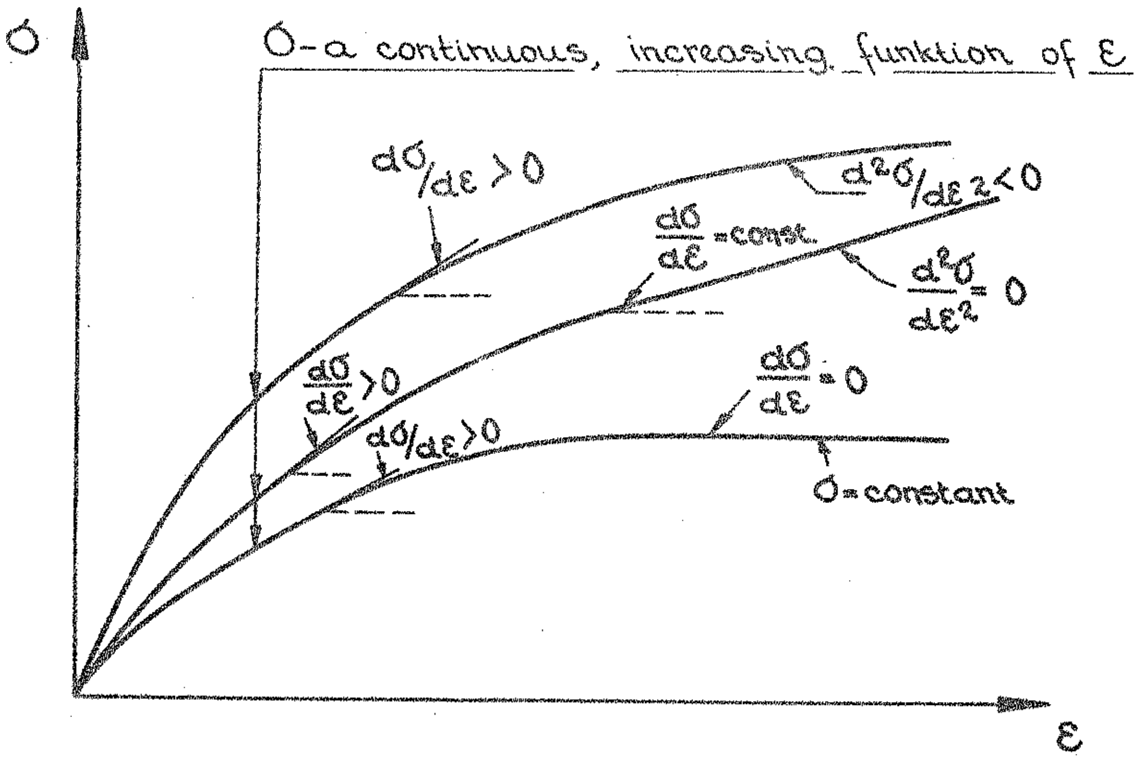


Fig. 25
Various alternative forms for a general stress-strain curve.

Having defined the mathematical properties of the general stress-strain curve, we proceed to determine all the variational possibilities of the unloading modulus of elasticity along any typical stress-strain diagram. We investigate the rate of change of $E_{a\varepsilon}$, with respect to the strain, at the point of unloading, e. Differentiating E_{ae}, given by Eq. (107), with respect to e, we get the following relation:

$$\frac{dE_{a\varepsilon}}{de} = \frac{2\sigma_e \cdot \frac{d\sigma_e}{de} \cdot 2\int_0^e \left[\sigma_\varepsilon + \sigma_{e\varepsilon}\cdot(e-\varepsilon)\cdot \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon}\right] d\varepsilon - \sigma_e^2 \left[2\sigma_e + 2\int_0^e \sigma_{e\varepsilon} \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon} \cdot d\varepsilon\right]}{4\left\{\int_0^e \left[\sigma_\varepsilon + \sigma_{e\varepsilon}\cdot(e-\varepsilon)\cdot \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon}\right] d\varepsilon\right\}^2} \tag{115}$$

The denominator of the above fraction is always positive. The numerator consits of three terms, and needs to be studied more closely. By observing that at the point of unloading, the expression

$$\int_0^e \left[\sigma_\varepsilon + \sigma_{e\varepsilon}(e-\varepsilon) \cdot \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon}\right] \cdot d\varepsilon$$

is equal to the elastic anergy, S_{ee}, and the expression $\frac{d\sigma_e}{de}$ is equal to the tangent modulus, E_t. The term

$$2\sigma_e \cdot \frac{d\sigma_e}{de} \cdot 2\int_0^e \left[\sigma_\varepsilon + \sigma_{e\varepsilon}(e-\varepsilon) \cdot \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon}\right] d\varepsilon$$

can be rewritten in the form:

$$2\sigma_e \cdot \frac{d\sigma_e}{de} \cdot 2\int_0^e \left[\sigma_\varepsilon + \sigma_{e\varepsilon} \cdot (e-\varepsilon) \cdot \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon}\right] \cdot d\varepsilon = 2\sigma_e \cdot E_t \cdot S_{ee} =$$

$$= 2\sigma_e \cdot E_t = \frac{\sigma_e^2}{\tan_\theta} = 2\sigma_e^3 \cdot \frac{E_t}{\tan_\theta} \qquad (116)$$

In the last subtitution S_{ee} has been replaced by the expression $\frac{\sigma_e^2}{2\tan\theta}$ from Eq. (28). The other two terms of the numerator, could be rewritten in the form:

$$\sigma_e^2 \left[2\sigma_e + 2\int_0^e \sigma_{e\varepsilon} \cdot \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon} \cdot d\varepsilon\right] = 2\sigma_e^3 \cdot \left[1 + \frac{\int_0^e \sigma_{e\varepsilon} \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon} \cdot d\varepsilon}{\sigma_e}\right] \qquad (117)$$

So, $\frac{dE_a}{de}$ can be written in the following simpler form:

$$\frac{dE_{ae}}{de} = \frac{\sigma_e^3\left[\frac{E_t}{\tan\theta} - \left(1 + \frac{\int_0^e \sigma_{e\varepsilon}\frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon}\cdot d\varepsilon}{\sigma_e}\right)\right]}{2\left\{\int_0^e\left[\sigma_\varepsilon + \sigma_{e\varepsilon}\cdot(e-\varepsilon)\cdot\frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon}\right]d\varepsilon\right\}^2} \tag{118}$$

In order to find out about the sign of $\frac{dE_a}{de}$, one must first find out an upper or lower limit for the integral

$$\int_0^e \sigma_{e\varepsilon}\cdot\frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon}\cdot d\varepsilon$$

What does this last integral represent physically? To answer this last question, we go back to Eq. (103):

$$\Delta\sigma_p = -\sigma_{e\varepsilon}\cdot\frac{\left[\frac{\tan\beta_{(\varepsilon+\Delta\varepsilon)} - \tan\beta_e}{\Delta\varepsilon}\right]}{\tan\beta_\varepsilon}\cdot\Delta\varepsilon$$

$\Delta\sigma_p$ represents the load carried by the plasticized, infintesimal area, between the points $(\varepsilon, \sigma_\varepsilon)$, and $(\varepsilon + \Delta\varepsilon, \sigma_\varepsilon + \Delta\sigma_\varepsilon)$. Thus, the term $(\sigma_{e\varepsilon}\frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon})$ represents the infinitesimal stress, carried by the infinitesimal area, plasticized between two points on the stress-strain diagram, situated an infinitesimal distance away from each other. Based on this knowledge, the integral, $\int_0^e \sigma_{e\varepsilon}\cdot\frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon}\cdot d\varepsilon$

can be expressed in the following way:

$$\int_0^e \sigma_{e\varepsilon} \cdot \frac{d^2\sigma_\varepsilon / d\varepsilon^2}{d\sigma_\varepsilon / d\varepsilon} \cdot d\varepsilon = - \sum_{\substack{i=1 \\ n \to \infty}}^{n} \sigma_{ei} \frac{\tan\beta_{(i-1)} - \tan\beta_i}{\tan\beta_{(i-1)}} \quad (119)$$

See Fig (26) below

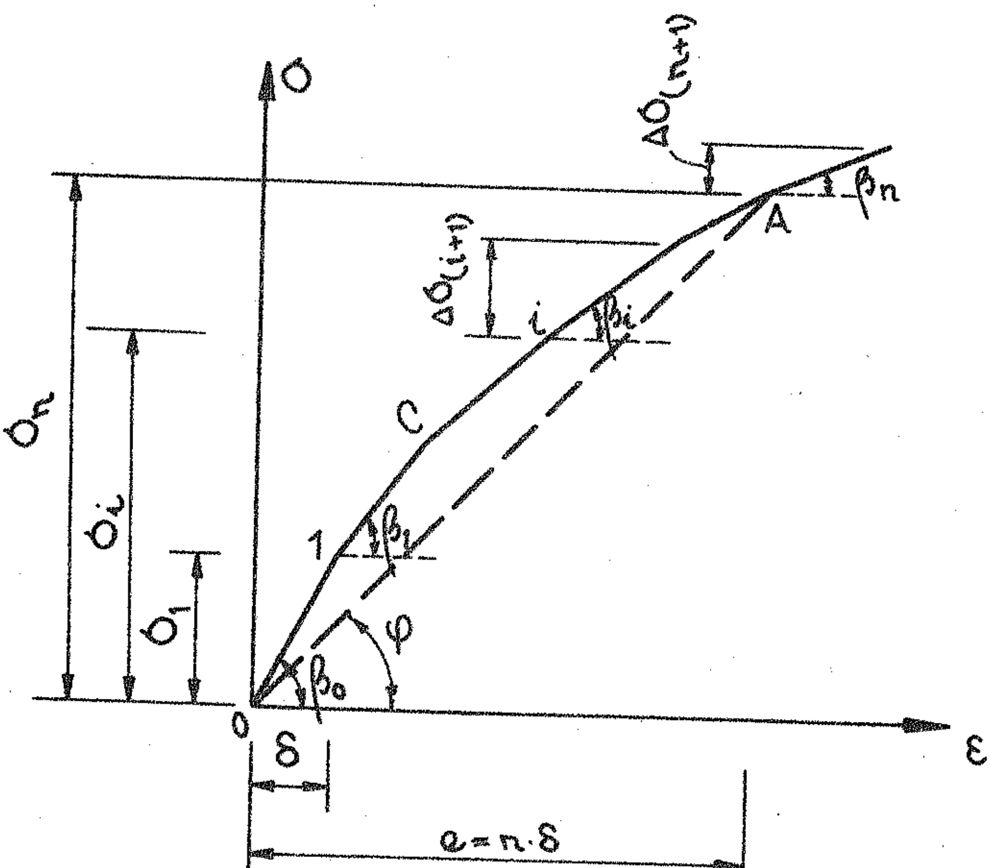


Fig. 26

From the origin of the coordinate axes to the point B on the stress-strain diagram, we choose two alternative paths for the evaluation of the summation given by the equation (119). For the first path we choose the actual stress-strain diagram OCA. For the second path we choose the straight line OA. We demonstrate here that the two proposed paths render indentical values to the summation given by Eq. (119) provided that after point A one follows the original stress-strain diagram for either of the two alternative paths.

I. Calculation along the path OCA:

$$-\sum_{i=1}^{n} \sigma_{ei} \frac{\tan\beta_{(i-1)} - \tan\beta_i}{\tan\beta_{(i-1)}} = -\sum \sigma_{ei} \frac{[\sigma_i - \sigma_{(i-1)}] - [\sigma_{(i+1)} - \sigma_i]}{\sigma_i - \sigma_{(i-1)}} \quad \text{See fig. 26}$$

Substituting from Eq. (113) for σ_{ei} and denoting the summation by S,

$$S = \sum_{i=1}^{n} i \left\{ \left[\sigma_i - \sigma_{(i-1)}\right] - \left[\sigma_{(i+1)} - \sigma_i\right] \right\}$$

$$S = \sum_{i=1}^{n} i \left[(\Delta\sigma_i) - (\Delta\sigma_{i+1}) \right]$$

$$S = (\Delta\sigma_1 - \Delta\sigma_2) + 2(\Delta\sigma_2 - \Delta\sigma_3) + \dots + i(\Delta\sigma_i - \Delta\sigma_{i+1}) + \dots + n(\Delta\sigma_n - \Delta\sigma_{n+1})$$

$$S = (\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3 + \dots \Delta\sigma_n) - n\left[\Delta\sigma_{(n+1)}\right] \qquad (120)$$

II. Calculation along the straight line path, OA.,

The second path goes, straight, from O till A without any plasticization. At point A, however, an abrupt plasticization takes place and the straight line OA is bent by angle β_n, after bending the straight line follows the original stress-strain diagram. If S′ dentoes the abrupt "plastic stress" at point A, the following relation holds:

$$S' = \sigma_n \cdot \frac{\tan\varphi - \tan\beta_n}{\tan\varphi} \qquad \text{(See Eq. 102)} \qquad (121)$$

$$\tan\varphi = \frac{\sigma_n}{e} = \frac{\sigma_n}{n \cdot \delta} \qquad \text{(See Fig. 26)} \qquad (122)$$

$$\sigma_n = \Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3 + \cdots\cdots \Delta\sigma_n \tag{123}$$

$$\tan\beta_n = \frac{\Delta\sigma_{(n+1)}}{\delta} \tag{124}$$

Substituting from Eqs. (122), (123) and (124) into Eq. (121)

$$S' = \sigma_n \cdot \frac{\frac{\sigma_n}{n\cdot\delta} - \frac{\Delta\sigma_{(n+1)}}{\delta}}{\frac{\sigma_n}{n\delta}} = \frac{\frac{\sigma_n}{n} - \Delta\sigma_{(n+1)}}{\frac{1}{n}}$$

$$S' = (\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3 + \cdots\cdots \Delta\sigma_n) - n\cdot\left[\Delta\sigma_{(n+1)}\right] \tag{125}$$

Comparison of Eqs. (120), and (125) shows that $S = S'$.

Hence we get the remarkable result that the integral $\int_0^e \sigma_{e\varepsilon} \cdot \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon} \cdot d\varepsilon$

is independent of the path followed from 0 to A provided that after point A the path coincides with the original stress-strain curve.

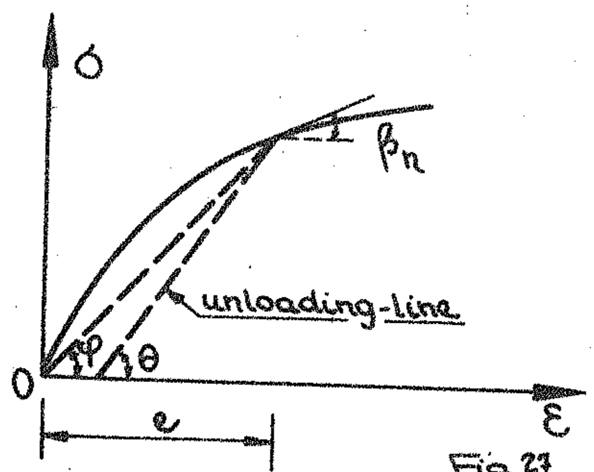


Fig. 27

Now, the results of this discovery can be used to determine the variational possibilities of the unloading modulus of elasticity. We go back to Eq. (118), and observe the following relations:

$$\int_0^e \sigma_{e\varepsilon} \cdot \frac{d^2\sigma_\varepsilon / d\varepsilon^2}{d\sigma_\varepsilon / d\varepsilon} \cdot d\varepsilon = -S' = -\sigma_e \cdot \frac{\tan\varphi - \tan\beta_n}{\tan\varphi} \quad \text{(See Eq. 121)}$$

$$\therefore \frac{dE_{a\varepsilon}}{de} = \frac{\sigma_e^3 \left[\frac{E_t}{\tan\theta} - \left(1 - \frac{\tan\varphi - \tan\beta_n}{\tan\varphi}\right) \right]}{\text{positive denominator}}$$

Considering that $\tan\beta_n$ is equal to the tangent modulus E_t at the point of unloading, we get the following:

$$\frac{dE_{a\varepsilon}}{de} = \frac{\sigma_e^3 \left(\frac{E_t}{\tan\theta} - \frac{E_t}{\tan\varphi} \right)}{\text{positive denominator}} \tag{126}$$

From Fig. (27) we observe that $\tan\varphi < \tan\theta$, and thus the following relation holds:

$$\therefore \frac{dE_a}{de} = \frac{\sigma_e^3 \left(E_t/\tan\theta - \frac{E_t}{\tan\varphi} \right)}{\text{positive denominator}} = \frac{\text{negative numerator}}{\text{positive denominator}} \tag{127}$$

This mathematical investigation proves, in the most general way, that the unloading modulus of elasticity always decreaes as the strain corresponding to the point of unloading increases. This means that by starting from the origin of coordinate axes with the initial modulus of elasticity, E_o, and moving along the typical stress-strain diagram, the unloading modulus of elasticity continually decreases. Eq. (126) shows that this conclusion is valid even if the tangent modulus gets a constant value. Eq. (126) further demonstrates that for the case when tangent modulus approaches zero, $\frac{dE_{a\varepsilon}}{d\varepsilon}$ also approaches zero; consequently, $E_{a\varepsilon}$ approaches a constant value.

The discovery that the integral $\int_0^e \sigma_{e\varepsilon} \cdot \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon} \cdot d\varepsilon$ is independent of the path followed can now be used to find out an analytic expression for the elastic stress, $\sigma_{e\varepsilon}$. Consider, any typical stress-strain diagram, such as the one shown in Fig. (28). At any point i on the curve with a strain ε_i less than the unloading strain e, the elastic stress, σ_{ei}, can be found by following the straight line Oi.

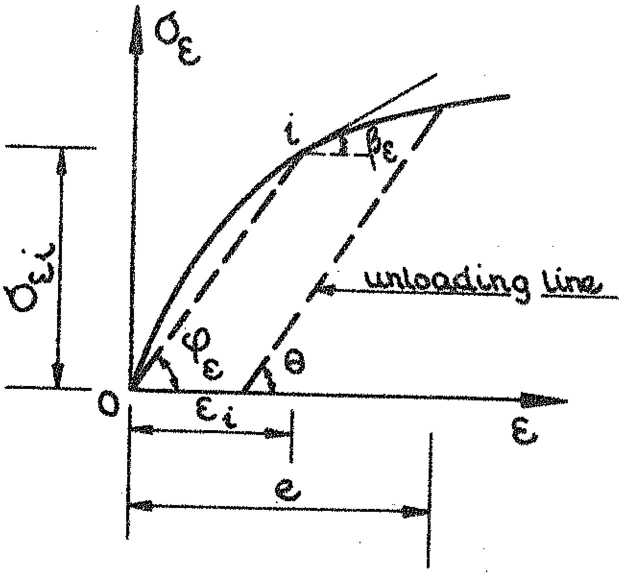


Fig. 28

$$\sigma_{ei} = \sigma_{\varepsilon i} - \sigma_{pi} \tag{128}$$

$$\sigma_{pi} = -\int_0^{\varepsilon_i} \sigma_{e\varepsilon} \cdot \frac{d^2\sigma_\varepsilon / d\varepsilon^2}{d\sigma_\varepsilon / d\varepsilon} \cdot d\varepsilon = \sigma_{\varepsilon i} \cdot \frac{\tan\varphi_\varepsilon - \tan\beta_\varepsilon}{\tan\varphi_\varepsilon} \tag{129}$$

See Fig (28) and Eq. (121).

Substitution in Eq. (128) gives

$$\sigma_{ei} = \sigma_{\varepsilon i}\left(1 - \frac{\tan\varphi_\varepsilon - \tan\beta_\varepsilon}{\tan\varphi_\varepsilon}\right) = \sigma_{\varepsilon i} \cdot \frac{\tan\beta_\varepsilon}{\tan\varphi_\varepsilon} \tag{130}$$

$\sigma_{ei} = \sigma_{e\varepsilon}$ at any point i; $\tan\beta_\varepsilon = \frac{d\sigma_\varepsilon}{d\varepsilon}$; $\tan\varphi_\varepsilon = \frac{\sigma_\varepsilon}{\varepsilon}$

Substitution in Eq. (130) gives

$$\sigma_{e\varepsilon} = \frac{\varepsilon}{\sigma_\varepsilon} \cdot \frac{d\sigma_\varepsilon}{d\varepsilon} \tag{131}$$

Finally substitution for $\sigma_{e\varepsilon}$ from Eq. (131), in Eq. (107) gives the following complete analytic solution for the unloading modulus of elasticity:

$$E_{a\varepsilon} = \frac{\sigma_e^2}{2\int_0^e \left[\sigma_\varepsilon + (e-\varepsilon)\cdot \frac{\varepsilon}{\sigma_\varepsilon} \cdot \frac{d^2\sigma_\varepsilon}{d\varepsilon^2}\right] d\varepsilon} \qquad (132)$$

38. A numerical method for calculation of the unloading-modulus.

Eq. (107) is used as the basis for the development of a numerical procedure for calculation of the unloading-modulus.

$$\frac{\sigma_e^2}{2\int_0^e \sigma_\varepsilon d\varepsilon + 2\int_0^e \left[(e-\varepsilon)\cdot\sigma_{e\varepsilon}\cdot \frac{d^2\sigma_\varepsilon/d\varepsilon^2}{d\sigma_\varepsilon/d\varepsilon}\right] d\varepsilon}$$

By observing that the integral, $\int_0^e \sigma_\varepsilon \cdot d\varepsilon$ represents the total area under the stress-strain curve and by using Eq. (119), the above equation can be written in the following form:

$$\frac{\sigma_e^2}{2\left[(S_t)_0^e - \sum_{i=1}^{n=e/\delta} (n\delta - i\delta)\sigma_{ei}\cdot \frac{\tan\beta(i-1) - \tan\beta_i}{\tan\beta_{(i-1)}}\right]} \qquad (133)$$

where

$$\sum_{i=1}^{n=e/\delta} (n\delta - i\delta)\cdot\sigma_{ei} \frac{\tan\beta_{(i-1)} - \tan\beta_i}{\tan\beta_{(i-1)}} = S_{fn} \qquad (134)$$

The summation (134) represents the total energy lost, up to the point of unloading n. The form of this summation indicates that the stress-strain diagram is divided into n strips, by dividing the strain axis into n equal intervals of length δ . The lines drawn from the end of intervals, parallel to the stress-axis, cut the stress-strain curve in n points. Any such intermediate point shall be referred to as the point i. When n approaches infinity, the summation, S_{fn}, coincides with the original integral. For any finite value of n, Eq. (134) gives an approximate value for S_{fn}. The approximation, however, can be carried through to any predetermined degree of accuracy by the proper choice of the interval length δ . The calculations can be guaranteed to be at least as accurate as those resulting from an analytical representation of the stress-strain curve. This is because any analytical expression for the stress-strain curve requires a numerical procedure with all the approximations involved.

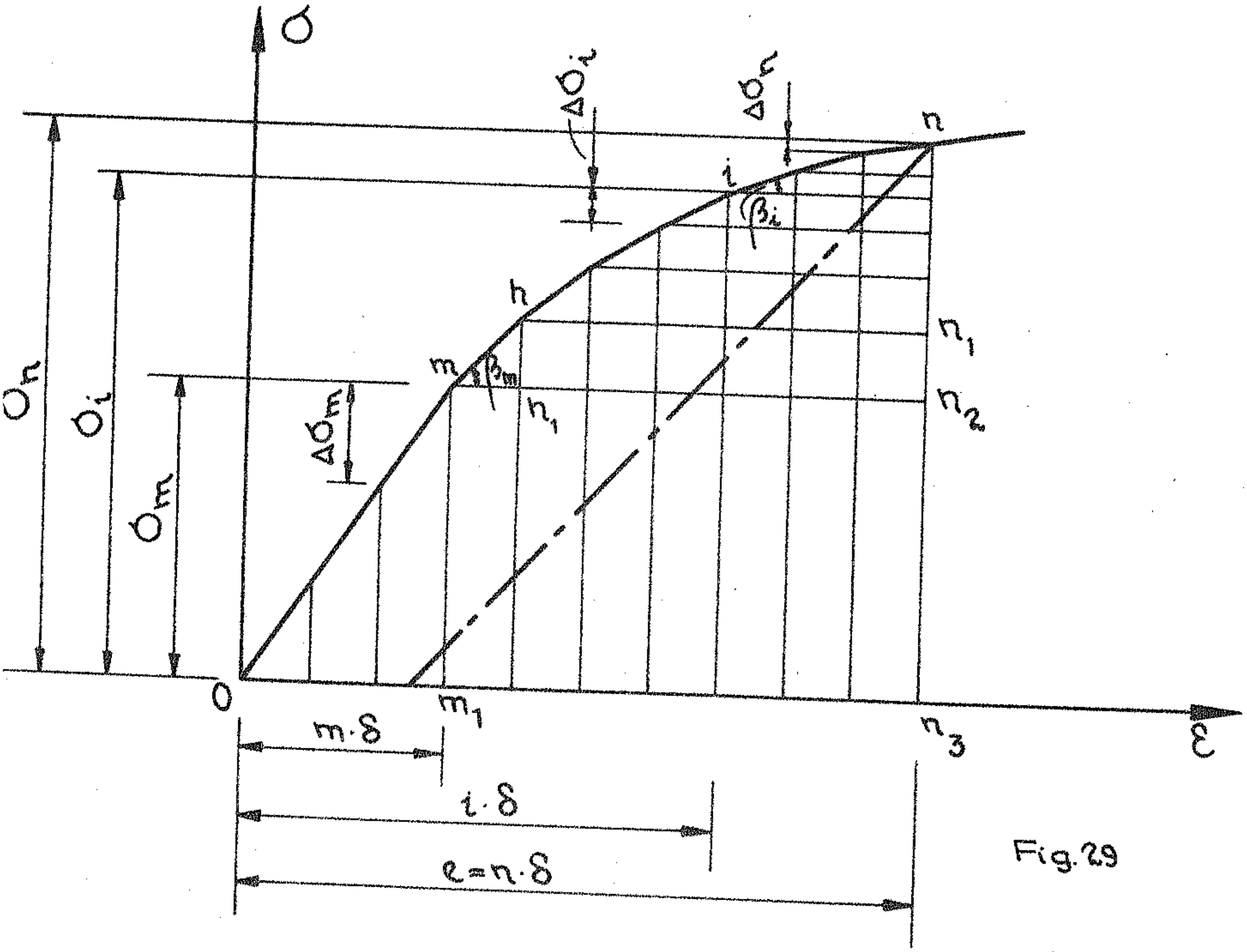


Fig.29

In Fig. (29) the unloading begins at the point n, while point i represents an intermediate point on the stress-strain curve. β_i is the angle which the stress-strain curve makes with the horizontal line at the point i. To give the numerical method a wider application, it is assumed that the stress-strain curve possesses a marked yield-point at the point m. This distinction makes the method applicable to all kinds of materials, whether they have a well-defined yiels-point or not. Thus, the material deforms elastically up to the yield-point m; after this point a continuous plasticization takes place.

By observing Fig. (29), the summation given by Eq. (134) can be written as follows:

$$S_{fn} = \sum_{i=m}^{n} (n\delta - i\delta) \cdot \sigma_{ei} \frac{\left[\sigma_i - \sigma_{(i-1)}\right] \left[\sigma_{(i+1)} - \sigma_i\right]}{\sigma_i - \sigma_{(i-1)}} \qquad (135)$$

Substitution from Eq. (113) for σ_{ei}, gives

$$S_{fn} = \sum_{i=m}^{n} \delta(n-i) \cdot i \left\{ \left[\sigma_i - \sigma_{(i-1)}\right] \left[\sigma_{(i+1)} - \sigma_i\right] \right\} \qquad (136)$$

Choosing the interval length δ, equal to unity and denoting $\left[\sigma_i - \sigma_{(i-1)}\right]$ by $\Delta\sigma_i$, the summation for S_{fn} can finally be written in the following form:

$$\boxed{S_{fn} = \sum_{i=m}^{n} (n-i) \cdot i \left[\Delta\sigma_i - \Delta\sigma_{(i+1)}\right]} \qquad (137)$$

The total area S_{tn} can be measured by planometer or by any other means of graphical integration. However, for the sake of systematization, a simple formula for the calculation of S_{tn}, will be derived as following. Considering Fig. (29) the total area will be regarded as the sum of four parts. The first part, the triangle Omm_1; the second part, the rectangle $mn_2n_3m_1$; the third part, the sum of rectangular strips of the type $hn_1n_2h_1$ above the level mn_2; and the fourth part, the sum of triangular areas of the type mhh_1 above the level mn_2. These areas will all be added together to form the total area S_{tn}.

$$S_{tn} = m \cdot \frac{\sigma_m}{2} + (n-m) \cdot \sigma_m + \Delta\sigma_{(m+1)} \left[n-(m+1)\right] +$$

$$+ \Delta\sigma_{(m+2)} \left[n-(m+2)\right] + \cdots + \Delta\sigma_{(n-1)} \left[n-(n-1)\right] +$$

$$+ \frac{\Delta\sigma_{(m+1)}}{2} + \frac{\Delta\sigma_{(m+2)}}{2} + \cdots + \frac{\Delta\sigma_n}{2}$$

$$S_{tn} = m \cdot \frac{\sigma_m}{2} + (n-m) \cdot \sigma_m + n \left[\Delta\sigma_{(m+1)} + \Delta\sigma_{(m+2)} + \cdots \Delta\sigma_{(n-1)}\right] +$$

$$+ 1/2 \left[\Delta\sigma_{(m+1)} + \Delta\sigma_{(m+2)} + \cdots \Delta\sigma_n\right] - \left[(m+1)\Delta\sigma_{(m+1)} + (m+2)\Delta\sigma_{(m+2)} + \cdots + (n-1)\Delta\sigma_{(n-1)}\right]$$

$$S_{tn} = m \cdot \frac{\sigma_m}{2} + (n-m) \cdot \sigma_m + (n+0{,}5)\left[\sigma_{(n-1)} - \sigma_m\right] - \\ - \sum_{i=m+1}^{n-1} i\,\Delta\sigma_i + \Delta\sigma_n/2 \qquad (138)$$

Sometimes it is desirable to calculate the unloading-modulus at different points on the same stress-strain diagram. In such cases the calculations can be facilitated by using the already known values of S_f and S_t for a preceding point k:

$$S_{fn} \quad S_{fk} + \sum_{i=k}^{n} (n-i) \cdot i\left[\Delta\sigma_i - \Delta\sigma_{(i+1)}\right] \qquad (139)$$

and similarly

$$S_{tn} = S_{t(k)} + (n-k)\sigma_k + (n+0{,}5)\left[\sigma_{(n-1)} - \sigma_k\right] - \\ - \sum_{i=k+1}^{n-1} i\,\Delta\sigma_i + \frac{\Delta\sigma_n}{2} \qquad (140)$$

With S_{fn} and S_{tn} calculated as above, the unloading-modulus at the point n can finally be calculated by the following formula which is identical to Eq. (28):

$$E_{an} = \frac{\sigma_n^2}{2\left(S_{tn} - S_{fn}\right)} \qquad (141)$$

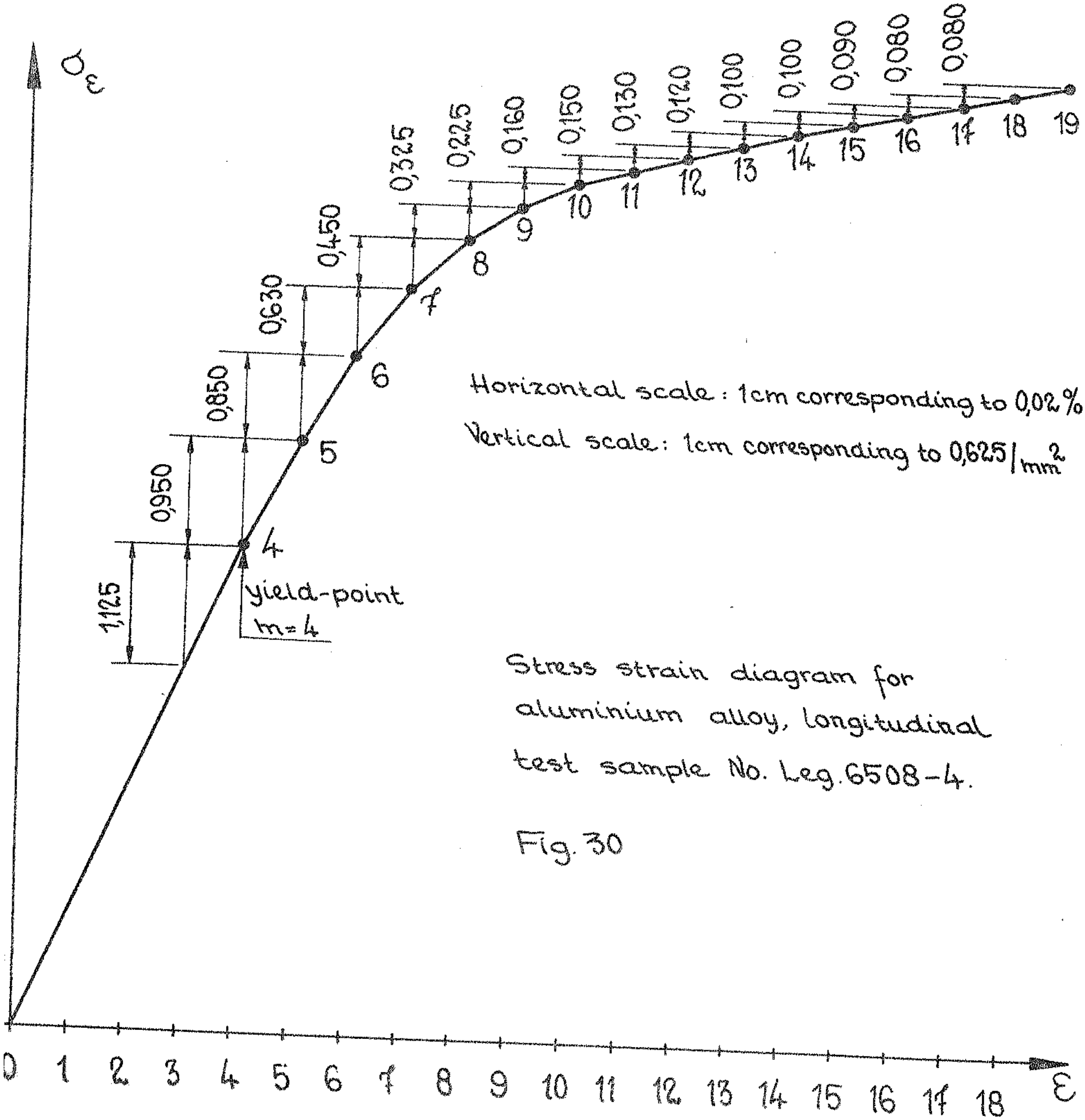
The equations (137) to (141) are put down in systematic simple forms, suitable for hand calculations, without any resort to digital computers.

An example of the applications of the method to an actual stress-strain diagram follows immediately.

39. A numerical example

The numerical method developed in the previous section is applied below to an actual stress-strain diagram, chosen from a series of experiments on the aluminium alloys[1].

The figure presented below is the result of graphical measurements on a larger scale diagram



Stress strain diagram for aluminium alloy, longitudinal test sample No. Leg.6508-4.

Fig. 30

1) The experimental stress-strain diagram is reproduced here, by courtesy of Techn. Lic. Rolf Baehre.

The calculations for the unloading-modulus at points 6, 8, 10, 12, 14, 16 and 18 will be carried out according to the following three tables: Based on equations (137) to (141), the first table gives all the intermediate steps when the point, i, is the variable; the second table gives all the intermediate steps, leading to the values of S_{tn} and S_{fn} when the point of unloading, n, is the variable; finally, the third table gives all the intermediate steps for determining the unloading-modulus, corresponding to any unloading-point, n. The tables lead to a systematic calculation and are suitable to be chosen as the standard for the hand-calculation of the unloading-modulus.

It should be mentioned that the horizontal scale of Fig. 30 has been reduced to half that of the original copy, while the vertical scale has been kept unchanged.

For the relative measurement of the areas S_{tn}, S_{fn} and S_{en} any arbitrary change of scale is immaterial.

In the numerical measurement of the vertical scale of Fig. (30) each 2 cm is taken as unity. Thus the actual values of stresses expressed in Kp/mm^2 can be obtained by multiplying the vertical scale by the factor 2 x 0.625 = = 1 · 25.

The unloading-lines drawn in relation to the stress-strain diagram of Fig. (30) are shown in Fig. (31).

TABLE 1: i, variable

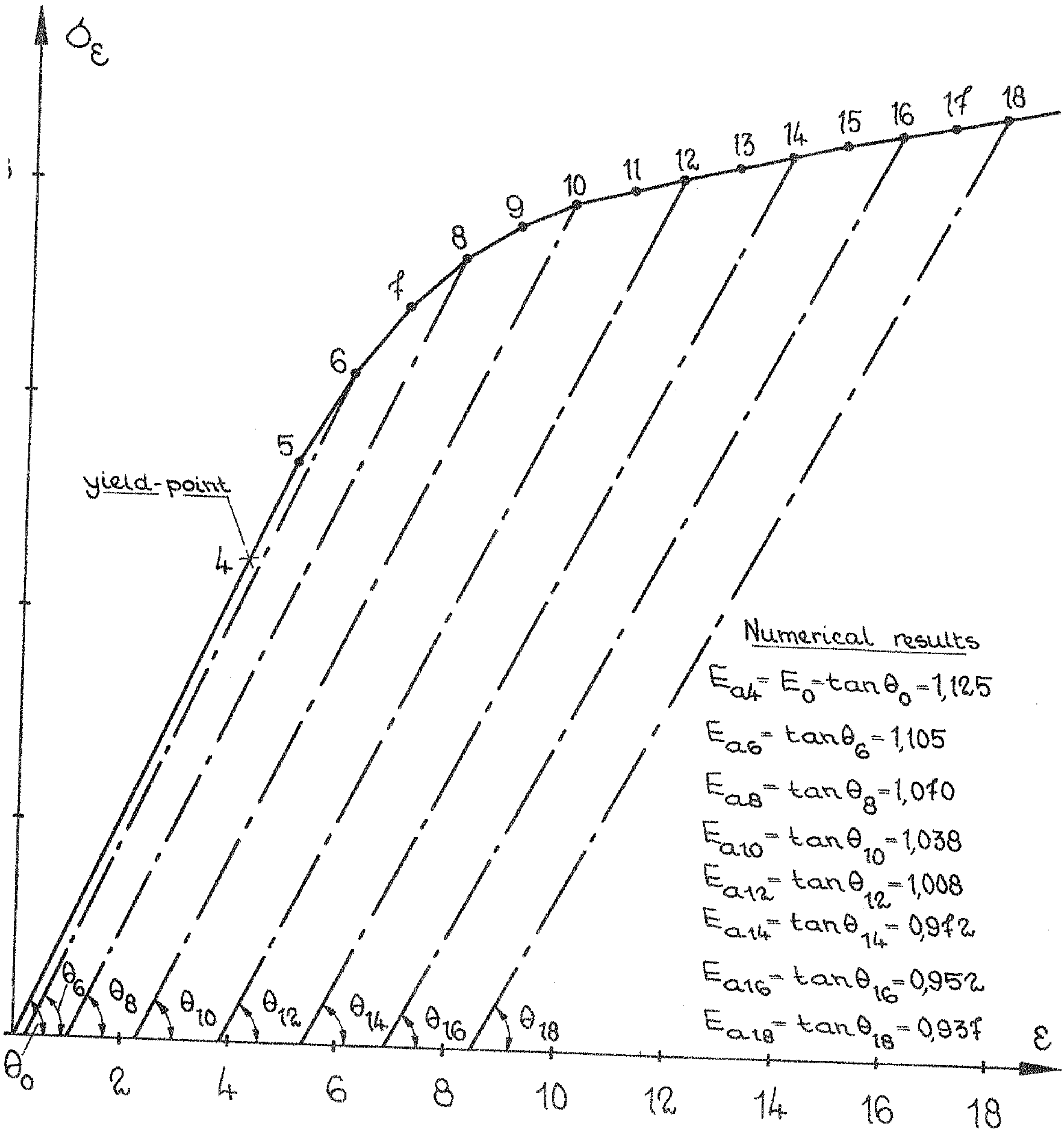
point i	σ_i	$\Delta\sigma_i$	$\Delta\sigma_i - \Delta\sigma_{(i+1)}$	$i \cdot [\Delta\sigma_i - \Delta\sigma_{(i+1)}]$	$i \cdot \Delta\sigma_i$	$(n-i)$							$(n-i) \cdot i \left[\Delta\sigma_i - \Delta\sigma_{(i+1)}\right]$						
						n=6	n=8	n=10	n=12	n=14	n=16	n=18	n=6	n=8	n=10	n=12	n=14	n=16	n=18
4	4.50	1.125	-	-	4.50	2	4	6	8	10	12	14	1.40	2.80	4.20	5.60	7.00	8.40	9.80
5	5.45	0,95	0,175	0.70	4.75	1	3	5	7	9	11	13	0.50	1.50	2.50	3.50	4.50	5.50	6.50
6	6.30	0.85	0.10	0.50	5.10	0	2	4	6	8	10	12	0	2.64	5.28	7.92	10.56	13.20	15.84
7	6.93	0.63	0.22	1.32	4.41		1	3	5	7	9	11		1.26	3.78	6.30	8.82	11.34	13.86
8	7.38	0.45	0.18	1.26	3.60		0	2	4	6	8	10		0	2.00	4.00	6.00	8.00	10.00
9	7.70	0.325	0.125	1.00	2.925			1	3	5	7	9			0.90	2.70	4.50	6.30	8.10
10	7.93	0.225	0.10	0.90	2.25			0	2	4	6	8			0	1.30	2.60	3.90	5.20
11	8.06	0.16	0.065	0.65	1.76				1	3	5	7				0.11	0.33	0.55	0.77
12	8.21	0.15	0.01	0.11	1.80				0	2	4	6				0	0.48	0.96	1.44
13	8.34	0.13	0.02	0.24	1.69					1	3	5					0.13	0.39	0.65
14	8.46	0.12	0.01	0.13	1.68					0	2	4					0	0.56	1.12
15	8.56	0.10	0.02	0.28	1.50						1	3						0	0
16	8.66	0.10	0	0	1.60						0	2						0	0.32
17	8.75	0.09	0.01	0.16	1.53							1							0.17
18	8.88	8.08	0.01	0.17	1.44							0							0
19	8.96	0.08	0	0	1.52														
												Σ	1.90	8.20	18.66	31.43	44.92	59.10	73.77

TABLE 2: n, variable $m=4,\ \sigma_m=4{,}50,\ m\cdot\frac{\sigma_m}{2}=4\cdot\frac{4{,}5}{2}=9$

point of unloading n	$(n-m)$	$(n-m)\cdot\sigma_m$	$[\sigma_{(n-1)}-\sigma_m]$	$(n+0{,}5)\cdot[\sigma_{(n-1)}-\sigma_m]$	$\sum_{i=m+1}^{n-1} i\cdot\Delta\sigma_i$	$\frac{\Delta\sigma_n}{2}$	$S_{tn}=m\cdot\frac{\sigma_m}{2}+(n-m)\cdot\sigma_m+(n+0{,}5)[\sigma_{(n-1)}-\sigma_m]-\sum_{i=m+1}^{n-1} i\cdot\Delta\sigma_i+\Delta\sigma_n/2$	$S_{fn}=\sum_{i=m}^{n}(n-i)\cdot i\cdot[\Delta\sigma_i-\Delta\sigma_{(i+1)}]$
6	2	9.00	0.95	6.17	4.75	0.43	19.85	1.90
8	4	18.00	2.43	20.65	14.26	0.23	33.62	8.20
10	6	27.00	3.20	33.60	20.79	0.11	48.93	18.66
12	8	36.00	3.56	44.50	24.80	0.08	64.78	31.43
14	10	45.00	3.84	55.70	28.29	0.06	81.47	44.92
16	12	54.00	4.06	67.00	31.47	0.05	98.58	59.10
18	14	63.00	4.25	78.70	34.60	0.04	115.84	73.77

TABLE 3: Evaluation of E_{an} $E_o = 1.125$

point of unloading n	σ_n	σ_n^2	S_{tn}	S_{fn}	$2[S_{tn}-S_{fn}]$	$E_{an} = \frac{\sigma_n^2}{2(S_{tn}-S_{fn})}$
6	6.30	39.70	19.85	1.90	35.90	1.105
8	7.38	54.40	33.62	8.20	50.84	1.070
10	7.93	62.90	48.93	18.66	60.54	1.038
12	8.21	67.30	64.78	31.43	66.70	1.008
14	8.46	71.50	81.47	44.92	73.10	0.979
16	8.66	75.00	98.58	59.10	78.96	0.952
18	8.88	78.80	115.84	73.77	84.14	0.937



Numerical results

$E_{a4} = E_0 = \tan\theta_0 = 1{,}125$

$E_{a6} = \tan\theta_6 = 1{,}105$

$E_{a8} = \tan\theta_8 = 1{,}070$

$E_{a10} = \tan\theta_{10} = 1{,}038$

$E_{a12} = \tan\theta_{12} = 1{,}008$

$E_{a14} = \tan\theta_{14} = 0{,}972$

$E_{a16} = \tan\theta_{16} = 0{,}952$

$E_{a18} = \tan\theta_{18} = 0{,}937$

Fig. 31 Experimental stress-strain diagram with the calculated unloading-moduli

4. New Theory of Elasto-Plastic Stability

41. The deduction of the new fundamental theorem of stability

In section 2 of this paper, it was proved that, assuming the tangent modulus as a critical load, all the possible alternatives of elasto--plastic buckling lead to unavoidable contradictions. The contradiction inherent in the tangent modulus theory had previously been discovered by earlier investigators, while the contradiction in Shanley's theory is pointed out in this paper for the first time. To generalize the argument, a new inelastic column theory is introduced and discussed in section 24 as a third alternative. This new alternative is based on the assumption that the column starts to bend at the tangent modulus load simultaneously with the increase of axial load, while the tangent modulus applies to the whole cross section up to a certain lateral deflection at which point the stresses due to bending become great enough to allow some stress reversal in the column. Thus the column attains extra rigidity after a certain amount of lateral deflection.

In section 24 where the third alternative is treated it is shown that the column can lose its stability at the tangent modulus load if and only if the increase of axial load compensates for the decrease of stresses on the convex side of the column in the bent position. This implies that the phenomenon of buckling with its instantaneous rapidity is caused by the time-dependent increase of loading, leading to the contradiction that the column in the infinitesimally-deflected position must be free from the effect of an increase of axial load, which increase is assumed to have caused that deflection.

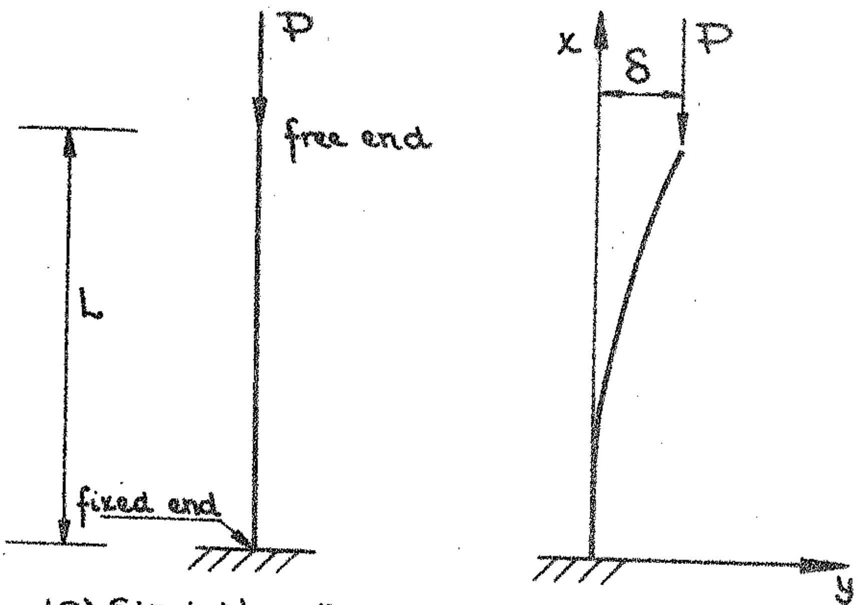
The contradiction in the third alternative can be generalized to include not only the tangent modulus load but also any load between zero and the true elasto-plastic buckling load. In fact no matter in what stage of loading you choose, the increase of axial load cannot cause the loss of stability, for such an action would lead to the same contradiction as the third alternative. This generalization leads to the conclusion that under any given load the stability of a column must be investigated under constant load since the simultaneous increase of axial load has no effect on the stability of the column. Since the above theory is so important it will be stated below as a fundamental theorem.

Fundamental theorem of stability: the stability of any column must be examined under the action of a constant load since the simultaneous increase of axial loading cannot affect the phenomenon of buckling.

The above fundamental theorem has not been verified in connection with the theory of elastic stability since the constant value of the modulus of elasticity under any given loading conditions does not require any such verification. In the case of elasto-plastic buckling, however, the theorem is extremely important, and the reason why it has not yet been deduced is due to the contradictions in the existing inelastic column theories and the accompanying scientific confusion that has prevailed ever since the tangent modulus theory was introduced.

The problem of elasto-plastic stability will be solved on the basis of the above proposed theorem in much the same way that Euler approached the problem of elastic stability.

42. Discussion of elasto-plastic instability



(a) Straight position. b) Deflected position.

Fig. 32 The column under centrally applied constant load P.

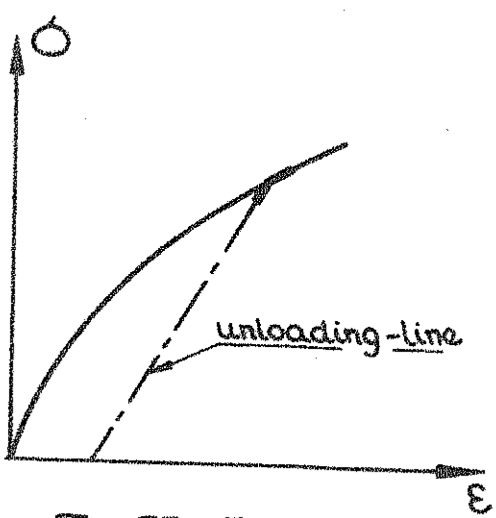
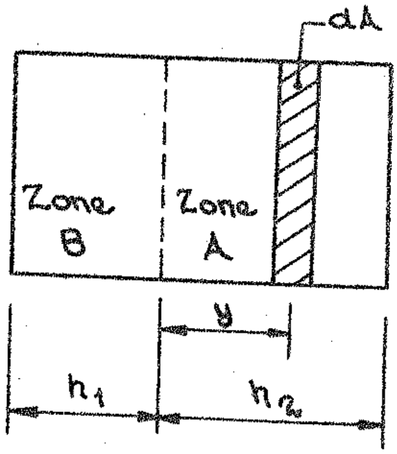


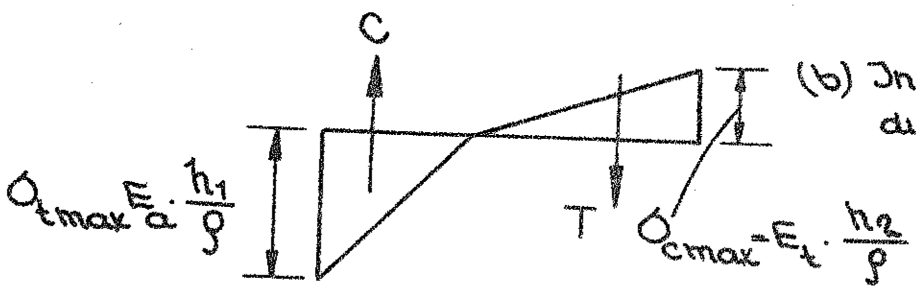
Fig. 33 Infinitesimal unloading.

We begin the discussion with a definition of the buckling load. An initially straight column, under the action of a centric load P, is given a small lateral displacement, δ . The smallest load at which the column stays in equilibrium in the displaced position is called the buckling load. For any load smaller than the buckling load the column does not stay in the displaced position but rather moves towards the original equilibrium position with the proviso that the cause of lateral bending is removed. Here, for the elasto-plastic column, in contrast to the elastic column, we use the expression, "moving towards" the equilibrium position, instead of the expression, "going back to" the original equilibrium position. The reason is because under any given load within the plastic range an elasto-plastic column never returns perfectly to the original equilibrium position after a lateral displacement. The reason for the choice of the expression "moving towards the original equilibrium position" will be explained in more detail immediately below.

The column in Fig. 32 is assumed to be rectangular in cross section (Fig. 34). It is further assumed that the column is initially straight and the external load P is applied to it centrally.



(a) As a result of bending the compression increases in zone A and decreases in zone B.



(b) Infinitesimal stresses due to bending.

Fig. 34

Suppose that at a certain stage of loading, below the buckling load, we give the column a small lateral deflection, δ . A cross section of the column, as shown in Fig. 34 (a), will be divided into two Zones, A and B. As a result of bending compressive stresses increase in Zone A and the extra deformations take place along the stress-strain diagram; while compressive stresses decrease in Zone B and the extra deformations take place along the unloading line (Fig. 33). In Zone A the additional deformations due to bending cause additional frictional losses, δW_f, as a result of the action of plastic stresses on the plasticized part of the effective area. Let us assume that lateral deflection of the column is prompted by the action of a horizontal force H, increasing from zero at the straight position of the column to the value, H, corresponding to the deflection, δ, of the free end of the column. The energy equation for the process of bending would consist of four terms: the first term would be the work done by the force, H, designated by δW_H; the second term, the work done by the load, P, shown by δW_P (caused by the short distance which the load falls); the third term, the elastic energy stored in the column, δ We; the fourth term, the energy lost, δW_f, as explained above. Thus the energy equation can be written as follows:

$$\delta W_H + \delta W_P = \delta W_e + \delta W_f \tag{142}$$

If we now reverse the process by gently decreasing the horisontal force from the value, H, to zero, the only recoverable energy would be δ We, which would be smaller than the sum $\delta W_H + \delta W_P$. Therefore, if the axial load P lies within the plastic range, the elasto-plastic column can never return freely to its original straight position. Hence, we conclude that the column "moves towards the original position". Such a property is significant because at any level below the elasto-plastic buckling load, falling within the plastic range, once the initially straight column is given a small lateral deflection, it will bend with a resulting permanent curvature.

Now suppose that the load, P at which the column is given a small lateral deflection corresponds to the elasto-plastic buckling load. We proceed to find a formula for the evaluation of the buckling load, using the condition of stability, which states that the column under the load P must remain in equilibrium in the new bent position. We now study the condition of elasto--plasticity as an element of area, dA, situated in Zone A of the cross section shown in Fig. 34 (a). The initial effective area corresponding to element, dA, would be designated by dA_o. According to the discussion in Section 36 the total, initial, effective area of the equivalent mathematical model can be taken to be equal to the nominal cross sectional area of the physical model. Thus, we can write: $A_o = A$ and $dA = dA_o$. Denoting the non-plasticized, effective part of the initial total effective area, A, by $A_{e\varepsilon}$ and the initial modulus of elasticity by E_o and observing that plasticization has taken place uniformly over the whole cross section we arrive at the following relation:

$$A_{e\varepsilon} = A \cdot a_{e\varepsilon} = A \cdot \frac{1}{E_o} \cdot \frac{d\sigma_\varepsilon}{d\varepsilon} \qquad \text{See Eq. (98)} \tag{143}$$

$a_{e\varepsilon}$ as defined in Section 36 is the elastic part of the unit effective area.

Substituting for $\frac{d\sigma_\varepsilon}{d\varepsilon} = E_t$ and writing the areas in differential form,

$$dA_{e\varepsilon} = \frac{dA}{E_o} \cdot E_t \qquad \text{or} \qquad \frac{dA}{dA_{e\varepsilon}} = \frac{E_o}{E_t} \tag{144}$$

Because of the bending of the column, the compressive stress on the element of the area dA, shown in Fig. 34 (a), increases by an amount denoted by σ_{dA}. The increase of stress can only be resisted by the non-plasticized effective part of dA, viz., $dA_{e\varepsilon}$, with the modulus of elasticity, E_o. This argument implies that we have to define another stress on $dA_{e\varepsilon}$, denoted by $\sigma_{dA_{e\varepsilon}}$, which is the actual stress on the non-plasticized part of the element of area dA, since the already plasticized part of the area cannot take up any extra stresses. For the stress, $\sigma_{dA_{e\varepsilon}}$, we have the following relation:

$$\sigma_{dA_{e\varepsilon}} = E_o \cdot \varepsilon_{dA} = E_o \cdot \frac{y}{\rho} \tag{145}$$

where ε_{dA} represents the strain at dA, caused by bending.

$\sigma_{dA_{e\varepsilon}}$, according to definition, is the force caused by bending per unit area of $dA_{e\varepsilon}$. Setting $dA_{e\varepsilon} = 1$ in Eq. (144),

$$dA = \frac{E_o}{E_t} \tag{146}$$

Measuring the intensity of stresses with respect to the initial effective area,

$$\sigma_{dA} = \frac{\sigma dA_{e\varepsilon}}{dA} = \frac{\sigma dA_{e\varepsilon}}{E_0/E_t} \qquad \text{when } dA_{e\varepsilon} = 1$$

Substituting for $\sigma_{dA_{e\varepsilon}}$ from Eq. (145),

$$\sigma_{dA} = \frac{y}{\rho} \cdot E_t \tag{147}$$

The stress defined by Eq. (145) is the actual stress measured with respect to the elastic area which does actually carry that stress. The stress defined by Eq. (147), on the other hand, is an average stress conveniently measured with respect to the initial effective area. The former definition of "elastic stress" was presented in connection with the study of the models in Sections 32 and 35, whereas the latter definition of "elastic stress" was first presented in Section 36.

From the above argument we conclude that throughout Zone A, the stress can be defined according to Eq. (147). The stresses on this part of the cross section of the column coincide with those obtained by Von-Kármán. The method of approach, however, is quite different. We must bear in mind that the stresses defined by Eq. (147) represent average stresses, while the actual values of stresses, defined by Eq. (145), are greater. Throughout Zone B, all elements of cross sectional area take part in the unloading process, with the unloading modulus of elasticity defined according to Eq. (132). The procedure for determining the buckling load will be similar to that followed by Von-Kármán with the initial modulus of elasticity in Von-Kármán's formula being replaced by the unloading modulus of elasticity defined according to Eq.(132).

Having discussed the various fundamental concepts of elasto-plastic instability, we proceed to present briefly the procedure for finding the elasto-plastic buckling load. For the rectangular cross section shown in Fig. (34) the requirement C = T gives $E_{a\varepsilon} \cdot h_1^2 = E_t \cdot h_2^2$;

$$h_1 + h_2 = h \;; \quad h_2 = h_1 \cdot \frac{\sqrt{E_{a\varepsilon}}}{\sqrt{E_t}} = h - h_1$$

$$\therefore h_1 = \frac{h\sqrt{E_t}}{(\sqrt{E_t} + \sqrt{E_{a\varepsilon}})} \;; \quad h_2 = \frac{h\sqrt{E_{a\varepsilon}}}{(\sqrt{E_t} + \sqrt{E_{a\varepsilon}})} \tag{148}$$

The bending moment, Mx, is calculated from Fig. 34 (b):

$$M_x = \frac{E_a \cdot h_1}{\rho} \cdot \frac{bh_1}{2} \cdot 2/3\,(h_1 + h_2)\ ;$$

$$M_x = \frac{E_a \cdot hb}{3\rho} \cdot \frac{h^2 \cdot E_t}{(\sqrt{E_t} + \sqrt{E_{a\varepsilon}})^2} = \frac{bh^3}{12\rho} \cdot \frac{4E_{a\varepsilon} \cdot E_t}{(\sqrt{E_t} + \sqrt{E_{a\varepsilon}})^2}\ ;$$

Let $\frac{4E_{a\varepsilon}E_t}{(\sqrt{E_t} + \sqrt{E_{a\varepsilon}})^2} = D$, $M_x = \frac{J \cdot D}{\rho}$, $\left(J = \frac{bh^3}{12}\right)$;

$$\frac{1}{\rho} = y'' = \frac{M_x}{D \cdot J}$$

By comparing the expression,

$$\frac{1}{\rho} = \frac{M_x}{DJ}$$

for the elasto-plastic column
and the expression,

$$\frac{1}{\rho} = \frac{M_x}{EJ}$$

for the elastic column
and by maintaining identical boundary conditions, we arrive at the following elasto-plastic buckling load, Pep, for the column shown in Fig. 32:

$$P_{ep} = \frac{\pi^2 \cdot DJ}{4L^2} \tag{149}$$

So far, nothing has been said about D except that it is a constant depending on the material properties, cross sectional form and cross sectional area of the column. There is, nevertheless, a similarity between the relation (149) and the corresponding relation for buckling load of an elastic column. For this reason, we replace D by the notation E_{ep}

that appears to have some familiarity with the modulus of elasticity. We suggest here that E_{ep} be called the "modulus of elasto-plasticity".

The other boundary conditions would similarly lead to the following general solution:

$$P_{ep} = \frac{\pi^2 \cdot E_{ep} J}{(\beta L)^2} \tag{150}$$

The modulus of elasto-plasticity for various cross sectional shapes can be obtained by a procedure similar to the one described above for the rectangular section. The expression, P_{ep}, for a rectangular section and an idealized I-section are determined by the following two equations:

$$E_{ep} = \frac{4 E_{a\varepsilon} \cdot E_t}{(\sqrt{E_t} + \sqrt{E_{a\varepsilon}})^2} \quad \text{For rectangular section ;} \tag{151}$$

$$E_{ep} = \frac{2 E_{a\varepsilon} E_t}{E_{a\varepsilon} + E_t} \quad \text{For idealized I-section} \tag{152}$$

where $E_{a\varepsilon}$ in the above equations is defined by Eq. (132).

A numerical comparison of various column theories follows immediately.

43. Numerical comparison of various column theories

The following general equation for the buckling load holds for all the column theories:

$$P_b = \frac{\pi^2 D J}{(\beta L)^2}$$

In the above equation the definition of D for various column theories is given below.

1. $D = E_o$ = initial modulus of elasticity for Euler load.
2. $D = E_t$ = tangent-modulus for the buckling load predicted by the tangent-modulus theory.

3. $D = E_r$ = reduced-modulus for the buckling load predicted by Von-Kármán's reduced-modulus theory;

$E_r = \frac{2 E E_t}{E + E_t}$ for an idealized I-section, where E is the initial modulus of elasticity.

4. $D = E_{ep}$ = modulus of elasto-plasticity for the buckling load calculated by the author's new theory of elasto-plastic stability;

$E_{ep} = \frac{2 E_{an} \cdot E_t}{E_{an} + E_t}$ for an idealized I-section, where E_{an} is the unloading-modulus at point n on the stress-strain diagram.

Using the experimental stress-strain diagram, shown in Fig. (31), and assuming an idealized I-section, the average buckling stress will be calculated as a function of the slenderness ratio, for all the above four column theories. Since the unloading-modulus varies along the stress-strain diagram, it is more convenient to calculate the slenderness ratio λ, as a function of the average stress σ_{bn} at the point of unloading, n. The general formula for the buckling load can be rewritten in the form,

$$\sigma_{bn} = \frac{\pi^2 \cdot D}{\lambda_n^2};$$

$$\lambda_n = \pi \cdot \sqrt{\frac{D}{\sigma_{bn}}}$$

The detailed calculations and the graphical representations are shown on the following pages.

TABLE 4 - Numerical comparison of various column theories

point of unloading n	σ_{bn}	$10^{-4} \cdot E_{an}$	$10^{-4} \cdot E_{tn}$	$10^{-8} \cdot 2E_o \cdot E_{tn}$	$10^{-4} \cdot (E_o + E_{tn})$	$10^{-4} \cdot E_{r(n)}$ $\frac{2E_o \cdot E_{tn}}{E_o + E_{tn}}$	$10^{-8} \cdot 2E_{an} \cdot E_{tn}$	$10^{-4} \cdot (E_{an} + E_{tn})$	$10^{-4} \cdot E_{ep(n)} = \frac{2E_{an} \cdot E_{tn}}{E_{an} + E_{tn}}$	λ_n Euler theory	λ_n Tangent-Modulus theory	λ_n Reduced Modulus theory	λ_n New theory of elasto-plastic stability
2	2.25	1.125	1.125			1.1125			1.125	222	222	222	222
4	4.50	1.125	1.125			1.125			1.125	157	157	157	157
6	6.30	1.105	0.850	1.912	1.975	0.968	1.880	1.955	0.962	133	115	123	122
8	7.38	1.070	0.450	1.012	1.375	0.737	0.964	1.520	0.633	123	77.5	99.5	92.0
10	7.93	1.038	0.225	0.506	1.350	0.375	0.466	1.263	0.369	118	53.0	68.3	67.7
12	8.21	1.008	0.150	0.338	1.275	0.265	0.302	1.158	0.261	116	42.5	56.5	56.0
14	8.46	0.979	0.100	0.225	1.225	0.1835	0.1958	1.079	0.1815	114.3	34.2	46.2	45.9
16	8.66	0.952	0.090	0.202	1.215	0.166	0.1715	1.042	0.1645	113.2	32.0	43.6	43.3
18	8.88	0.937	0.080	0.180	1.205	0.1493	0.150	1.017	0.1475	112.0	29,8	40.8	40.5

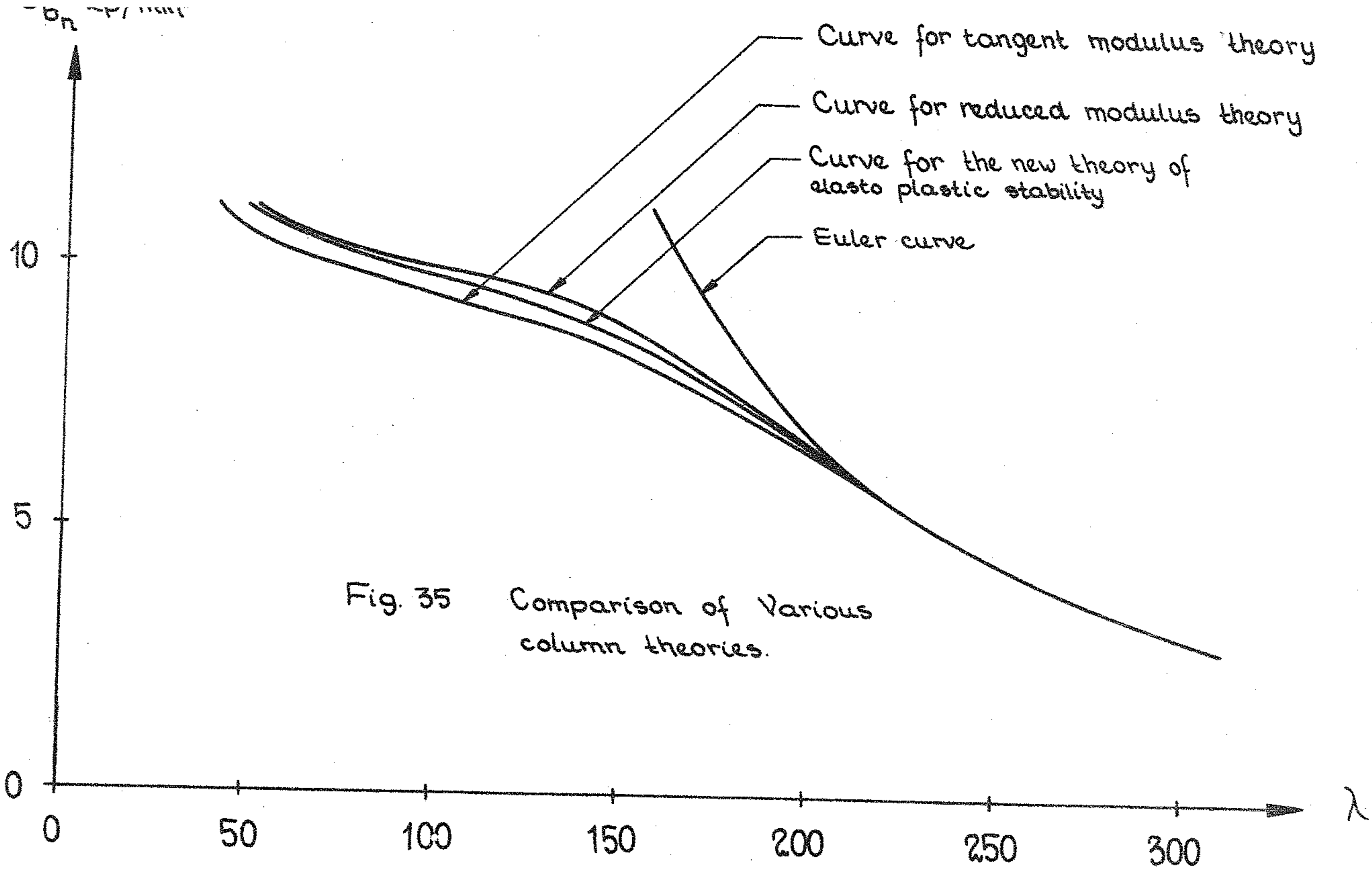


Fig. 35 Comparison of Various column theories.

Note: In order to get the actual values of stresses and slenderness ratios for the stress-strain diagram of Fig. 30, the relative values of σ_{bn} and λ in Table 4 are multiplied by the factors 1.25 and $\sqrt{2} = 1.414$ respectively.

CONCLUSIONS

We have arrived at the following conclusions which are based on the preceding discussions, mathematical analyses and theoretical investigations:

1. The paradoxical idea of <u>bending</u> or <u>buckling</u> at the tangent modulus load for an initially straight, centrally loaded elasto-plastic column is not correct.

2. For any physical model, constructed of any arbitrary typical elasto-plastic material, the unloading-modulus is a function of the shape of the stress-strain curve and the position of the point of unloading. The unloading-modulus is in general less than the initial modulus of elasticity and it decreases as the strain corresponding to the point of unloading increases. The unloading modulus can be equal to the initial modulus of elasticity in two cases: (a) when the deformable medium is ideal elastic; (b) when the deformable medium is ideal elasto-plastic, viz., when the deformations up to a certain loading level are ideal elastic and after that level continue under constant load. When the tangent modulus approaches zero, the unloading-modulus tends to a constant. However, when the tangent modulus approaches a constant value the unloading-modulus continues to decrease as the value of strain for the point of unloading increases.

3. The fundamental theorem of stability, which has been deduced from the previous statements in this paper, asserts that the stability of any column must, as a general rule, be investigated under the influence of a constant load because the simultaneous increase of axial loading cannot affect phenomenon of stability.

4. If we were to give a lateral displacement to an initially straight, centrally loaded elasto-plastic column under the influence of any load below the elasto-plastic buckling load and within the range of plastic deformations, the column then, upon the removal of the cause of lateral displacement, would " move towards the original equilibrium position" but would never perfectly return to its original state of equilibrium. Thus the column would bend with resulting permanent curvature.

5. When an initially straight centrally loaded elasto-plastic column under the action of its buckling load is given a small lateral displacement, the column in the bent position would acquire extra rigidity; such an added rigidity would mean that the proposed "modulus of elasto-plasticity" would always be greater than the tangent modulus. However, since the unloading-modulus is always less than the initial modulus of elasticity, the proposed "modulus of elasto-plasticity" would always be less than Von-Kármán's reduced modulus. Thus, the new theory of elasto-plastic stability would result in a buckling inelastic column load which would lie somewhere between the tangent modulus load and the value predicted by the reduced modulus theory.

A summary of this paper in English, German and French with references to relevant figures and equations follows immediately.

SUMMARY

The investigations presented in this dissertation treat the problem of stability of an initially straight centrally load elasto-plastic column. The paper is divided into four chapters; each chapter in turn is divided into a number of sections. Chapter 1 is devoted to a general introduction with a short history of the development of the various column theories as recorded in literature up to the present date. Chapter 2 covers a full discussion of the contradictions in the inelastic column theories that assume the tangent-modulus load as a critical load. Chapter 3 is devoted to the development of a new theory of elasto-plasticity for the determination of the unloading-modulus of elasticity. Finally Chapter 4 is devoted to the development of the new theory of elasto-plastic stability.

Following the above short introduction, the original contributions of the investigations appearing in this dissertation will be explained in brief.

Discussion of the Contradictions in the Inelastic Column Theories that Assume Tangent Modulus Load as a Critical Load.

The discussions begin with a review of Shanley's theory including an examination of Shanley's remarks, assumptions, test data, conclusions, and his mathematical analysis. The discussion continues with the interpretations of the theory by Timoshenko and Von-Kármán. Timoshenko is mentioned because his book, Theory of Elastic Stability, is available at almost any technical library throughout the world, even where there is no direct access to Shanley's original literature. Von-Kármán is mentioned because it was just the shortcomings in Von-Kármán's original reduced modulus theory which led Shanley to the development of his theory. After a review of Shanley's theory the discussion is generalized to show the contradictions in all the inelastic column theories that assume the tangent modulus load to be critical load. The correct alternative for the elasto-plastic buckling load is finally pointed out.

Discussion of the Shanley's theory

Shanley's remarks, appeared for the first time in the article, "The Column Paradox", in Journal of The Aeronautical Sciences, Vol. 13, no. 12, Dec. 1946.

A few months later his inelastic column theory was published in the same periodical Vol. 14, no. 5, May 1947. These two articles are referred to here as Reference 1 and 2. Shanley's remarks and assumptions, his test data, and his mathematical analysis are discussed here in full detail. All the equations and figures reproduced from Ref. 2 are given new numbers in this paper.

After the appearance of Considère-Engesser and Von-Kármán's theory, it was asserted by many investigators, including Shanley, that the true value of the inelastic buckling-load must lie somewhere between the tangent modulus load and the value predicted by the reduced-modulus theory. Shanley reasoned that such a result could be obtained by assuming that the column starts bending at the tangent modulus load while at the same time the axial load increases. Shanley reasoned further that, in order to permit loading beyond the tangent modulus load a certain type of stress-distribution according to Fig. 4 is needed. This type of stress-distribution allows strain-reversal from the very inception of bending. However, in the discussion following Fig. 5, it is demonstrated that the extra stresses produced at the start of bending are infinitesimal in comparison with stresses due to the tangent modulus load. Therefore, even if the neutral axis of the column has shifted considerably for the infinitesimal stresses due to bending, the displacement of the neutral axis for the total stresses can be neglected provided that the values of the bending stresses remain infinitesimal with respect to the uniform stresses due to tangent-modulus load. Thus, while the column is still centrally loaded and while the bending stresses are infinitesimal, the increase of axial load can accompany stress-reversal if and only if the stresses due to an increase of axial load are smaller than any infinitesimal value, that is to say, if they are equal to zero. Therefore, at the start of bending before the deformations become large enough, it is impossible to get stress-reversal simultaneously with an increase of axial load. This fundamental error is further reflected in Shanley's mathematical analysis.

Eq. (3) is considered by Shanley in his mathematical analysis to represent the complete theory of column action. The derivation of this equation is based upon three assumptions. In the first assumption the column starts bending at the tangent modulus load. In the second assumption the axial load continues to increase as the bending proceeds. In the third assumption the stress-reversal takes place as soon as the bending starts. The second and third assumptions are contradictory to each other and cannot be fulfilled simultaneously. In fact, Shanley's first assumption that the column starts bending at the tangent modulus load loses its significance because of the theory's failure to achieve its goal.

The discussion can be generalized by examining all the alternative theories of the elasto-plastic buckling, assuming the tangent modulus load as a critical load. The tangent modulus theory has been rejected by previous investigators because of the theory's failure to take into consideration the extra rigidity possessed by the column in the bent position. Shanley's theory, as shown in this paper, cannot be accepted because of the contradiction arising out of Shanley's second and third assumptions. There remains only one more alternative to be investigated.

<u>The third alternative</u> the column starts bending at the tangent modulus load simultaneously with the increase of axial load, while the tangent modulus applies to the whole cross section, up to a certain lateral deflection where the stresses due to bending become large enough to allow some stress reversal in the column to take place. This action gives the column extra rigidity after a certain amount of lateral deflection.

The third alternative has not been even discussed or mentioned by previous investigators. The discussion of this alternative is important because it is the last possibility which predicts the start of bending at the tangent modulus load. This theory avoids the contradiction inherent in Shanley's theory and allows equilibrium sequences in the bent position. This theory, however, leads to the following contradiction.

The tangent modulus load is the smallest load if and only if the tangent modulus applies to the whole column. This is true, if and only if the increase of axial load compensates for the decrease in stresses on the convex side of the column in the bent position. In reality this means that the cause of lateral deflection is the increase of axial load. However, the increase of axial load beyond the tangent modulus load is dependent on time, whereas the assumed loss of stability at the tangent modulus load is independent of time. The reason is that the increase of axial load P from an infinitesimal value below the critical load to an infinitesimal value above the critical load is accompanied by a change in the state of energy of the system, which is consisted of the load and the column. Once the load has reached the critical value, the change in the state of energy of the system is independent of time. Independence from time means that for the infinitesimal range of lateral deflections, where the formula for the critical load is theoretically correct, buckling takes place instantaneously. This argument leads to the conclusion that no matter how fast the rate of loading beyond the tangent modulus load, the rate of assumed lateral deflection must be faster. This last point leads to the contradiction that the column in the bent position must be free from the effect of increase of axial loading.

We conclude that the tangent modulus load is not a critical load and that an inelastic column has only one critical load which always exceeds the tangent modulus load. The elasto-plastic buckling load is, on the other hand, always less than the value predicted by the reduced modulus theory. The proof of the last statement is left to the result of the investigations in Chapters which follow.

New Theory of Elasto-Plasticity for the Determination of the Unloading-Modulus.

The essential ideas and concepts of elasto-plasticity are presented and defined in Section 32 in connection with a detailed discussion of load deformation properties of a mathematical, structural model built up in cross section by three distinct layers of homogeneous, ideal elasto-plastic media with the same modulus of elasticity but with different yield-point stresses. The stress-strain diagram for this model is shown in Fig. 12, where the three breaking points 1, 2 and 3 indicate the various loading levels at which the three different elements of area of the cross section get plasticized. The concept of "effective or elastic area" refers to the non-plasticized area at a given loading-level. The plasticized part of the "initial total effective area" is called the "plastic area". The force carried by the elastic area is called the "elastic force" and the force carried by the plastic area is called "the plastic force".
As the external load increases, the effective area decreases. This structural behaviour gives the impression of a continuously decreasing modulus of elasticity. However, the modulus of elasticity remains unchanged. It is the continuously diminishing effective area that causes this effect.

The work done by the external load on a nominal unit area of the element of unit length is equal to the area under the stress-strain diagram. Part of this energy is converted into potential energy, stored in the sample through the action of the elastic stresses. The other part is converted into heat energy through the action of the plastic-stresses. The first part is reversible, while the second part is irreversible.

During the process of unloading all the potential energy stored in the model, in the form of elastic energy will be recovered. No further plasticization takes place during this period. The path of the unloading will be a straight line because the energy to be recovered is the sum of all the elastic energy due to the action of the elastic stresses during the loading process. So far as unloading is concerned, in the absence of any plasticization during this period, there will be no difference shatsoever in the structural behaviour of an elasto-plastic, or an ideal-elastic material. Because of behaviour of the unloading process the tangent of the unloading-line is called the <u>unloading modulus of elasticity</u> or simply the <u>unloading-modulus</u>.

With precise information about the various elements of area and the loading levels at which they get plasticized one can determine the load-deformation properties of the model for both loading and unloading processes (Fig. 15).

In Section 33 the generalized model is presented by its cross sectional area consisting of n layers of incremental area, ΔA_i, constructed of homogeneous ideal elasto-plastic media with the same modulus of elasticity but with varying yield-point stresses. Eq. (44) and Inequality (45) define the mathematical model precisely. By the proper choice of the incremental areas of Eq. (44) and by the proper assignment of values to the elements of the Inequality (45) we could construct a mathematical model which would be equivalent to any given stress-strain diagram and vice versa. The load-deformation equivalence between the mathematical model and the physical or experimental model holds as long as creep deformations or thermal effects do not begin to change the form of the experimental stress-strain diagram. One may conceive of the incremental areas of the generalized model spreading out over the whole cross section uniformly. Thus, we get uniform plasticization over the whole cross sectional area at all loading levels. For an actual physical model the stress distribution over the cross section can be conceived in the manner descriebed immediately below.

An elasto-plastic material is thought to be made up of tiny molecular elements which constitute the building stones inside the crystal units. The position and the inclination of the molecular elements define the geometric configuration of the crystal structure. All the molecular elements are supposed to have the same modulus of elasticity which can be considered to be an average value for the material.

When a continuously increasing external static load is applied centrically to a test sample of an elasto-plastic material with the nominal cross sectional area, A, all the molecular elements crossing a certain cross section of the test-sample take part in carrying the load, even if they are not stressed identically. (Different elements have different inclinations with respect to the direction of the external force.) The stress distribution is a function of the geometric configuration of the crystal units and their orientation with respect to the direction of the force. At this point the concept of effective area for the physical model is introduced. For all the elements which cross a certain cross section of the test-sample, one element parallel to the direction of the external force is singled out. All the other elements are replaced by immaginary ones parallel to the direction of the force with fictitious cross sectional areas chosen in such a way that the deformation properties of the test-sample remain unchanged. The total cross sectional area presented in this way is called the total effective area.

Based on the above basic ideas and fundamental concepts, a space-truss, built up of one vertical element and three identical inclined elements, is subjected to a detailed study in Section 35. The truss shown in Fig. (17) is assumed to be made of an ideal elasto-plastic material. The choice of such a model represents very simplified form of a crystal unit. As a continuously increasing vertical load P is applied at the crown-point 0, all the four truss elements deform elastically up to a certain limit-load, P_1. Beyond this load, complete plasticization takes place in the vertical element, while the inclined elements are still stressed within the range of their elasticity. Any extra load must be carried only by the inclined elements. The load-deformation diagram for this model is shown in Fig (19). Point 1 on this diagram corresponds to a stage of loading where plasticization in the vertical element begins to take place. At an infinitesimal distance to the right of point 1, the load P_1' represents the total load;

the load carried by the vertical element represents the plastic load; while the load carried by the inclined-elements together represents the elastic load. These terms are analogous to the total stress, plastic stress, and elastic stress, already defined.

The work done, due to the action of the plastic stress, beyond point 1, is lost in the form of heat energy. The calculation for the unloading-modulus is carried out in detail. (Eq. 90). A numerical evaluation demonstrates that the deviation between the real unloading-line and what has (previously) been assumed to be the right one can be appreciable. (Fig. 21).

Derivation of the unloading-modulus for the most general case of a visco-elastic material with any stress-strain diagram is mathematically more involved because of the continuously-changing curvature of the stress-strain diagram. However, in Section 36, the calculations are carried out, step by step, by the systematic application of the fundamental concepts introduced in the theory. Eq. (107) is an analytic expression for the unloading-modulus. The term, $\sigma_{e\varepsilon}$, defined as the elastic-stress, appears in Eq. (107). The functional relationship for the elastic stress as a continuously decreasing function of strain is determined in Section 37. Using the basic definitions and dividing the stress-strain diagram into a series of straight lines as shown in Fig. (24), we introduce a numerical method for the calculation of the elastic-stress. The recursive relation (112) gives the elastic-stress at any point on the stress-strain diagram, provided the elastic stress at the preceding neighbouring point is known. Using this recursive relation, we develop a general simple formula for the elastic stress at any point on the stress-strain curve. The simple formula for this is given by Eq. (114).

Investigations of the variational possibilities of the unloading modulus along any stress-strain curve.

In Section 37 the properties of a general typical stress-strain curve is defined; the curve is then diagramatically presented in Fig. (25). The mathematical investigation of the rate of change of the unloading-modulus with respect to the strain at the point of unloading results in Eq. (115). The terms appearing in the fraction, to the right of Eq. (115), are very general in nature. In the absence of any specific relationship between stress and strain, these terms cannot be directly evaluated. For a mathematical investigation of such generality, the physical meaning of the integrals appearing in the numerator of the fraction has to be determined. The denominator of the fraction is always positive. It is shown that the most decisive term is the integral, $\int_0^e \sigma_{e\varepsilon} \cdot \frac{d^2\sigma_\varepsilon / d\varepsilon^2}{d\sigma_\varepsilon / d\varepsilon} \, d\varepsilon$

The detailed calculations lead to the remarkable result that this integral is independent of the path followed from the origin of the coordinate axes, O, to any point, A, on the stress-strain curve, provided that after point A, the path coincides with the original stress-strain curve. This discovery is used, first of all, to determine the variational possibilities of the unloading-modulus and secondly to determine a relationship for the elastic stress, $\sigma_{e\varepsilon}$, as a function of stress and strain given by Eq. (131). Substitution for $\sigma_{e\varepsilon}$ from Eq. (131) in Eq. (107) results in Eq. (132) which is the final analytic solution for the unloading modulus of elasticity.

Investigations for the variational possibilities of the unloading-modulus lead to Eq. (127) which demonstrates, in the most general way, that $\frac{dE_{a\varepsilon}}{de}$ is always negative. This result proves that the unloading-modulus decreases as the strain corresponding to the point of unloading increases. This means that by starting from the origin of coordinate axes with the initial modulus of elasticity and by moving along the typical stress-strain diagram defined according to Fig. (25), the unloading-modulus continually decreases. Eq. (126) shows that this is true even if the tangent modulus gets a constant value. Eq. (126) further shows that in the case where the tangent modulus approaches zero, $\frac{dE_{a\varepsilon}}{de}$ also approaches zero. Consequently, $E_{a\varepsilon}$ approaches a constant value.

<u>Numerical method for the calculation of the unloading-modulus.</u>

Eq. (107) for the unloading-modulus can be rewritten in the form given by Eq. (133). By dividing the stress-strain curve into n strips with equal intervals $\delta = 1$ along the strain axis and using the familiar concepts of the theory combined with the geometric properties of the stress-strain curve, as shown in Fig. (29), we arrive at Eq. (137) for S_{fn}, which gives the total energy lost up to the point of unloading, Ω. Eq. (138) is an expression for the total area under the stress-strain curve. Eqs. (139) and (140) give the values of S_f and S_t at the point n, while the values of these quantities is already calculated at the point k on the stress-strain diagram. If the values of S_{fn} and S_{tn} are known the unloading-modulus can be determined by Eq. (141). A numerical example is calculated in Tables 1, 2 and 3.

New Theory of Elasto-Plastic Stability.

In Section 41 a fundamental theorem of stability is deduced which asserts that the stability of any column must be investigated under the influence of constant load because the simultaneous increase of axial loading cannot affect the phenomenon of bucking.

In Section 42 by starting from the very primary definition of buckling, and on the basis of the fundamental theorem of stability, an initially straight column under the action of a centric load, P, is given a small lateral displacement, δ. The smallest load at which the column stays in the displaced position is called the buckling load. For any load smaller than the buckling-load, the column moves towards the original position, provided that the cause of lateral bending is removed. Here, for the elasto-plastic column, in contrast to the elastic column, the phrase "moving towards the original equilibrium position", is used instead of the phrase, "going back to the original equilibrium position". The reason for this is that an elasto-plastic column under any given load in the plastic range will never return perfectly to the original equilibrium position after a lateral displacement.

By giving the column a small lateral displacement, at the buckling load, the column bends. As a result of bending a cross section of the column is divided into two zones. In Zone A the compression increases infinitesimally, while in the Zone B the compression decreases infinitesimally. Calculations show that the stresses on Zone A of the column's cross section, defined through Eq. (147), coincide with those obtained by Von-Kármán. The difference is that the stresses given by Eq. (147) are the average stresses, while the actual stresses are greater. The stresses given by Von-Kármán, however, are left without any further explanation as to whether they are average or actual stresses. This difference arises out of the different approaches to the problem. While Von-Kármán considers a changing modulus of elasticity as the basis, the theory introduced here presupposes a constant modulus of elasticity but a changing effective area. Throughout Zone B all the elements of the cross sectional area take part in the unloading process with the unloading-modulus defined according to Eq. (132). Therefore, the procedure for determining the buckling load will be similar to that followed by Von-Kármán, with the initial modulus of elasticity in the Considère-Engesser's or Von-Kármán's theory being replaced by the unloading-modulus, (Eq. 132).

Eq. (150) is the general formula for the buckling-load of an elasto-plastic column. The new term introduced here is the "modulus of elasto-plasticity", designated by E_{ep}, read (E - ep). Eqs. (151) and (152) define E_{ep} for a regtangular section and an idealised I-section.

In this paper it has been proved mathematically, in the most general case, that the unloading modulus is always less than the initial modulus of elasticity; therefore the modulus of elasto-plasticity is always less than Von-Kármán's reduced modulus. The modulus of elasto-plasticity is, obviously, always greater than the tangent modulus. <u>These results lead to the conclusion that the true value of the elasto-plastic buckling load lies between the tangent modulus load and the value predicted by Von-Kármán's reduced-modulus theory</u>[1].

1) A numerical comparison of various column theories is shown in Table 4, and Fig. 35.

ZUSAMMENFASSUNG

Die in dieser Dissertation vorgelegten Untersuchungen behandeln das Stabilitätsproblem eines anfänglich geraden, zentrisch gedrückten, elasto-plastischen Stabes. Die Arbeit ist in vier Kapitel eingeteilt. Jedes von diesen zerfällt noch in eine Anzahl Abschnitte.

Das erste Kapitel behandelt die Entwicklungsgeschichte der Theorien über elasto-plastische Stäbe soweit aus der Literatur zu entnehmen bis zum heutigen Tage. Das zweite Kapitel enthält eine vollständige Diskussion der Widersprüche in den Theorien über Stäbe, die von der Tangentenmodul-Last als der kritischen Kraft ausgehen. Das dritte Kapitel behandelt die Entwicklung einer neuen Elasto-Plastizitäts-Theorie über die Bestimmung des Elastizitätsmoduls bei Entlastung. Das vierte Kapitel ist der Entwicklung der neuen Theorie der elasto-plastischen Stabilität gewidmet.

Nach der obigen kurzen Einführung werden die Originalbeiträge der in dieser Dissertation vorgelegten Untersuchungen in Kürze erklärt.

Diskussion der Widersprüche in den Theorien über Stäbe, die die Tangentenmodul-Last als kritische Kraft voraussetzen.

Die Diskussion beginnt mit einem Überblick über Shanleys Theorie, einschl. einer Überprüfung von Shanleys Anmerkungen und Voraussetzungen, seiner Testdaten und Rückschlüsse, sowie seiner mathematischen Analyse. Die Diskussion erstreckt sich weiterhin auf die Interpretation der Theorie durch Timoshenko und Von-Kármán. Timoshenko wird erwähnt, weil sein Buch Theory of Elastic Stability in fast allen technischen Bibliotheken in der ganzen Welt verfügbar ist, auch dort, wo man keinen direkten Zugang zu Shanleys Originalwerken hat. Von-Kármán wird erwähnt, weil es gerade die Schwächen in Von-Kármáns originaler Doppelmodultheorie sind, die Shanley zu der Entwicklung seiner Theorie geführt haben. Nach einem Überblick über Shanleys Theorie wird die Diskussion verallgemeinert, um die Widersprüche in allen Theorien über Stäbe aufzuweisen, die die Tangentenmodul-Last als kritische Kraft voraussetzen. Die korrekte Alternative für die elasto-plastische Knickkraft wird schliesslich hervorgehoben.

Diskussion der Shanley-Theorie

Shanleys Anmerkungen, unter dem Titel The Column Paradox, erscheinen erstmalig in The Journal of the Aeronautical Sciences, Band 13, No. 12, im Dezember 1946. Seine Stabtheorie erschien in derselben Zeitschrift, Band 14, No. 5, im Mai 1947. Auf diese beiden Artikel wird hier als Referenz 1 und 2 hingewiesen. Shanleys Anmerkungen und Voraussetzungen, seine Testdaten und mathematische Analyse werden hier eingehend diskutiert. Allen Gleichungen und Abbildungen aus Referenz 2 werden in dieser Diskussion neue Nummern gegeben.

Nach dem Erscheinen der Considère-Engesser und Von-Kármán-Theorie wurde angenommen, auf Grund der experimentellen Resultate vieler Forscher, einschl. Shanley, dass der richtige Wert der unelastischen Knickkraft irgendwo zwischen der Tangentenmodul-Last und dem durch die Doppelmodul-Theorie vorausgesagten Wert liegen muss. Shanley nahm an, dass ein solches Ergebnis durch die Annahme erzielt werden könnte, dass der Stab bei der Tangentenmodul-Last auszubiegen beginnt, während die Druckkraft gleichzeitig ansteigt. Shanley folgert weiter, dass, um eine grössere Belastung als die Tangentenmodul-Last zu erlauben, ein gewisser Typ von Spannungsverteilung, wie in Abb. (4), nötig wird. Dieser Typ von Spannungsverteilung erlaubt eine Spannungsänderung vom ersten Anfang der Krümmung an. Jedoch wird hier in der Diskussion zu Abb. (5) gezeigt, dass die Extraspannungen, die am Anfang einer Krümmung entstehen, infinitesimal sind im Vergleich zu den Spannungen auf Grund der Tangentenmodul-Last. Auch wenn sich daher die neutrale Achse des Stabes bedeutend auf Grund der infinitesimalen Krümmungsspannungen verschoben hat, so braucht diese Verschiebung der neutralen Achse für die Gesamtspannungen nicht berücksichtigt zu werden, vorausgesetzt, dass die Werte der Krümmungsspannungen infinitesimal bleiben in Bezug auf die einförmigen Spannungen auf Grund der Tangentenmodul-Last. Während also die Summe der Spannungen im Stabquerschnitt noch zentrisch ist und die Krümmungsspannungen infinitesimal sind, kann die Steigerung der Druckkraft die Spannungsentlastung begleiten, wenn und nur wenn die Spannungen auf Grund erhöhter Druckkraft geringer sind als irgendein infinitesimaler Wert, d.h., wenn sie gleich Null sind. Daher ist es am Anfang einer Krümmung, bevor die Deformationen gross genug werden, unmöglich, eine Entlastung gleichzeitig mit einer Steigerung der Druckkraft zu erhalten. Dieser grundlegende Irrtum zeigt sich weiterhin in Shanleys mathematischer Analyse.

Shanley nimmt in seiner mathematischen Analyse an, dass die Gl. (3) die vollständige Theorie der Stabwirkungsweise darstellt. Die Ableitung dieser Gleichung basiert sich auf drei Voraussetzungen. Die erste Voraussetzung besagt, dass sich der Stab bei der Tangentenmodul-Last zu biegen beginnt. Die zweite Voraussetzung besagt, dass die Druckkraft im gleichen Masse ansteigt, wie die Krümmung fortschreitet. Die dritte Voraussetzung besagt, dass Entlastung eintritt, sobald die Krümmung beginnt. Die zweite und die dritte Voraussetzung stehen im Widerspruch zueinander, und können nicht gleichzeitig zutreffen. Tatsächlich ist es so, dass Shanleys erste Voraussetzung, dass nämlich der Stab sich bei der Tangentenmodul-Last zu biegen beginnt, ihre Bedeutung verliert, da diese Theorie nicht ihr Ziel erreicht.

Die Diskussion kann verallgemeinert werden, indem alle alternativen Theorien über elasto-plastische Knickung überprüft werden, unter der Annahme, dass die Tangentenmodul-Last die kritische Kraft ist. Die Tangentenmodul-Theorie ist von früheren Forschern abgelehnt worden, weil sie nicht die extra Steifigkeit berücksichtigte, die der Stab in gekrümmter Form besitzt. Shanleys Theorie, wie hier bewiesen, kann nicht wegen der Widersprüche in seiner zweiten und dritten Voraussetzung akzeptiert werden. Es bleibt also nur noch eine Alternative zu untersuchen, und zwar

<u>die dritte Alternative</u>: der Stab beginnt, sich bei der Tangentenmodul-Last zu biegen, unter gleichzeitigem Ansteigen der Druckkraft, während der Tangentenmodul den gesamten Querschnitt bestimmt, bis zu einer gewissen seitlichen Ausbiegung, wo die Krümmungsspannungen gross genug werden, um eine gewisse Entlastung im Stabe eintreten zu lassen. Der Stab erhält also extra Steifigkeit nachdem eine gewisse seitliche Ausbiegung stattgefunden hat. Diese Alternative ist nicht von früheren Forschern erwähnt oder behandelt worden. Die Diskussion dieser alternativen Theorie ist wichtig, weil sie die letzte Möglichkeit darstellt, um den Anfang einer Krümmung bei Tangentenmodul-Last vorauszusagen. Diese Theorie vermeidet weiterhin den Widerspruch, der sich in Shanleys Theorie zeigt, und erlaubt Gleichgewichtszustände in gekrümmter Lage. Jedoch führt diese Theorie zu dem folgenden Widerspruch:

Die Tangentenmodul-Last ist dann und nur dann die geringste kritische Kraft, wenn der Tangentenmodul für den gesamten Stab zutrifft. Dieses trifft dann und nur dann zu, wenn die Steigerung der Druckkraft für den Spannungsnachlass auf der konvexen Seite des Stabes in gekrümmter Lage kompensiert. Dieses bedeutet in Wirklichkeit, dass die Ursache seitlicher Ausbiegung in der Steigerung der Druckkraft liegt. Jedoch hängt die Steigerung der Druckkraft, über die Tangentenmodul-Last hinaus, von einem Zeitfaktor ab, während der angenommene Stabilitätsverlust bei der Tangentenmodul-Last von einem Zeitfaktor unabhängig ist. Die Ursache hierfür ist, dass die Steigerung der Druckkraft P, von einem infinitesimalen Wert unterhalb der kritischen Kraft bis zu einem infinitesimalen Wert oberhalb der kritischen Kraft, von einer Veränderung im Energieverhältnis in dem System begleitet wird, das sich aus Last und Stab zusammensetzt. Wenn die Last einmal den kritischen Wert erreicht hat, ist die Veränderung im Energieverhältnis des Systems unabhängig vom Zeitfaktor. Dieses bedeutet, dass für den infinitesimalen Bereich seitlicher Ausbiegungen, für welche die Formel für die kritische Kraft theoretisch korrekt ist, Knickung augenblicklich auftritt. Dieses Argument führt zu dem Schluss, dass, gleichgültig wie gross die Belastungsgeschwindigkeit ausserhalb der Tangentenmodul-Last auch ist, die Geschwindigkeit der angenommenen seitlichen Ausbiegung grösser sein muss, was zu dem Widerspruch führt, dass der Stab in ausgebogener Lage frei vom Effekt einer Laststeigerung sein muss.

Wir folgern hieraus, dass die Tangentenmodul-Last keine kritische Kraft ist, und dass ein Stab nur eine kritische Kraft hat, die immer grösser ist als die Tangentenmodul-Last. Die elasto-plastische Knickkraft ist andererseits immer geringer als der durch die Doppelmodul-Theorie vorausgesagte Wert. Das Ergebnis der Untersuchungen, die in den folgenden Abschnitten behandelt werden, ergibt den Beweis für diese letzte Feststellung.

Neue Elasto-Plastizitätstheorie zur Bestimmung des Elastizitätsmoduls bei Entlastung

Die wesentlichen Begriffe und Gedanken über die Elasto-Plastizität sind im Abschnitt 32, in Verbindung mit einer detaillierten Erörterung über die Eigenschaften der Lastdeformation eines mathematischen Strukturmodells, dargestellt und erläutert. Dieses Modell ist in einem Querschnitt von drei deutlichen Schichten, mit gleichen Elastizitätsmoduln, aber mit verschiedenen Streckgrenzspannungen, aufgebaut.

Das Spannung-Dehnungsdiagramm für dieses Modell wird in Abb. 12 gezeigt, wo die drei Diskontinuitätspunkte 1, 2 und 3 die verschiedenen Lastniveaus zeigen, bei denen in die drei verschiedenen Flächenelementen der Querschnittsfläche völlige Plastifizierung eintritt. Die Vorstellung von "Effektivfläche" oder "Elastischfläche" bezieht sich auf den Teil der Querschnittsfläche in welchem, bei einem gegebenen Lastniveau, keine Plastifizierung stattgefunden hat. Der plastische Teil der "Anfangseffektivfläche" wird die "Plastischfläche" genannt.

Die Kraft, die von der elastischen Fläche getragen wird, nennt man "elastische Kraft" und die Kraft, die von der plastischen Fläche getragen wird, "plastische Kraft".

Die Arbeitsleistung der externen Kraft auf eine nominale Einheitsfläche des Probestückes von Einheitslänge ist gleich der Fläche unter der Spannungs-Dehnungs-Kurve. Ein Teil dieser Energie wird als potentielle Energie im Probestück gespeichert, durch die Wirkungsweise der elastischen Spannungen. Der andere Teil wird durch die plastischen Spannungen in Wärmeenergie verwandelt. Der erste Vorgang ist umkehrbar, während der zweite Vorgang nicht umkehrbar ist.

Während des Entlastungsablaufs wird alle im Probestück gespeicherte potentielle Energie, d.h. die elastische Energie, zurückgewonnen. Keine weitere Plastifizierung findet in dieser Zeitspanne statt. Der Weg der Entlastung wird eine gerade Linie sein, denn die zurückzubildende Energie ist die Summe der gesamten elastischen Energie, die sich auf Grund der elastischen Spannungen während des Belastungsablaufs gebildet hat. Im Hinblick auf Entlastung, wenn keine Plastifizierung inzwischen stattgefunden hat, wird sich überhaupt kein Unterschied im strukturellen Verhalten eines elasto-plastischen, oder eines ideal-elastischen Materials, zeigen. Wegen dieses elastischen Verhaltens im Entlastungsablauf nennen wir die Tangente der Entlastungslinie den <u>Elastizitätsmodul bei Entlastung</u> (unloading modulus of elasticity), oder einfach <u>Entlastungsmodul</u>.

Mit genauen Informationen über die verschiedenen Flächenelemente und der genauen Kenntnis der Lastniveaus, in denen in den Flächenelementen Plastifizierung eintritt, können die Lastdeformationseigenschaften des Modells sowohl für Belastung als auch Entlastung bestimmt werden.

Im Abschnitt 33 wird das allgemeine Modell gezeigt dessen Querschnittsfläche aus <u>n</u> Flächenelementen, ΔA_i, besteht. Diese Flächenelemente sind aus idealen elasto-plastischen Medien, mit gleichen Elastizitätsmoduln, aber mit verschiedenen Streckgrenzspannungen, gebaut.

Die Gleichung (44) und die Ungleichung (45) bestimmen das mathematische Modell genau.

Bei passender Auswahl der Flächenelemente der Gleichung (44) und bei passender Zuweisung der Werte für die Elemente der Ungleichung (45), könnten wir ein mathematisches Modell konstruieren, das jedem gegebenen Spannungs-Dehnungsdiagramm gleichwertig ist oder umgekehrt.

Die Zusammengehörigkeit der Lastdeformation zwischen einem mathematischen Modell und einem physikalischen oder experimentellen Modell stimmt solange wie die Kriechdeformation oder die Thermaleffekte nicht die Form des experimentellen Diagrammes verändern.

Man kann sich vorstellen, dass die Flächenelemente des allgemeinen Modells sich über den ganzen Querschnitt gleichmässig verteilen. Daher haben wir gleichmässige Plastifizierung über die ganze Querschnittsfläche auf allen Lastniveaus.

Für ein tatsächliches physikalisches Modell kann die Spannungsverteilung über den Querschnitt in folgender Weise erklärt werden:

Es wird angenommen, dass sich ein elasto-plastisches Material aus winzigen molekularen Elementen zusammensetzt, die die Bausteine innerhalb der Kristalle darstellen. Die Lage und Inklination der molekularen Elemente bestimmen den geometrischen Aufbau der Kristallstruktur. Man nimmt weiter an, dass alle molekularen Elemente denselben Elastizitätsmodul haben, der ein Durchschnittswert des Materials ist.

Wenn eine ständig zunehmende, externe Last zentrisch auf ein Probestück eines elasto-plastischen Materials mit der nominalen Querschnittsfläche A aufgetragen wird, nehmen alle molekularen Elemente, die sich in einem gewissen Querschnitt des Probestücks befinden, am Tragen der Last teil, auch wenn sie nicht identisch gespannt sind, denn verschiedene Elemente haben unterschiedliche Inklinationen in Bezug auf die Richtung der externen Kraft. Die Spannungsverteilung ist eine Funktion der geometrischen Struktur der Kristalle und ihrer Ausrichtung hinsichtlich der Richtung der Kraft.

Hier wird der Begriff der Effektivfläche für ein physikalisches Modell eingeführt. Von allen Elementen, die sich in einem gewissen Querschnitt des Probestückes befinden, wird ein Element, das sich parallel zu der Richtung der externen Kraft befindet, herausgenommen. Alle anderen werden durch imaginäre Elemente ersetzt, parallel zu der Richtung der Kraft, und imaginäre Querschnittsflächen werden in der Art ausgewählt, dass die Deformationseigenschaften des Probestückes unverändert bleiben. Die so dargestellte gesamte Querschnittsfläche nennen wir die totale Effektivfläche. In dem Masse, in dem die externe Kraft zunimmt, vermindert sich die Effektivfläche. Dieses strukturelle Verhalten gibt den Eindruck eines ständig abnehmenden Elastizitätsmoduls. Jedoch bleibt in Wirklichkeit der Elastizitätsmodul unverändert. Der Eindruck wird durch die ständig kleiner werdende Effektivfläche verursacht.

Unter Zugrundelegung der obigen fundamentalen Gedanken und Begriffe wird jetzt ein Raumfachwerk, das aus einem vertikalen und drei identisch schrägen Elementen aufgebaut ist, eingehend studiert. Wir gehen von der Annahme aus, dass das Fachwerk (Abb. 17) aus einem idealen elasto-plastischen Material besteht. Die Wahl des Modells in dieser Bauart stellt die stark vereinfachte Form eines Kristalls dar. Während eine ständig zunehmende vertikale Kraft P am Scheitelpunkt 0 aufgetragen wird, deformieren sich alle vier Fachwerkelemente elastisch bis zu einer gewissen Grenzlast P_1. Bei einer noch stärkeren Last tritt im vertikalen Element völlige Plastifizierung ein, während die schrägen Elemente noch innerhalb ihres Elastizitätsvermögens belastet sind. Jede zusätzliche Last muss allein von den schrägen Elementen getragen werden. Die Last-Deformationskurve für dieses Modell zeigt Abb. 19. Punkt 1 auf dieser Kurve entspricht dem Belastungsstadium, in dem die Plastifizierung im vertikalen Element gerade beginnt. In einem infinitesimalen Abstand rechts von Pkt. 1 entspricht die Last P_1 der Totallast; die von dem vertikalen Element getragene Last ist die plastische Last, während die insgesamt von den schrägen Elementen getragene Last die elastische Last ist. Diese Begriffe entsprechen den bereits definierten Begriffen wie Totalspannung, plastische Spannung und elastische Spannung.

Die Arbeitsleistung der plastischen Last, über Pkt. 1 hinaus, geht als Wärmeenergie verloren. Die Berechnung des Entlastungsmoduls wird im Einzelnen ausgeführt (Gl. 90). Eine numerische Auswertung zeigt, dass die Abweichung zwischen der wirklichen Entlastungslinie und der, die man früher als die richtige ansah, nicht übersehen werden darf (Abb. 21).

Die Ableitung des Entlastungsmoduls, für den allgemeinsten Fall eines viskoelastischen Materials mit beliebiger Spannungs-Dehnungs-Kurve ist mathematisch komplizierter wegen der ständig sich verändernden Krümmung der Spannungs-Dehnungs-Kurve. Jedoch werden die Berechnungen Schritt für Schritt - unter der systematischen Anwendung der in der Theorie eingeführten grundlegenden Begriffe - ausgeführt. Gl (107) ist ein analytischer Ausdruck für den Entlastungsmodul. Das Glied $\sigma_{e\varepsilon}$, als elastische Spannung definiert, erscheint in Gl. (107). Die funktionelle Beziehung der elastischen Spannung n als ständig abnehmende Dehnungsfunktion ist in Abschnitt 37 bestimmt. Unter Anwendung der grundlegenden Definitionen und bei Unterteilung der Spannungs-

Dehnungs-Kurve in eine Reihe gerader Linien (Abb. 24), wird eine numerische Methode für die Berechnung der elastischen Spannung eingeführt. Die Rekursionsformel (112) ergibt die elastische Spannung für jeden Punkt der Spannungs-Dehnungs-Kurve, vorausgesetzt, dass die elastische Spannung für den vorhergehenden Nachbarpunkt bekannt ist. Mit dieser Rekursionsformel ist eine einfache Formel für die elastische Spannung bei jedem Punkt der Spannungs-Dehnungs-Kurve ausgearbeitet worden. Diese einfache Formel ergibt sich aus Gl. (114).

Untersuchungen der Variationsmöglichkeiten des Entlastungsmoduls entlang beliebiger Spannungs-Dehnungs-Kurven

Die Eigenschaften einer allgemeinen Spannungs-Dehnungs-Kurve werden in Abb. 25 definiert und als Diagramm dargestellt. Die mathematische Untersuchung der Änderung des Entlastungsmoduls, in Bezug auf die Dehnung beim Entlastungspunkt, führt zu Gl. (115). Die Glieder, die in den Bruch rechts von Gl. (115) erscheinen, sind in ihrer Art sehr allgemein. Wenn kein spezifisches Verhältnis zwischen Spannung und Dehnung besteht, können diese Glieder nicht direkt berechnet werden. Für eine so allgemein gehaltene mathematische Untersuchung muss die physikalische Bedeutung der Integrale im Zähler des Bruchs herausgefunden werden. Der Nenner des Bruchs ist immer positiv. Es zeigt sich, dass das entscheidendste Glied das Integral

$$\int_0^a \sigma_{e\varepsilon} \cdot \frac{d^2\sigma_e/d\varepsilon^2}{d\sigma_e/d\varepsilon} \quad \text{ist}$$

Die eingehenden Berechnungen führen zu dem überraschenden Ergebnis, dass dieses Integral unabhängig ist von dem Weg, der sich vom Nullpunkt des Koordinatensystems 0 an bis zu irgendeinem Punkt A auf der Spannungs-Dehnungs-Kurve ergibt, vorausgesetzt, dass dieser Weg nach Punkt A mit der ursprünglichen Spannungs-Dehnungs-Kurve zusammenfällt. Diese Entdeckung dient erstens zur Bestimmung der Variationsmöglichkeiten des Entlastungsmoduls und, zweitens, um eine Formel für die elastische Spannung $\sigma_{e\varepsilon}$ zu bestimmen, als eine Funktion von Spannung und Dehnung wie in Gl. (131) angegeben. Substitution von $\sigma_{e\varepsilon}$, aus der Gl. (131) in Gl. (107) ergibt Gl. (132), welche die endgültige analytische Lösung für den Elastizitätsmodul bei Entlastung darstellt.

Die Untersuchungen der Variationsmöglichkeiten des Entlastungsmoduls führen zu Gl. (127), welche auf höchst allgemeine Art zeigt, dass $\frac{dE_{ae}}{de}$ immer negativ ist. Dieses Ergebnis beweist, dass der Entlastungsmodul abnimmt, wenn die Dehnung, die dem Entlastungspunkt entspricht, zunimmt. Dies bedeutet, dass, wenn man beim Nullpunkt des Koordinatensystems mit dem anfänglichen Elastizitätsmodul beginnt und sich auf der typischen Spannungs-Dehnungs-Kurve - wie in Abb. 25 definiert - bewegt, der Entlastungsmodul stetig abnimmt. Gl. (126) beweist, dass dieses zutrifft, auch wenn der Tangentenmodul einen konstanten Wert erhält. Gl. (126) zeigt weiterhin, dass, wenn der Tangentenmodul sich Null nähert, $\frac{dE_{ae}}{de}$ sich Null nähert; folglich nähert sich also E_{ae} einem konstanten Wert.

Numerische Methode für die Berechnung des Entlastungsmoduls

Die Gl. (107) für den Entlastungsmodul kann auch in der Form der Gl. (133) wiedergegeben werden. Durch Aufteilung der Spannungs-Dehnungs-Kurve in n Streifen mit gleichen Abständen, $\delta = 1$, entlang der Dehnungsachse, und unter Anwendung der herkömmlichen Begriffe der Theorie, verbunden mit den geometrischen Eigenschaften der Spannungs-Dehnungs-Kurve wie in Abb. (29), gelangten wir zu der Gleichung (137) für S_{fn}. Diese gibt den gesamten Energieverlust bis zum Entlastungspunkt n an. Gl. (138) ist ein Ausdruck für die gesamte Fläche unter der Spannungs-Dehnungs-Kurve. Gln. (139) und (140) geben die Werte für S_f und S_t bei Pkt. n an, sofern die Werte für diese Ausdrücke bereits im Pkt k auf der Spannungs-Dehnungs-Kurve bekannt sind. Sind die Werte für S_{fn} und S_{tn} bekannt, kann der Entlastungsmodul schliesslich durch Gl. (141) bestimmt werden. Die Berechnung eines numerischen Beispiels wird in Tabellen 1, 2 und 3 gegeben.

Die neue Theorie über elasto-plastische Stabilität

Im Abschnitt (41) wird ein grundlegendes Theorem der Stabilität abgeleitet, welches erklärt, dass die Stabilität jedes Stabes, unter dem Einfluss einer konstanten Last, untersucht werden muss, da die simultane Vergrösserung der Axialkraft nicht das Phänomen der Knickung beeinflussen kann.

Im Abschnitt 42 auf der Basis des grundlegenden Theorems der Stabilität wird ein anfänglich gerader Stab einer geringen seitlichen Verschiebung δ ausgesetzt. Die kleinste Druckkraft, bei welcher der Stab in der verschobenen Lage verbleibt, heisst Knickkraft. Bei jeder geringeren Kraft als der Knickkraft bewegt sich der Stab zu seiner Ausgangslage hin, vorausgesetzt, dass die Ursache der seitlichen Ausbiegung entfernt worden ist. In unserem Fall, für den elasto-plastischen Stab im Gegensatz zu dem elastischen Stab, wird der Ausdruck Hinbewegung zur Gleichgewichtslage gebraucht, statt Rückgang zur ursprünglichen Gleichgewichtslage. Der Grund hierfür ist, dass ein elasto-plastischer Stab bei beliebiger Belastung im Bereich der Plastizität niemals völlig seine ursprüngliche Gleichgewichtslage wieder einnimmt, nachdem eine seitliche Verschiebung stattgefunden hat.

Wenn der Stab eine geringe seitliche Verschiebung, bei Knickkraft, erhält, biegt sich der Stab. Als Folge einer solchen Krümmung teilt sich ein Querschnitt des Stabes in zwei Zonen ein. In Zone A steigt die Druckkraft infinitesimal an, während sie in Zone B infinitesimal abnimmt. Die Berechnungen ergeben, dass die Spannungen in Zone A des Stabquerschnitts, in Gl. (147) definiert, mit den Berechnungen Von-Kármáns übereinstimmen. Der Unterschied ist, dass die in Gl. (147) gegebenen Spannungen Durchschnittsspannungen sind, während die wirklichen Spannungen grösser sind. Dagegen fehlt den Spannungen Von-Kármáns jede weitere Erklärung darüber, ob es sich um Durchschnittsspannungen oder wirkliche Spannungen handelt. Dieser Unterschied beruht auf den verschiedenartigen Methoden, das Problem zu lösen. Während Von-Kármán von einem veränderlichen Elastizitätsmodul ausgeht, setzt die hier eingeführte Theorie einen konstanten Elastizitätsmodul, jedoch eine veränderliche Effektivfläche, voraus. In der gesamten Zone B sind alle Elemente am Entlastungsprozess beteiligt, mit einem Entlastungsmodul wie in Gl. (132) definiert. Daher wird der Berechnungsvorgang für die Knickkraft Ähnlichkeit mit Von-Kármáns Vorgang haben, nur dass der anfängliche Elastizitätsmodul der Considère-Engesser oder Von-Kármán-Theorie durch den Entlastungsmodul wie in Gl. (132) definiert, ersetzt worden ist.

Gl. (150) ist die allgemeine Formel für die Knickkraft eines elasto-plastischen Stabes. Der hier eingeführte neue Ausdruck ist der Elasto-

Plastizitätsmodul, geschrieben E_{ep}, gelesen (E - ep). Gln. (151) und (152) definieren E_{ep} für einen rechtwinkligen Schnitt und einen idealisierten I-Schnitt.

In dieser Arbeit ist mathematisch bewiesen worden, für den allgemeinsten Fall, dass der Entlastungsmodul immer kleiner ist als der anfängliche Elastizitätsmodul. Daher ist der Elasto-Plastizitätsmodul immer kleiner als Von-Kármáns Doppelmodul. Der Elasto-Plastizitätsmodul ist offensichtlich immer grösser als der Tangentenmodul.

Diese Ergebnisse führen zu der Schlussfolgerung, dass der korrekte Wert der elasto-plastischen Knickkraft zwischen der Tangenten-Modul-Last und dem durch Von-Kármáns Doppelmodul-Theorie vorausgesagten Wert liegt[1].

1. Der numerische Vergleich verschiedener Stabtheorien ist aus Tabelle 4 und Abb. 35 ersichtlich.

RESUME

Les recherches présentées dans cette dissertation traitent du problème de la stabilité d'une poutre élasto-plastique, originellement droit et chargé concentriquement.

Cette étude est divisée en quatre chapitres; chacune d'entre elles est à son tour divisée en un certain nombre de paragraphes. Le premier chapitre traite de l'histoire du développement des théories sur la stabilité de la poutre inélastique, comme il est mentionné dans les ouvrages jusqu'à ce jour. Le second chapitre couvre une discussion complète des contradictions dans les théories sur la stabilité de la poutre inélastique, théories qui affirment la charge tangente-module être une charge critique. Le troisième chapitre a trait au développement d'une nouvelle théorie sur l'élasto-plasticité pour la détermination du module décharge d'élasticité. Finalement le quatrième chapitre est consacré à l'exposé de la nouvelle théorie de la stabilité élasto-plastique.

Après cette courte introduction ci-dessus, les contributions originales des recherches apparaissant dans cette dissertation seront expliquées en bref.

Discussion sur les contradictions dans les théories sur la stabilité de la poutre qui affirment la charge tangente-module en tant que charge critique.

Les discussions commencent par une revue de la théorie de Shanley, y compris un examen des remarques et hypothèses de Shanley, les données et conclusions de ses expériences et son analyse mathématique. La discussion continue plus loin sur les interprétations de la théorie par Timoshenko et Von-Kármán. Timoshenko est mentionné parce que son livre "Theory of Elastic Stability" peut être obtenu dans presque n'importe quelle bibliotèque technique de par le monde, même lorsqu'il n'y a pas accès direct à la littérature originale de Shanley. Von-Kármán est mentionné, parce que c'est justement les lacunes dans la théorie originale de Von-Kármán sur le module réduit[1] qui a conduit Shanley au développement de sa théorie. Après révision de la théorie de

1. La théorie du module réduit est mentionnée parfois sous le nom de théorie du module double

Shanley, la discussion est généralisée pour montrer les contradictions dans toutes les théories sur la stabilité de la poutre inélastique, théories qui affirment que la tangente module est une charge critique. La correcte alternative pour la charge élasto-plastique flambante est finalement identifiée.

Discussion sur la théorie de Shanley.

Les remarques de Shanley, sous le titre, "The Column paradox", (Le paradoxe du pilier), ont paru pour la première fois, dans le "Journal of The Aeronautical Sciences", Vol. 13, no. 12, Déc. 1946. Sa théorie sur le pilier inélastique a paru dans le même journal, Vol. 14, no. 5, Mai 1947. Ces deux articles sont référés ici par les mentions: Référence 1 et Référence 2 respectivement. Les remarques et hypothèses de Shanley, les résultats de ses expériences et ses analyses mathématiques sont discutées ici dans les plus grand détails. À toutes les figures et équations dans "Référence 2" sont données des chiffres de référence nouveaux dans cette discussion.

Après l'apparition de la théorie de Considère-Engesser et Von-Kármán, on affirma grâce aux résultats expérimentaux de nombreux chercheurs, y compris Shanley, que la vraie valeur de la charge inélastique flambante doit se situer quelque part entre la charge tangente module et la valeur prédite par la théorie du module réduit. Shanley estime qu'un tel résultat pourrait être obtenu en faisant l'hypothèse que la poutre commence à plier sous la charge tangente module, tandis que la charge coaxiale augmente simultanément. Shanley estime par ailleurs, qu'afin de permettre une charge par delà la charge tangente module, un certain genre de distribution de la contrainte selon la figure 4 est nécessaire. Ce genre de distribution de la contrainte permet un renversement de la contrainte, dès le début de la flexion. Cependant, dans la discussion qui suit la figure 5, il est démontré que les contraintes supplémentaires produites au commencement de la flexion, sont infinitésimales en comparaison avec les flexions dues à la charge tangente module. C'est pourquoi même si l'axe neutre de la poutre s'est déplacé considérablement à cause des contraintes infinitésimales dues à la flexion, le déplacement de l'axe neutre pour les contraintes totales peux être négligé pourvu que les valeurs de ces contraintes de flexions demeurent infinitésimales en ce qui concerne les contraintes uniformes dues à la charge tangente module. Ainsi quand la résultante des contraintes sur la section de la poutre est encore centrale, et quand les contraintes de flexion sont infinitésimales, l'augmentation de la charge coaxiale peut accompagner un renversement de la contrainte seulement et seulement si les contraintes dues à l'augmentation de la charge coaxiale sont plus petites que n'importe quelles valeurs infinitésimales, ce qui revient à dire, si elles sont égales à zéro. C'est pourquoi au commencement de la flexion, avant que les déformations deviennent trop larges, il est impossible d'obtenir un renversement de la contrainte, simultanément avec une augmentation de la charge coaxiale. Cette erreur fondamentale est d'ailleurs exprimée dans l'analyse mathématique de Shanley.

L'Equation (3), est considérée par Shanley dans son analyse mathématique, comme représentant la théorie complète de l'action de la poutre. La détermination de cette équation est basée sur trois hypothèses. La première exprime que la poutre commence à plier à la charge tangente module. La seconde affirme que la charge coaxiale continue à augmenter dans le processus de flexion. La troisième exprime que le renversement de la contrainte se manifeste aussitôt que la flexion commence. La deuxième et la troisième hypothèse se contredisent l'une l'autre et ne peuvent s'accomplir simultanément. En fait, la première hypothèse de Shanley, que la poutre commence à fléchir à la charge tangente module, perd sa signification, parce que la théorie a manqué son but.

La discussion peut être généralisée en examinant toutes les autres théories sur le flambement élasto-plastique qui affirment que la charge tangente module doit être la charge critique. La théorie tangente module a été rejetée par les chercheurs précédents, à cause de la lacune de la théorie qui ne tient pas compte de l'extra rigidité possédée par la poutre dans la position fléchissante. La théorie de Shanley, comme le montre cet exposé ne peux être acceptée en raison de la contradiction découlant de la deuxième et de la troisième hypothèse de Shanley. Il reste maintenant une seule possibilité qui peux être élaborée comme suit.

<u>La troisième possibilité</u>: La poutre commence à fléchir à la charge tangente module, simultanément avec l'augmentation de la charge coaxiale, tandis que la tangente module s'applique à la totalité de la section, jusqu'à une certaine déflection latérale là où les contraintes dues aux flexions deviennent suffisamment larges pour permettre certains renversements de contrainte de

prendre place dans la poutre, donnant ainsi à la poutre l'extra rigidité après une certaine déflection latérale. Cette possibilité n'a pas été mentionnée ou discutée par les chercheurs précédents. La discussion de cette autre possibilité quant à la théorie est importante, parce que c'est la dernière possibilité qui prédit le commencement de la flexion à la charge tangente module. Cette théorie évite la contradiction inhérente à la théorie de Shanley et permet des séquences équilibrées dans la position de flexion. Cette théorie, cependant, conduit aux contradictions suivantes:

La tangente module est la plus petite charge seulement et seulement si la tangente module est applicable pour la poutre toute entière. Cela est vrai seulement et seulement si l'augmentation de la charge coaxiale compense la diminution dans les contraintes sur le coté convexe de la poutre dans la position de flexion. Cela signifie en réalité que la cause de la déflection latérale est l'augmentation de la charge coaxiale. Par delà la charge tangente module, en effet, l'augmentation de la charge coaxiale est dépendante du temps, tandis que la perte assumée de stabilité à la charge tangente module est indépendante du temps, parce que l'augmentation de la charge coaxiale P d'une valeur infinitésimale en dessous de la charge critique vers une valeur infinitésimale au dessus de la charge critique est accompagnée par un changement dans l'état d'énergie du système composé à la fois de la charge et de la poutre. Une fois que la charge a atteint sa valeur critique, le changement dans l'état de l'énergie du système est indépendant du temps, ce qui signifie que pour la valeur infinitésimale de déflections latérales où la formule pour la charge critique est théoriquement correcte, un flambement prend place instantanément. Cet argument conduit à la conclusion que, si rapide que soit la vitesse de charge par delà la charge tangente module, la vitesse de déflection latérale supposée doit être encore plus rapide, ce qui entraine la contradiction que la poutre dans la position fléchissante doit être libre de l'effet de l'augmentation de la charge coaxiale.

Nous concluons que la charge tangente module n'est pas une charge critique et que la poutre inélastique a seulement une charge critique qui toujours excède la charge tangente module. La charge flambante élasto-plastique est, d'autre part, toujours moindre que la valeur prédite par la théorie du module réduit. La preuve de cette dernière affirmation est laissée aux résultats des recherches dans les paragraphes suivants.

<u>Nouvelle théorie de l'élasto-plasticité pour la détermination du module de décharge de l'élasticité.</u>

Les idées essentielles et les concepts de base de l'élasto-plasticité sont présentés et définis dans le paragraphe 32 en vue d'une discussion détaillée des propriétés charge-déformation d'un modèle structurel mathématique représenté en section par trois rangées d'un matériau élasto-plastique idéal homogène, avec le même module d'élasticité mais avec différents contraintes de fluage. Le diagramme $\sigma - \varepsilon$ pour ce modèle est montré dans la figure 12, où les trois points de discontinuité 1, 2 et 3 indiquent les différents niveaux de charge auxquels les trois différents éléments de superficie de la section sont plastifiés. Les concepts de superficie "effective" ou "élastique" se réfèrent à la superficie non-plastifiée à un niveau de charge donné. La partie plastifiée de la "superficie effective initiale totale" est appelée "superficie plastique". La force portée par la superficie élastique est appelée "force élastique" et la force portée par la superficie plastique est appelée "force plastique".

Le travail fait par la charge externe sur une élément unitaire de superficie et de longueur est égale à la superficie sous la coube $\sigma - \varepsilon$. Une partie de cette énergie est convertie en potentielle énergie, accumulée dans l'échantillon, par l'action des contraintes élastiques; l'autre partie est convertie en énergie calorifique par l'action des contraintes plastiques. La première partie est réversible, tandis que la seconde est irréversible.

Durant le processus de décharge, toute l'énergie potentielle accumulée dans l'échantillon d'expérimentation dans la forme d'énergie élastique sera récupérée. Une interruption s'effectue dans la plastification pendant cette période. Le chemin de décharge sera une ligne droite, parce que l'énergie qui sera récupérée est la somme de toute l'énergie élastique due à l'action de toutes les contraintes élastiques pendant le processus de charge. En ce qui concerne une décharge en l'absence de quelconque plastification pendant cette période, il n'y aura aucune différence dans le comportement de la structure d'un élasto-plastique ou d'une matière plastique idéale. A cause du comportement du processus de décharge, la tangente de la ligne de décharge est appelée le <u>module-décharge d'élasticité</u> ou simplement le <u>module-décharge</u>.

Avec des informations précises sur les différents éléments de superficie et sur les niveaux de charge auxquels ils sont plastifiés on peut déterminer les propriétés charge-déformation d'un modèle pour les processus de charge aussi bien que de décharge (Fig. 15).

Dans le paragraphe 33 le modèle général est représenté par sa superficie de section, consistant en n éléments de surfaces ΔA_i faits de matériaux élasto-plastiques idéaux homogènes avec le même module d'élasticité mais avec différents contraintes de fluage. L'égalité 44 et l'inégalité 45 définissent le modèle mathématique avec précision. Par un choix approprié des surfaces élémentaires de l'égalité 44, et par des valeurs appropriées données aux éléments de l'inégalité 45, nous pouvons construire un modèle mathématique qui serait équivalent à n'importe quel $\sigma - \varepsilon$ diagramme et vice-versa. L'équivalence charge-déformation entre un modèle mathématique et un modèle physique ou expérimental est valable aussi longtemps que les déformations différées (fluage) ou les effets thermiques ne commencent à changer le diagramme expérimental $\sigma - \varepsilon$. On peut concevoir que les surfaces élémentaires du modèle général recouvrent la section entière uniformément, de sorte que l'on obtienne une plastification uniforme de toute la superficie de section à tous les niveaux de charge. Pour un modèle physique réel la distribution de la contrainte sur la section peut être expliquée de la manière indiquée ci-dessous.

Il est supposé ici naturellement qu'une matière élasto-plastique est constituée de minuscules éléments moléculaires qui constituent les pierres de construction à travers les unités du crystal. La positions et l'inclinaison de ces éléments moléculaires définissent la configuration géométrique de la structure du crystal. Tous ces éléments moléculaires sont supposés avoir le même module d'élasticité qui a une valeur moyenne pour la matière.

Quand une charge extérieure augmentant continuellement est appliquée centralement sur un échantillon d'expérience d'une matière élasto-plastique avec une section d'une superficie nominale A, tous les éléments moléculaires traversant une certaine section d'un échantillon d'expérience prennent part en portant la charge même s'ils ne possèdent pas une contrainte identique, parce que différents éléments ont des inclinaisons différentes en ce qui concerne la direction de cette force extérieure. La distribution de contrainte est une fonction de la configuration géométrique de ces unités de crystal, et de leur orientation par rapport à la direction de la force.

La conception de superficie effective pour un modèle physique est introduite. Parmi tous les éléments qui traversent une certaine section de l'échantillon d'expérience, un élément parallèle à la direction de la force externe est mis à part. Tous les autres éléments sont remplacés par des éléments imaginaires parallèles à la direction de la force, avec des superficies de section également imaginaires, choisies de telle manière que les propriétés de déformation de l'échantillon d'expérience demeurent inchangées. La superficie totale de section présentée de cette manière est appelée la superficie totale effective. Au fur et à mesure que la charge augmente, la superficie effective diminue. Ce comportement de structure donne l'impression d'un module d'élasticité diminuant continuellement. Cependant, le module d'élasticité demeure inchangé. C'est la superficie diminuant continuellement qui produit cet effet.

Reposant sur des idées primordiales et des conceptions fondamentales, un treillis dans l'espace construit d'un élément vertical et de trois éléments inclinés identiques est soumis à une étude détaillée dans le paragraphe 35. On assume que le treillis montré dans la figure 17 est construit d'une matière idéale élasto-plastique. Le choix du modèle de cette manière signifie la forme vraiment simplifiée d'une unité-crystal. Comme une charge P verticale augmentant continuellement est appliquée au sommet 0, tous les quatres éléments du treillis se déforment élastiquement jusqu'à une certaine charge limite P_1. Au delà de cette charge, la complète plastification s'éffectue dans l'élément vertical, tandis que les éléments inclinés sont encore sous contrainte dans le domaine de leur élasticité. N'importe quelle charge supplémentaire est alors seulement supportée par les éléments inclinés. Le diagramme charge-déformation pour ce modèle est montré dans la figure 19. Le point I dans ce diagramme correspond à cette phase de décharge où la plastification est en train de commencer dans l'élément vertical. A une distance infinitésimale sur la droite du point I, la charge P_1 représente la charge totale; la charge supportée par l'élément vertical représente la charge plastique, alors que la charge supportée par les éléments inclinés représente la charge élastique. Ces termes sont analogues à la terminologie déjà définie: contrainte totale, contrainte élastique et contrainte plastique.

Le travail accompli du à l'action de la charge plastique au delà du point I est perdu en forme d'énergie calorifique. Le calcul pour le module de décharge est réalisé en détail (eq. 90). Une évaluation numérique démontre, que la déviation entre la ligne réelle de décharge et ce qui a été supposé auparavant être la ligne correcte peut être non négligeable (Fig. 21).

Le dérivation du module de décharge pour le cas le plus général d'une matière visco-élastique avec n'importe quel diagramme $\sigma - \varepsilon$ est mathématiquement plus complexe à cause de la courbe diagramme continuellement changente. Cependant dans le paragraphe 36 les calculs sont entrepris pas à pas, au moyen de l'application systématique des conceptions fondamentales introduites dans la théorie. L'équation (107), est une expression analytique pour le module de décharge. Le terme $\sigma_{e\varepsilon}$ défini en tant que contrainte élastique apparait dans l'équation (107). La relation fonctionnelle pour la contrainte élastique, en tant que fonction diminuant continuellement quant à la dilatation, ε, est précisée dans le paragraphe 37. Utilisant les définitions de base et divisant le diagramme $\sigma - \varepsilon$ en une série de lignes droites, comme on le montre dans la figure (24), une méthode numérique pour le calcul de la contrainte élastique est introduite. La relation récursive (112) donne la contrainte élastique à n'importe quel point du diagramme $\sigma - \varepsilon$ pourvu que la contrainte élastique au point voisin précédent soit connue. Utilisant cette relation récursive, une formule simple générale pour la contrainte élastique, à n'importe quel point de la courbe $\sigma - \varepsilon$, est élaborée. La formule simple en est donnée par l'équation (114).

Études sur les possibilités de variation du module de décharge le long de la courbe $\sigma - \varepsilon$.

Dans le paragraphe 37 les propriétés d'une courbe typique généralisée $\sigma - \varepsilon$ sont définies et graphiquement représentées dans la figure (25). L'étude mathématique de la dérivation du module-décharge, par rapport à la dilatation au point de décharge, aboutit à l'équation (115). Les termes apparaissant dans la fraction sur la droite de l'équation (115) sont de nature très générale. Dans l'absence d'aucune relation spécifique entre la contrainte et la dilatation ces termes ne peuvent être directement évalués. Pour une étude mathématique d'une telle généralité, la signification physique des intégrales apparaissant dans le numérateur de la fraction doit être découverte. Le dénominateur de la fraction est toujours positif. Il est prouvé que le terme le plus décisif est l'intégrale,

$$\int_0^e \sigma_{e\varepsilon} \cdot \frac{d^2\sigma_\varepsilon / d\varepsilon^2}{d\sigma_\varepsilon / d\varepsilon} \cdot d\varepsilon$$

Les calculs détaillés mènent au remarquable résultat que cette intégrale est indépendante du chemin parcouru du point d'origine de la courbe $\sigma - \varepsilon$, jusqu'á n'importe quel point A sur la courbe, pourvu que après le point A, le chemin parcouru coincide avec la courbe originale $\sigma - \varepsilon$ Cette découverte est utilisée en premier lieu pour déterminer les possibilités de variation du module de décharge et en deuxième lieu pour déterminer une relation pour la contrainte élastique $\sigma_{e\varepsilon}$, en tant que fonction de σ et ε, donnée par l'équation (131). La substitution pour $\sigma_{e\varepsilon}$ de l'équation (131) dans l'équation (107) aboutit à l'EQUATION (132) qui est la solution analytique finale pour le module de décharge de l'élasticité.

Les études sur les possibilités de variation du module de décharge nous amène à l'équation (127) qui démontre de la facon la plus générale que $\frac{dE_{a\varepsilon}}{de}$ est toujours négatif. Ce résultat prouve que le module-décharge diminue, pendant que la dilatation correspondant au point de décharge augmente, ce qui signifie que, en commencant de l'origine de la courbe $\sigma - \varepsilon$, avec le module initial d'élasticité, et en se déplacant le long du diagramme typique $\sigma - \varepsilon$ selon la figure (25), le module de décharge diminue continuellement. L'équation (126) démontre que cela est vrai même si la tangente module garde une valeur constante. L'équation (126) démontre par ailleurs que au cas où la tangente module tend vers zéro, $\frac{dE_{a\varepsilon}}{de}$ tend aussi vers zéro; en conséquence $E_{a\varepsilon}$ tend vers une valeur constante.

L'équation (107) pour le module-décharge peut être réécrite dans la forme donnée par l'équation (133). En divisant la courbe $\sigma - \varepsilon$ en n morceaux, d'intervalles égaux, $\delta = 1$, le long de l'axe de dilatation, ε , et en utilisant les conceptions de la théorie combinées avec les propriétés géométriques de la courbe σ-ε comme démontré dans la figure (29), on arrive à l'équation (137) pour S_{fn}, qui donne l'énergie totale perdue jusqu'au point de décharge n. L'équation (138) est une expression pour la superficie totale sous la courbe σ-ε. L'équation (139), et l'équation (140), donnent les valeurs de S_f et S_t, au point n, tandis que les valeurs de ces quantités sont déjà calculées au point k sur le diagramme $\sigma-\varepsilon$. Connaissant les valeurs de S_{fn} et S_{tn}, le module de décharge peux finalement être déterminé par l'équation (141). Un exemple numérique est calculé, comme le montre les tables 1, 2 et 3.

La nouvelle théorie de la stabilité élasto-plastique.

Dans le paragraphe 41 un théorème fondamental de stabilité est démontré. Ce théorème affirme que la stabilité de toute poutre doit être étudiée sous l'influence d'une charge constante, parce que l'augmentation simultanée de charge axiale ne peut effectuer le phénomène de flambement. Dans le paragraphe 42, d'après le théorème fondamental de la stabilité une poutre initialement droite, sous l'action d'une charge concentrique P, effectue un léger déplacement latéral δ. La plus petite charge, par laquelle la poutre demeure dans la position déplacée est appelée, charge flambante. Pour n'importe quelle charge plus petite que la charge flambante, la poutre se déplace vers la position d'origine pourvu que la cause de la déflection latérale soit éliminée. En ce qui concerne ici la poutre élasto-plastique, en contraste avec la poutre élastique, la phrase: "allant vers la position d'équilibre", est utilisée au lieu de "retournant à la position originale d'équilibre". C'est parce que, une poutre élasto-plastique, sous n'importe quelle charge dans le domaine de plasticité, ne peut jamais parfaitement retourner à sa position originale d'équilibre après un déplacement latéral.

En donnant à la poutre un léger déplacement latéral au moment de la charge flambante, la poutre fléchit. Le résultat de ce pliement ou flexion, c'est qu'une section de la poutre est divisée en deux zones. Dans la zone A la compression augmente de manière infinitésimale, tandis que dans la zone B la compression diminue de manière infinitésimale. Des calculs montrent que les contraintes de la zone A de la section de la poutre, définie par l'équation (147), coincident avec celles obtenues par Von-Kármán. La différence est que les contraintes données par l'équation (147) sont les contraintes moyennes, alors que les contraintes réelles sont plus grandes. Mais, les contraintes données par Von-Kármán par contre sont laissées sans aucune explication, à savoir si elles sont contraintes moyennes ou réelles. Cette différence vient de la différence dans les méthodes d'approche du problème. Alors Von-Kármán considère un module changeant d'élasticité comme base, la théorie introduite ici suppose un module constant d'élasticité, mais une superficie effective changeante. A travers la zone B, tous les éléments de la superficie de section prennent part dans le processus de décharge avec le module de décharge défini selon l'équation (132). C'est pourquoi la procédure pour trouver la charge flambante sera similaire à celle suivie par Von-Kármán avec le module initial d'élasticité dans la théorie de Considère-Engesser ou Von-Kármán remplacé par le module de décharge selon l'équation (132).

L'équation (150) est la formule générale pour la charge flambante d'une poutre élasto-plastique. Le nouveau terme introduit ici est le module d'élasto-plasticité désigné par E_{ep}, lire (E - ep). Les équations (151) et (152) définissent E_{ep}, pour une section rectangulaire et une section I idéalisée. Dans ce traité, il a été prouvé mathématiquement, dans le cas le plus généralisé, que le module de décharge est toujours moindre que le module initial d'élasticité, c'est pourquoi le module d'élasto-plasticité est toujours moindre que le module réduit de Von-Kármán. Le module d'élasto-plasticité est évidemment toujours plus grand que la tangente-module.

Ces résultats conduisent à la conclusion que la vraie valeur de la charge flambante élasto-plastique se situe entre la charge tangente module et la valeur prédite par la théorie du module réduit de Von-Kármán[1].

1. La comparaison numérique des diverses théories sur la stabilité de la poutre est faite dans le tableau 4 et la figure 35.