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Published in:
PAMM

2012

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We consider positivity preserving model order reduction of SISO linear systems. Whereas well-established model reduction methods usually do not result in a positive approximation, we show that a symmetry characterization of balanced truncation can be used to preserve positivity after performing balanced truncation. As a consequence, the method is independent of the initial realization and always returns a symmetric reduced model.

1 Introduction

Models in biology, economics and physics (cf. [2, 8]) often lead to positive systems (see Def. 1). If the order of the system is very large, one is interested in a low-order approximation, which is also positive. Standard methods for model order reduction (cf. [1, 6, 10]) focus on the approximation error e.g. in the H_∞- and H_2-norms, but typically do not preserve positivity. Recently, some positivity-preserving methods have been developed (cf. [3, 7, 11]), however with quite conservative H_∞-errors and exceeding computational costs already for relatively small dimensions.

In this paper we show how balanced truncation can be used to obtain a reduced model which is guaranteed to be positive. We will see, that balanced truncation to order one is always positive. The key ideas to obtain higher order approximations are then given by the sign-symmetry of balanced realizations (cf. [4]) and the positive realizability of symmetric systems.

2 Preliminaries

Let $A = (a_{ij})$ be a matrix. Then we write $A \geqq 0$ if $\forall(i, j) : a_{ij} \geq 0$ and $|A| = (|a_{ij}|)$ for the componentwise absolute value. We consider asymptotically stable linear systems (A, B, C, D) given in the standard form

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \sigma(A) \subset \mathbb{C}_- \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1}$$

with state variable $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $y \in \mathbb{R}^p$.

Definition 1 (Internal Positivity) *The system in (1) is called (internally) positive if and only if its state and output are nonnegative for every nonnegative input and every nonnegative initial state. [2]*

Theorem 1 (Positive Linear System) *The linear system (1) is positive if and only if A is Metzler and $B, C, D \geqq 0$. [8]*

Theorem 2 (Sign-symmetry) *Let $G(s)$ be the transfer function of an arbitrary SISO-system. Then there exists a balanced realization (A, B, C, D) of $G(s)$, such that $|A| = |A^T|$ and $|B| = |C^T|$. [4]*

3 Symmetric Balanced Truncation

The idea of Symmetric Balance Truncation (SBT) is to obtain a symmetric approximation by balance truncation and then apply Lanczos algorithm to regain a positive realization [5]. By the following theorem we conclude, that this procedure always results in a non-trivial approximation.

Theorem 3 (Positive First Order Balanced Truncation) *Let (A_1, B_1, C_1, D_1) be a first order system obtained by balanced truncation of a positive system given in (1). Then (A_1, B_1, C_1, D_1) has always a positive, asymptotically stable realization $(A_1, |B_1|, |C_1|, D_1)$.*

P r o o f. If (A, B, C, D) is positive then the reachability and observability Gramians $P, Q \geqq 0$. By the Perron-Frobenius theorem, PQ possesses nonnegative left and right eigenvectors w_1^T and v_1 corresponding to the largest Hankel singular value. Balanced truncation to order one then gives the matrices $A_1 = w_1^T A v_1 < 0$, $B_1 = w_1^T B \geqq 0$, $C_1 = C v_1 \geqq 0$ and $D_1 = D \geqq 0$. □

Notice, the theorem presents a necessary condition on the positivity of a MIMO-system. However, it is not sufficient as shown in [5].

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4 Example

This section demonstrates the properties of SBT by a numerical comparison with the results of the method proposed in [11]. We refer to the latter method as Generalized Balanced Truncation (GBT).

We start with the same water reservoir configuration as in [11]. The system consists of n connected water reservoirs $R_1, \dots, R_n$, located on the same level. By a_i and h_i, we denote the base area and fill level of reservoir R_i, respectively. The connections of R_i and R_j is established by a pipe of diameter $d_{ij} = d_{ji} \geq 0$ and a flow f_{ij} from R_i to R_j, which is assumed to be linear dependent on the pressure difference at both ends. Input and output of the system are given by the external inflow to reservoir R_1 and the sum of all outflows $f_{o,i}$ of R_i through a pipe with diameter $d_{o,i}$, respectively. Hence, by Pascal's law the flows are described by $f_{ij}(t) = d_{ij}^2 \cdot k \cdot (h_i(t) - h_j(t))$ and $f_{o,i}(t) = d_{o,i}^2 \cdot k \cdot (h_i(t) - h_j(t))$, where k is a constant representing gravity as well as viscosity and density of the medium. These equations represent a SISO-system with $B = \left(\frac{1}{a_1}, 0, \dots, 0\right)^T$, $C = k \begin{pmatrix} d_{o,1}^2 & \cdots & d_{o,n}^2 \end{pmatrix}$ and an A-matrix with entries

$$a_{ij} := \frac{k}{a_i} \begin{cases} -d_{o,i}^2 - \sum_{m=1}^{n} d_{im}^2, & i = j \\ d_{ij}^2 & i \neq j, \end{cases} \quad \text{with } d_{ii} := 0.$$

The system in [11] consist of two substructures, each with five reservoirs. In both substructures all reservoirs are mutually connected by a pipe of diameter $d_{ij} = 1$. The substructures are connected by a pipe of diameter $d_{1,10} = d_{10,1} = 0.2$ between reservoir 1 and 10. For simplicity, $a_i = 1$ and $k = 1$. Applying SBT results in a first order system with $(A, B, C) = (-1, 1, 1)$ and zero error. In contrast, since GBT does generally not return a minimal realization, it gives a first order system $(A, B, C) = (-3.0395, 1, 3.0395)$ with a relative H_∞-error of 0.5014.
Next, we modify the system by setting $d_{o,i} = 0.01 \cdot i$ and by only admitting a flow from R_1 to R_j, but not vice versa. The new system is minimal and has a non-symmetric A-matrix. Let each substructure consist of 50 reservoirs, then SBT gives a reduced symmetric model of order 2 ,

$$A_2 = \begin{pmatrix} -0.1305 & 0.0914 \\ 0.0914 & -0.2676 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.0457 \\ 0 \end{pmatrix},$$
$$C_2 = \begin{pmatrix} 0.0457 & 0 \end{pmatrix},$$

with error 0.0032. To get the same error with GBT, we would have to reduce the system to order 91. We notice that SBT performs fairly well even for systems with non-symmetric A-matrix.

5 Conclusion

Besides a positivity preserving model reduction method for SISO systems, the paper shows a necessary condition for positivity of a MIMO-system. This condition is preferable over considering the nonnegativity of the impulse response (cf. [2]). Moreover, the method does not require a positive realization and hence by the help of methods such as the Iterative Rational Krylov algorithm (cf. [6]), it is possible to deal with large-scale systems. Beside this, the method preserves and provides the symmetry of a system.

Acknowledgements This work was supported by the Swedish Research Council through the LCCC Linnaeus Center.

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