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Linear Quadratic Performance Criteria for Cascade Control

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Abstract— **In this paper we consider the problem of linear quadratic regulator (LQR) performance for cascade control structures of series coupled systems. The necessary and sufficient condition for the linear quadratic performance of a cascade control structure to achieve the same performance as any given centralized LQR is obtained. The cascade problem could be seen as a special case of the network control problem with a special network structure. The results presented show that decentralized control is possible with some natural fundamental limitations in this particular case. Also, a constructive proof is given for the inverse LQR problem that can be used to calculate the decentralized LQR from a given centralized one.**

I. INTRODUCTION

The systems to be controlled are getting more and more complex. Especially, for an interconnection between different dynamical systems, it is desired that the controllers are separated in such a way that engineers responsible for different subsystems can independently find a *good* controller for each subsystem, and still, when putting things together, the overall interconnected system performance is optimized in some sense. It is interesting to examine what fundamental limitations are faced when we try to find decentralized controllers on networks in terms of analysis and synthesis. Therefore, examining some special cases of interconnection structures, can give a good insight into the more general problem. Moreover, some special structures can have practical use of their own.

In this paper, we study the case of decentralized cascade control over the class of linear quadratic regulators (LQR) for a series connection structure. For series coupling of an arbitrary number of LTI systems, we obtain necessary and sufficient conditions for the performance of a cascade control structure to be as good as for any given centralized LQR.

For instance, consider a vehicle control problem, where it is desired to control the position and the velocity. To do this, one would like to affect the vehicle with some forces and torques. This is done through the system actuators, like wheels, rudders, etc (see Figure 1). But since the actuators have dynamics, they are themselves dynamical systems that need to be controlled.

One way of finding an optimal linear quadratic regulator (LQR) is to consider the series connected systems, that is the
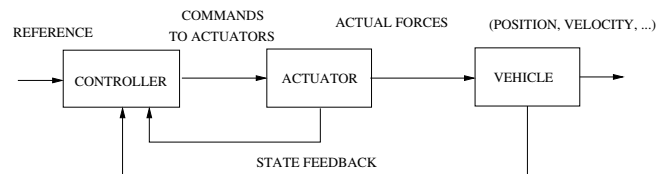

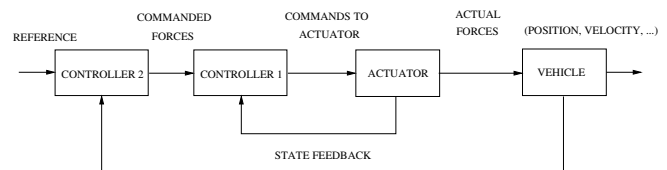Fig. 1.   Vehicle example. Centralized controller.

Fig. 2.   Vehicle example. Decentralized controllers.

connection between the actuators and the vehicle dynamics, as one system and find an LQR, which can be considered as a centralized controller. The question now is if we instead can separately find LQRs for the actuators and the vehicle, that still give the same performance index as the centralized one (see Figure 2). This is of potential practical importance since one would like to have two modules for controller design, one for the actuators, and the other for the vehicle in terms of the actual outputs of the actuators (like resulting forces and torques affecting the vehicle). Tuning can be done for both modules separately, rather than for the whole system. An important issue is the case when we get actuator failure, which implies a change in the actuator dynamics. Having separate controllers makes it easier to adapt to such changes in the dynamics.

Another issue is that there can be constraints on the size of the forces and torques that affect the vehicle. A centralized LQR for the series connection optimizes with respect to the inputs to the actuators, and not the actual forces and torques. Controller separation gives more understanding of the behaviour of each subsystem in terms of the optimization with respect to the inputs and outputs of every subsystem, and such constraints can be handled easier.

There has been previous work on similar problems. In Härkegård[6], the control allocation problem is considered as LQ optimization combined with static optimization for the control allocation. In Aoki[4], the decentralized control

problem is discussed where the agents may exchange their control values but not the state vector observation value. In Krtolica *et al.*[8], interconnection measure is introduced to compare the decentralized controller's performance index with that of a centralized controller. Rotkowitz *et al.*[9] gives a parameterization of stabilizing linear quadratic controllers that are equivalent to a decentralized one. Also, it is worth mentioning the standard reference book in LQ-design, namely Anderson and Moore[3], where the LQ problem is investigated in depth.

The outline of the paper is as follows: In section II we introduce some basic notation. The main results are presented in section III. We start by the *inverse* LQR problem, where we give a constructive proof for the existence of a weighting matrix that corresponds to a given linear static feedback law. Then, we consider the problem of decentralized LQR of series connection consisting of two systems, and then we generalize to an arbitrary number of series connected systems. A flight control example is solved in section IV, using the results in section III. In the Appendix, we state the standard LQR problem.

II. NOTATION

For a vector $v \in \mathbb{R}^n$, we define

$$\text{size}(v) = n.$$

Let two LTI systems Σ_1 and Σ_2 be given by:

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u_1 \\ y_1 &= C_1 x_1, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \dot{x}_2 &= A_2 x_2 + B_2 u_2 \\ y_2 &= C_2 x_2. \end{aligned} \tag{2}$$

If we connect Σ_1 and Σ_2 such that the input to Σ_1 is the output of Σ_2, i.e. $u_1 = y_2$, then we write the resulting series connected system as

$$\Sigma = \Sigma_2 \bullet \Sigma_1.$$

In general we write a series connection of n systems Σ_i

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i, \end{aligned} \tag{3}$$

for $i = 1, 2, \ldots, N$, as

$$\Sigma = \Sigma_N \bullet \cdots \bullet \Sigma_2 \bullet \Sigma_1,$$

which means that the systems are connected in series such that the input of Σ_i is the output of Σ_{i+1}, i.e. $u_i = y_{i+1}$, for $i = 1, \ldots, N-1$.

III. MAIN RESULTS

A. *The inverse LQR problem.*

The inverse LQR problem is if there is a symmetric and positive definite weighting matrix

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}, \tag{4}$$

that corresponds to a given static linear controller $u = Lx$ (see the Appendix for the standard LQR problem). A frequency domain condition that is sufficient for the existence of such a weighting matrix was shown by Kalman[7] for the SISO case. Here, we show that it always exists, and we show *how* to find the corresponding matrix.

Theorem 1 (Inverse LQR): Consider the standard LQR problem. For any given stabilizing feedback law

$$u = Lx,$$

there exists a symmetric matrix

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} > 0, \tag{5}$$

such that the optimal LQR with respect to this weighting matrix gives $u = Lx$. In particular, we can find a symmetric matrix $P > 0$, such that

$$(A+BL)^T P + P(A+BL) < 0,$$

and we can choose a symmetric matrix $R > 0$ large enough, such that

$$||R|| > \frac{||B^T P||^2}{2||P(A+BL)||}, \tag{6}$$

choose Q such that

$$Q = -(A+BL)^T P - P(A+BL) + L^T RL + L^T B^T P + PBL,$$

and set

$$S = -(B^T P + RL)^T,$$

to obtain the desired optimal feedback law $u = Lx$, and (5) holds.

Proof: The proof is given in the Appendix. ∎

B. *Decentralized control of series connection.*

Consider two LTI systems Σ_1 and Σ_2 given by

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u_1 \\ y_1 &= C_1 x_1, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \dot{x}_2 &= A_2 x_2 + B_2 u_2 \\ y_2 &= C_2 x_2, \end{aligned} \tag{8}$$

respectively, where $y_1 \in \mathbb{R}^l$, $y_2, u_1 \in \mathbb{R}^m$, $u_2 \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^k$, $x_2 \in \mathbb{R}^t$.
Let the input of Σ_1 be the output of Σ_2, that is, $u_1 = y_2$. Assume that (A_1, B_1) and (A_2, B_2) are stabilizable.
Introduce

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then we can write the coupled systems as a larger system Σ:

$$\dot{x} = Ax + Bu_2$$
$$y_1 = Cx, \tag{9}$$

where

$$A = \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix},$$

and

$$C = \begin{pmatrix} C_1 & 0 \end{pmatrix}.$$

Now consider the total system Σ (see Figure 3). The standard LQR problem for Σ is that one would like to find an optimal control law that minimizes the cost

$$\min_u \int_0^\infty \begin{pmatrix} \Delta x^T & \Delta u^T \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta u \end{pmatrix} dt, \tag{10}$$

where $\Delta x = x - x^*$, $\Delta u = u_2 - u^*$, x^*, u^* are the reference trajectories, and

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix},$$

is a symmetric positive definite matrix(see the Appendix for more details). Then, we can find an optimal control law:

$$u_2 = Lx + L_r r$$
$$= \begin{pmatrix} l_1 & l_2 \end{pmatrix} x + L_r r \tag{11}$$
$$= l_1 x_1 + l_2 x_2 + L_r r,$$

corresponding to some given choices of Q, R and S.

On the other hand, if we separately try to find LQRs for system Σ_1 and Σ_2 respectively (compare with Figure 4), we get the control law for Σ_1

$$u_{c1} = L_1 x_1 + L_{r1} r, \tag{12}$$

and for Σ_2

$$u_{c2} = L_2 x_2 + L_{r2} u_{c1}$$
$$= L_2 x_2 + L_{r2} L_1 x_1 + L_{r2} L_{r1} r. \tag{13}$$

for some choices of the weighting matrices Q_1, R_1, S_1, and Q_2, R_2, S_2 corresponding to Σ_1 and Σ_2, respectively.

The controller for the whole system Σ is given by equation (11). The combination of the controllers for Σ_1 and Σ_2 is given by equation (13). Equivalence between the two controller designs is obtained if $u_{c2} = u_2$, that is if

$$L_{r2} L_1 = l_1, \tag{14}$$

$$L_2 = l_2, \tag{15}$$

$$L_{r2} L_{r1} = L_r. \tag{16}$$

Equation (14) and (16) are solvable for the matrices L_1, L_2, L_{r1} and L_{r2} iff

$$\text{rank}(L_r) \leq m, \tag{17}$$

and

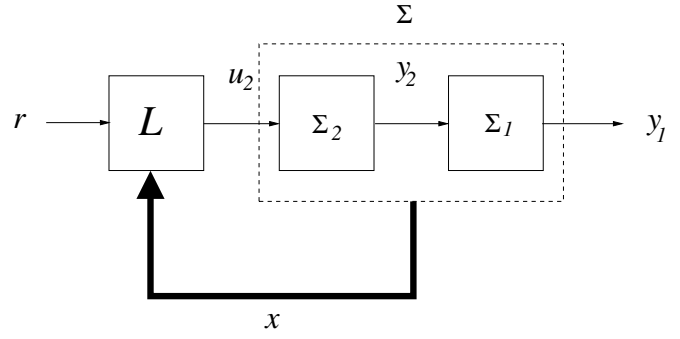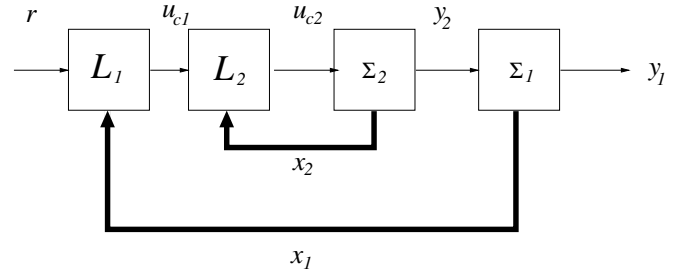$$\text{rank}(l_1) \leq m. \tag{18}$$

Fig. 3. The total system.

Fig. 4. The system with cascade control.

This is seen by noting that the rank of $L_{r2} L_1$ and $L_{r2} L_{r1}$ is always less than or equal to m, since L_{r2} is of size $n \times m$. Now if this condition holds we can solve for L_1, L_2, L_{r1}, L_{r2} equations (14)-(16). Now we are ready to state:

Theorem 2: The cascade LQR given by equation (13), for the series coupled systems Σ_1 and Σ_2 given by equation (1) and (2) respectively, can be made equal to any given centralized LQR for $\Sigma = \Sigma_2 \bullet \Sigma_1$, iff one of the conditions $n \leq m$, or $l, k \leq m$ holds.

Proof: For the decentralized cascade LQR to be equal to any given centralized LQR, we must be able to solve equations (14)-(16) even in the case where l_1 and L_r have full rank, that is

$$\text{rank}(l_1) = \min(n, k),$$

$$\text{rank}(L_r) = \min(n, l).$$

Hence, it is easy to see that equation (14)-(16) is solvable iff one of the conditions $n \leq m$ or $l, k \leq m$ holds. Now we must show that there exists weighting matrices Q_1, R_1, S_1, and Q_2, R_2, S_2 corresponding to Σ_1 and Σ_2, respectively, such that the feedback matrices L_1 and L_2 that solve (14)-(16) are obtained. But according to Theorem 1, it is always possible. This concludes the proof. ∎

REMARK. These conditions simply mean that u_2 must be large enough to carry all required information about the states x_1 to system Σ_1, in order to achieve the same performance after controller separation.

Now we give a generalization of Theorem 2. Consider N LTI systems Σ_i, $i = 1,\ldots,n$, connected in series as

$$\Sigma = \Sigma_N \bullet \cdots \bullet \Sigma_2 \bullet \Sigma_1.$$

(see Figure 5). The next theorem gives necessary and sufficient conditions for the controller separation of the series connected system Σ (see Figure 6).

Theorem 3: Consider a network consisting of N linear systems $\{\Sigma_i : i = 1, 2, \ldots, N\}$, defined according to equation (3), where the output of system Σ_{i+1} is the input to Σ_i, for $i = 1, 2, \ldots, N-1$. Then we can separately find LQRs for every system Σ_i such that the network is equivalent to any LQ-optimal centralized controller, iff

$$\text{size}(u_N) \leq \text{size}(u_i),$$

or

$$\text{size}(x_i), \text{size}(y_i) \leq \text{size}(u_i),$$

for $i = 1, 2, \ldots, N-1$.

Proof: The statement is easily proved by induction using Theorem 2. Note that if we write

$$\Sigma' = \Sigma_N \bullet \cdots \bullet \Sigma_2,$$

then

$$\Sigma = \Sigma' \bullet \Sigma_1.$$

Also, the input to Σ' is u_N and the output is y_2. We can separate the controllers between Σ' and Σ_1 iff

$$\text{size}(u_N) \leq \text{size}(u_1),$$

or

$$\text{size}(x_1), \text{size}(y_1) \leq \text{size}(u_1),$$

according to Theorem 2. Now let

$$\Sigma'' = \Sigma_N \bullet \cdots \bullet \Sigma_3,$$

and

$$\Sigma' = \Sigma'' \bullet \Sigma_2.$$

Again, the controller over Σ' can be separated over the systems Σ'' and Σ_2 iff

$$\text{size}(u_N) \leq \text{size}(u_2),$$

or

$$\text{size}(x_2), \text{size}(y_2) \leq \text{size}(u_2).$$

Continuing with this procedure, we see that any LQR can be split over the systems $\{\Sigma_i\}$ iff

$$\text{size}(u_N) \leq \text{size}(u_i),$$

or

$$\text{size}(x_i), \text{size}(y_i) \leq \text{size}(u_i),$$

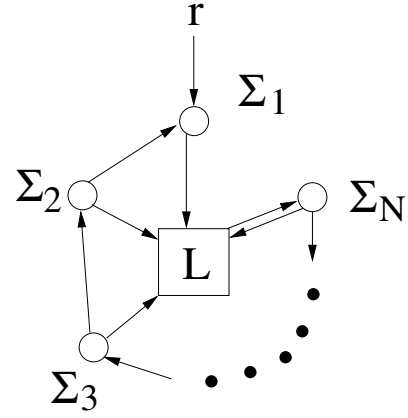for $i = 1, 2, \ldots, N-1$, which concludes the proof. ∎
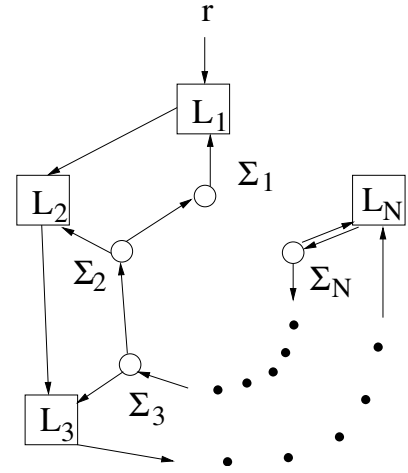

Fig. 5.   Network with series connection structure.

Fig. 6.   The network with controller separation.

IV. FLIGHT CONTROL EXAMPLE

In this section we Consider a flight control example that was used as a motivation in Härkegård[6]. The example is based on the ADMIRE model [1]. ADMIRE describes a small single engine fighter with a delta-canard configuration. The linearized model is given by

$$
\begin{aligned}
x &= \begin{pmatrix} \alpha & \beta & p & q & r \end{pmatrix}^T - x_0 \\
y &= \begin{pmatrix} \alpha & \beta & p \end{pmatrix}^T - y_0 \\
\delta &= \begin{pmatrix} \delta_c & \delta_{re} & \delta_{le} & \delta_r \end{pmatrix}^T - \delta_0 \\
u &= \begin{pmatrix} u_c & u_{re} & u_{le} & u_r \end{pmatrix}^T - u_0
\end{aligned}
$$

$$
\begin{pmatrix} \dot{x} \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} A & B_x \\ 0 & -B_\delta \end{pmatrix} \begin{pmatrix} x \\ \delta \end{pmatrix} + \begin{pmatrix} 0 \\ B_\delta \end{pmatrix} u, \tag{19}
$$

where α = angle of attack, β = side slip angle, p = roll rate, q = pitch rate, r = yaw rate, are the aircraft state variables, δ and u contain the actual and the commanded deflections of the canard wings, the right and left elevons, and the rudder, respectively. x_0, y_0, δ_0 and u_0 are the points of linearization. This model is actually a model of a connection in series
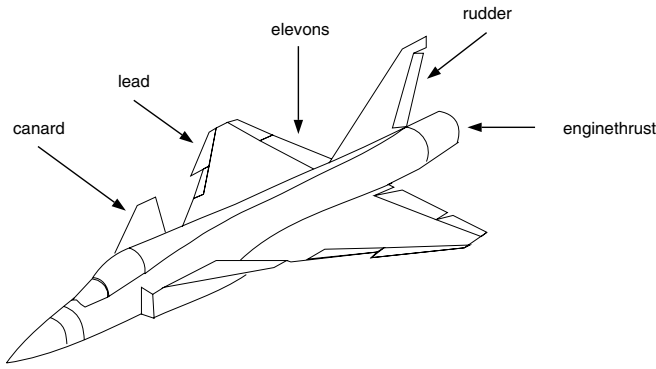
Fig. 7. The actuators affecting a fighter in the ADMIRE model.

between two systems, as Figure (3) shows. In our example Σ_1 describes the flight dynamics

$$\dot{x} = Ax + B_x\delta$$

and Σ_2 the actuator dynamics

$$\dot{\delta} = -B_\delta\delta + B_\delta u.$$

In Härkegård[6], the problem of cascade LQR separation is solved by approximating the rudder's dynamics to be static. We will solve the same problem without making any relaxations.

The numerical values of the system matrices are

$$A = \begin{pmatrix} -0.5432 & 0.0137 & 0 \\ 0 & -0.1179 & 0.2215 \\ 0 & -10.5128 & -0.9967 \\ 2.6221 & -0.0030 & 0 \\ 0 & 0.7075 & -0.0939 \end{pmatrix}$$

$$\begin{pmatrix} 0.9778 & 0 \\ 0 & -0.9661 \\ 0 & 0.6176 \\ -0.5057 & 0 \\ 0 & -0.21277 \end{pmatrix},$$

$$B_x = \begin{pmatrix} 0.0069 & -0.0866 & -0.0866 & 0.0004 \\ 0 & 0.0119 & -0.0119 & 0.0287 \\ 0 & -4.2423 & 4.2423 & 1.4871 \\ 1.6532 & -1.2735 & -1.2735 & 0.0024 \\ 0 & -0.2805 & 0.2805 & -0.8823 \end{pmatrix},$$

$$B_\delta = \begin{pmatrix} 20 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 20 \end{pmatrix}.$$

Consider the coupled systems as one system. First we try to find an optimal LQR for the total system in equation (19) with respect to the weighting matrices

$$Q = \text{diag}(10, 10, 2, 1, 10, 1, 1, 1, 1),$$
$$R = \text{diag}(10, 10, 10, 10),$$
$$S = 0.$$

This design gives the state feedback matrix

$$L = \begin{pmatrix} 1.1491 & 0.0065 & 0.0004 & 0.6844 & -0.0015 \\ -1.0027 & 0.3160 & -0.1884 & -0.5851 & -0.1212 \\ -1.0007 & -0.3272 & 0.1877 & -0.5839 & 0.1239 \\ 0.0043 & 0.3190 & 0.1251 & 0.0024 & -1.1311 \end{pmatrix}$$

$$\begin{pmatrix} 0.1001 & -0.0438 & -0.0437 & 0.0002 \\ -0.0438 & 0.1247 & -0.0010 & -0.0085 \\ -0.0437 & -0.0010 & 0.1245 & 0.0082 \\ 0.0002 & -0.0085 & 0.0082 & 0.1042 \end{pmatrix}.$$

First, we see that the conditions of Theorem 2 hold. Using Theorem 1 we see that for the aircraft controller, we can use the weighting matrices

$$R_1 = \text{diag}(1, 1, 1, 1),$$

$$Q_1 = -(A + B_x l_1)^T P_1 - P_1(A + B_x l_1) + l_1^T R_1 l_1 + l_1^T B_x^T P_1 + P_1 B_x l_1 > 0,$$

where

$$P_1 = \begin{pmatrix} -0.094 & -0.000 & 0.015 & -0.492 & -0.000 \\ -0.000 & -0.860 & -3.666 & 0.008 & -0.221 \\ 0.015 & -3.666 & -60.433 & 0.050 & -4.835 \\ -0.492 & 0.008 & 0.050 & 1.411 & 0.003 \\ -0.000 & -0.221 & -4.835 & 0.003 & -1.081 \end{pmatrix}$$

solves the Lyapunov equality

$$(A + B_x l_1)^T P_1 + P_1(A + B_x l_1) = -I,$$

and

$$S_1 = -(B_x^T P_1 + R_1 l_1),$$

and for the actuators, we can use

$$R_2 = \text{diag}(10, 10, 10, 10),$$

$$Q_2 = -(-B_\delta + B_\delta l_2)^T P_2 - P_2(B_\delta + B_\delta l_2) + l_2^T R_2 l_2 + l_2^T B_\delta^T P_2 + P_2 B_\delta l_2,$$

where

$$P_2 = \begin{pmatrix} 0.0279 & -0.0014 & -0.0014 & 0.0000 \\ -0.0014 & 0.0286 & 0.0000 & -0.0003 \\ -0.0014 & 0.0000 & 0.0286 & 0.0003 \\ 0.0000 & -0.0003 & 0.0003 & 0.0279 \end{pmatrix}$$

solves the Lyapunov equality

$$(-B_\delta + B_\delta l_2)^T P_2 + P_2(-B_\delta + B_\delta l_2) = -I,$$

and

$$S_2 = -(B_\delta^T P_2 + R_2 l_2).$$

It is easy to verify that the optimal feedback law obtained from the separated LQR-design above gives the same design obtained from considering the total system.

V. Conclusions

For a series connection structure of an arbitrary number of subsystems, necessary and sufficient conditions are presented for controller separation for the LQR problem. If the given conditions hold, designing a controller for every subsystem separately gives the same performance as if trying to design a centralized controller.

A flight control example is solved using the results of this paper.

VI. Acknowledgements

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References

[1] ADMIRE ver. 3.4h. *Aerodata Model in Research Environment (AD-MIRE), version 3.4h.* Swedish Defence Research Agency (FOI), 2003.
[2] S. Boyd, V. Balakrishnan, E. Feron, L. ElGhaoui. *Control System Analysis and Synthesis via Linear Matrix Inequalities.* In Proc. ACC, pp. 2147-2154, 1993.
[3] B. D. O. Anderson and J. B. Moore. *Optimal Control: Linear Quadratic Methods.* Prentice Hall International, Inc., 1989.
[4] Masanao Aoki. *On Decentralized Linear Stochastic Control Problems with Quadratic Cost.* IEEE Tans. Automatic Control, Vol. 18, No. 3, pp. 243-250, June 1973.
[5] T. Glad, L. Ljung. *Control Theory: Multivariable and Nonlinear Methods.* Taylor & Francis, 2000.
[6] O. Härkegård. *Backstepping and Control Allocation with Applications to Flight Control.* PhD Thesis, 2003. Department of Electrical Engineering, Linköping University, Linköping, Sweden.
[7] R. E. Kalman. *"When Is a Linear Control System Optimal?".* Trans. ASME, J. Basic Eng., ser. D, pp.1-10, March 1964.
[8] R. Krtolica and D. D. Siljak. *Suboptimality of Decentralized Stochastic Control and Estimation.* IEEE Trans. Automatic Control, Vol. 25, No. 1, pp. 76-83, February 1980.
[9] M. Rotkowitz and S. Lall. *Decentralized Control Information Structures Preserved Under Feedback.* IEEE Conference on Decision and Control, pp. 569-575, December 2002.

Appendix

The following proposition states the standard LQR design:

Proposition 1: Consider the following LTI system:

$$\dot{x} = Ax + Bu$$
$$y = Cx.$$

For the positive definite and symmetric matrix

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix},$$

(A, B) is stabilizable, the stationary optimal control law that minimizes the cost

$$\min_u \int_0^\infty \begin{pmatrix} \Delta x^T & \Delta u^T \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta u \end{pmatrix} dt, \quad (20)$$

where $\Delta x = x - x^*$, $\Delta u = u - u^*$, and x^*, u^* solve

$$\min_{x,u} \quad u^T Ru$$
$$\text{subject to} \quad Ax + Bu = 0 \quad (21)$$
$$Cx = r,$$

is given by

$$u = Lx + L_r r,$$
$$L = -R^{-1}(B^T P + S^T). \quad (22)$$

Here, P is the unique positive definite and symmetric solution to the algebraic Riccati equation (ARE):

$$A^T P + PA + Q - (PB + S)R^{-1}(B^T P + S^T) = 0. \quad (23)$$

Proof: For a proof consult *Glad and Ljung*[5]. ∎

Proof of Theorem 1

First, taking the Schur complement of equation (5), we see that (5) is equivalent to $R > 0$ and

$$Q - SR^{-1}S^T > 0.$$

Then, we want to show the existence of symmetric and positive definite matrices P, Q and R and a matrix S such that the ARE (23) is fulfilled, and

$$L = -R^{-1}(B^T P + S^T), \quad (24)$$

which is equivalent to

$$-B^T P - RL = S^T. \quad (25)$$

The ARE (23) can be rewritten as

$$Q = -(A + BL)^T P - P(A + BL) + L^T RL + L^T B^T P + PBL. \quad (26)$$

First, pick any symmetric $P > 0$, such that

$$(A + BL)^T P + (A + BL)P < 0.$$

The existence of such a P follows from the fact that L is stabilizing. Now set

$$S = -(B^T P + RL)^T. \quad (27)$$

Then we see that the choice of S according to equation (27) implies that

$$L = -R^{-1}(B^T P + S^T).$$

It remains to check that the condition of equation (5) hold. We first note that

$$SR^{-1}S^T = (B^T P + RL)^T R^{-1}(B^T P + RL)$$
$$= L^T RL + L^T B^T P + PBL + PBR^{-1}B^T P. \quad (28)$$

Then, equation (26) can be rewritten as

$$Q = -(A + BL)^T P - P(A + BL) + SR^{-1}S^T - PBR^{-1}B^T P,$$

or equivalently

$$Q - SR^{-1}S^T = -(A + BL)^T P - P(A + BL) - PBR^{-1}B^T P. \quad (29)$$

Since

$$-(A + BL)^T P - P(A + BL) > 0,$$

we see that if R is large enough such that

$$||R|| > \frac{||B^T P||^2}{2||(P(A + BL)||}, \quad (30)$$

the right hand side of equation (29) would be positive definite, which implies that

$$Q - SR^{-1}S^T > 0.$$

Hence, we have shown in a constructive way how to choose the matrices Q, R, S that give the desired LQR. ∎