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DIRECT AND INVERSE SCATTERING FOR TRANSIENT ELECTROMAGNETIC WAVES IN NONLINEAR MEDIA

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Abstract: This paper discusses a time domain approach to the solution of the inverse scattering problem in nonlinear materials. Two different reconstruction algorithms are presented.

PREREQUISITES

In a source-free environment the Maxwell equations in one dimension are

$$\begin{aligned} \partial_z E(z,t) + \partial_t B(z,t) &= 0 \\ \partial_z H(z,t) + \partial_t D(z,t) &= 0, \end{aligned}$$

where E, B, H and D denotes the field amplitudes. In this paper we assume that the nonlinear constitutive relations are

$$\begin{aligned} D(z,t) &= \varepsilon_0 F_{\rm e}(E(z,t)) \\ B(z,t) &= \frac{1}{c_0} F_{\rm m}(\eta_0 H(z,t)). \end{aligned}$$

The functions $F_{\rm e}$ and $F_{\rm m}$ are assumed to model a passive medium. For a non-magnetic material, a sufficient condition on $F_{\rm e}$ is [1]

$$F_{\rm e}'(x) \geq a > 0.$$

In this paper we also assume that $F_{\rm m}'(x) \geq b > 0$. It is convenient to transform the dynamics into

$$\partial_t \begin{pmatrix} u^+ \\ u^- \end{pmatrix} + c(u^+, u^-) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_z \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = 0.$$

where the Riemann invariants $u^\pm$ are

$$\begin{aligned} u^+ &= \frac{1}{2}\left(g_{\rm e}(E) + g_{\rm m}(\eta_0 H)\right) = \frac{1}{2}\left(\int_0^E \sqrt{F_{\rm e}'(x)}\,dx + \int_0^{\eta_0 H} \sqrt{F_{\rm m}'(x)}\,dx\right) \\ u^- &= \frac{1}{2}\left(g_{\rm e}(E) - g_{\rm m}(\eta_0 H)\right) = \frac{1}{2}\left(\int_0^E \sqrt{F_{\rm e}'(x)}\,dx - \int_0^{\eta_0 H} \sqrt{F_{\rm m}'(x)}\,dx\right). \end{aligned}$$

and the wave speed c is

$$c(u^+, u^-) = \frac{c_0}{g_{\rm e}'(g_{\rm e}^{-1}(u^+ + u^-))\, g_{\rm m}'(g_{\rm m}^{-1}(u^+ - u^-))}.$$

PROPAGATION ALONG CHARACTERISTICS

We can solve the propagation problem via the method of characteristics. A characteristic curve is defined by $(z,t) = (\zeta^\pm(\tau), \tau)$, where $\zeta^\pm$ satisfy

$$\frac{d\zeta^\pm}{d\tau} = \pm\, c(u^+, u^-)\big|_{(z,t)=(\zeta^\pm(\tau),\tau)}.$$

The fields $u^\pm(z,t)$ are constant along these curves $\zeta^\pm$. The characteristic curve ζ^+ is a straight line provided the field u^- is constant in the slab, and vice versa.

A numerical example of one-way wave propagation is depicted in Figure 1. Notice that the pulse steepens at the trailing edge, as a consequence of a wave speed which decreases with increasing field strength.

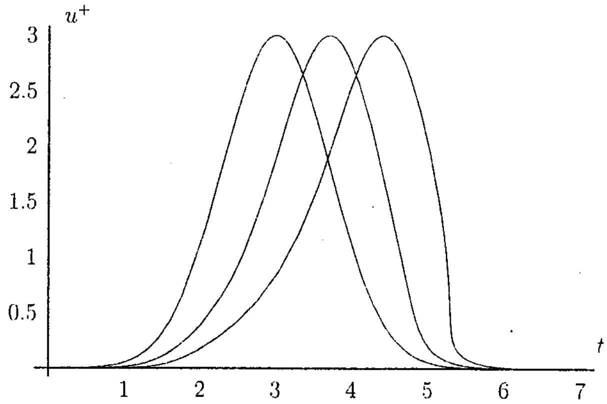

Figure 1: The propagation of an incident pulse $u^+(0,t) = 3H(t)\exp\{-(t-3)^2\}$. The material model is a Kerr saturation model $F(E) = E + 4E^3/(E^2+1)$. Three different propagation distances, $z = \ell$, values are shown, with $\ell = 0, 0.3$, and $\ell = 0.6$. In this example, we are using a numerically scaled field, which is the reason for large parameter values.

RECONSTRUCTIONS USING INTERNAL FIELDS

Provided the medium only supports one type of waves, *e.g.*, u^+ waves, the wave speed depends on only one field, and an explicit reconstruction algorithm can be found for pulses with no shocks [1]. The explicit result is

$$c(\xi) = \ell/\left(\overline{u}^{-1}(\xi) - h^{-1}(\xi)\right), \quad t > \ell/c(\xi_0).$$

Here, the incident field at $z = 0$ is denoted $u^+(0,t) = h(t)$ and the internal field at $z = \ell$ is denoted $u^+(\ell, t) = \overline{u}(t)$. Moreover, the time t and the parameter ξ are related by $t = \overline{u}^{-1}(\xi)$ and $\xi_0 = h(0)$.

RECONSTRUCTIONS USING EXTERNAL FIELDS

In the previus section the reconstruction of the wave speed from the internal field was presented. The underlying assumption there was the one-way wave propagation. In this section, we consider a slab located between $z = 0$ and $z = \ell$.

The internal fields, $u^\pm$, on the boundary of the slab can be related to the external incident, reflected, and transmitted fields, E^i, E^r, and E^t, respectively. The explicit expressions when the field impinges from the left are, for $z = 0$ and $z = \ell$, respectively

$$\begin{cases} E^i + E^r = g_{\mathrm{e}}^{-1}(u^+ + u^-) \\ E^i - E^r = g_{\mathrm{m}}^{-1}(u^+ - u^-) \end{cases} \qquad \begin{cases} E^t = g_{\mathrm{e}}^{-1}(u^+ + u^-) \\ E^t = g_{\mathrm{m}}^{-1}(u^+ - u^-). \end{cases}$$

Provided the left-going field inside the slab can be neglected, *i.e.*, $u^- \approx 0$, we essentially have wave propagation along straight characteristics. In this section we assume that this is the case, which is reasonable if the scattering properties at the edges are small, and it also seems to be connected to the strength of the nonlinearity. An important consequence of this assumption is that equal amplitudes of the u^+ field travel with equal speeds. Since there is a one to one correspondence between internal and external fields, this observation can be used to obtain the wave speed from transmission data.

If we denote the measurable quantities $E^i + E^r$ and $E^i - E^r$ by e and h, and neglect the left propagating field at the left boundary ($0 = u^- = g_{\mathrm{e}}(e) - g_{\mathrm{m}}(h)$ at $z = 0$), we can experimentally

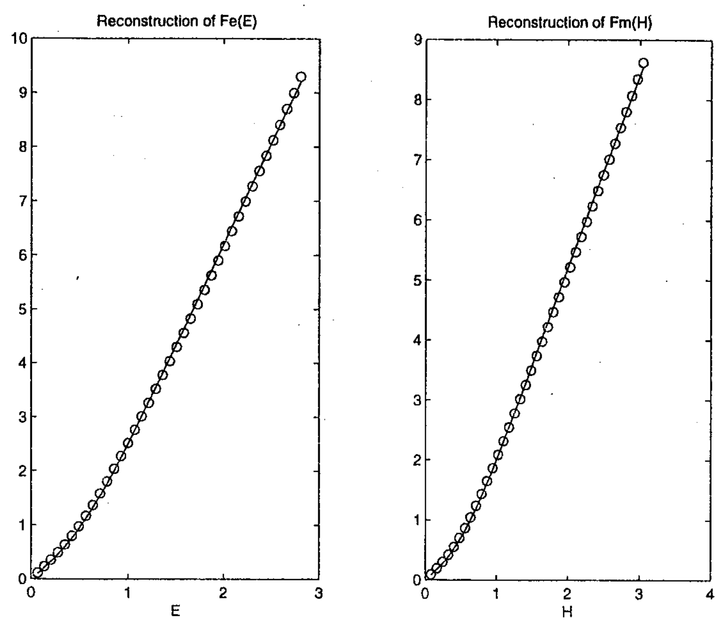

Figure 2: Reconstructed functions, from external fields. The circles are the reconstructed values, and the lines are the true functions.

determine the functions

$$
\begin{aligned}
e(h) &= g_e^{-1}(g_{\mathrm{m}}(h)) \\
c(e, h) &= \frac{c_0}{g_{\mathrm{e}}'(e) g_{\mathrm{m}}'(h)}.
\end{aligned}
$$

The derivative of e with respect to h is $\frac{de}{dh} = \frac{g_{\mathrm{m}}'(h)}{g_{\mathrm{e}}'(e)}$, corresponding to the wave impedance. We can thus find $g_{\mathrm{e}}'(e)^2 = F_{\mathrm{e}}'(e)$ and $g_{\mathrm{m}}'(h)^2 = F_{\mathrm{m}}'(e)$ by combining these relations:

$$
\begin{aligned}
F_{\mathrm{e}}'(e) &= \frac{\frac{dh}{de}}{c(e, h(e))} &\Rightarrow\quad F_{\mathrm{e}}(e) &= \int_0^{h(e)} \frac{dh'}{c(e(h'), h')} \\
F_{\mathrm{m}}'(h) &= \frac{\frac{de}{dh}}{c(e(h), h)} &\Rightarrow\quad F_{\mathrm{m}}(h) &= \int_0^{e(h)} \frac{de'}{c(e', h(e'))}.
\end{aligned}
$$

The inverse problem is thus seen to be well posed, and an explicit numerical illustration showing the performance of the inverse algorithm is depicted in Figure 2.

REFERENCES

[1] Kristensson, G. and Wall, D. J. N., "Direct and inverse scattering for transient electromagnetic waves in nonlinear media," *Inverse Problems*, **vol. 14**, 1998, pp. 113–137.