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OPTIMAL TIME–FREQUENCY OCCUPANCY OF FINITE PACKET OFDM

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***Abstract*— In this paper we consider the least time–frequency product necessary to transmit a small finite symbol packet such that the symbols can be independently detected. The system model assumed is offset QAM-OFDM, based on a finite duration pulse shape. The outcome is that the optimal pulse shape is of very short duration and that the optimal symbol allocation strategy is often to use as many subcarriers as there are symbols to transmit. Symbol packets up to 150 symbols are considered.**

I. INTRODUCTION

In this paper we consider a fundamental problem in digital communication theory: what is the least time–frequency product needed to transmit a finite number of data symbols over an additive white Gaussian noise (AWGN) channel, such that the symbols can be independently detected?

When the number of symbols, hereinafter referred to as the blocklength, tend to infinity, the answer is well known: 0.5 Hz-s must (at least) be spent per symbol. This is a classical result which underlies Shannon's capacity results for bandlimited channels, however, the result seems to have been first speculated by Nyquist in [1]. Both Shannon and Nyquist assumes perfectly bandlimited systems. Slepian [2] went further and considered systems where a fraction δ of the power is allowed to escape outside the official bandwidth and time supports. The outcome is that, as the blocklength tends to infinity, the needed product is 0.5 Hz-s.

Practical systems assume a specific signal generation form; this can only increase the needed product. Halpern [3] considered single carrier linear modulation based on finite duration pulses, and found pulses that minimize the fractional out of band energy (FOBE) outside of a certain bandwidth. In [4] optimal pulses for multicarrier (OFDM) setups are derived, again with the FOBE constraint. In [5], pulses that minimize bandwidth with root-mean-square (RMS) and minimum-maximum-magnitude criterions are derived. In [6] a simple method for designing time-limited orthogonal pulses for multicarrier modulation, without any optimality constraint, is given. Common for the above mentioned papers is that the blocklength is not considered. In [7], a more general problem is considered: what is the least bandwidth needed in order to support K orthogonal signals of finite time duration; several different bandwidth measures are used. The outcome of [3]– [5] can theoretically never be any better than [7], because particular system models are assumed. But from a practical point of view, OFDM systems are frequently used and it is intresting to investigate their ultimate performance. Moreover, there is only a single pulse shape needed instead of K shapes in [7].

This paper attacks the problem of designing pulses that are optimal in an OFDM scheme, implemented with offset quadrature amplitude modulation (OQAM), when the blocklength is small; blocklengths in the range 16–150 symbols are considered. In this case it is no longer true that the signaling can be supported with only 0.5 Hz-s time–frequency product.

In this paper we are not interested in the benefits of OFDM for multipath fading etc. We are investigating the capabilities of OFDM to transmitt finite blocklengths over the Gaussian channel with small time–frequency occupancy.

II. OPTIMUM TIME–FREQUENCY PRODUCT DERIVATION

A. System Model

Assume that the blocklength is K symbols. The signal generation form in this paper is OFDM/OQAM [8], [9]. The transmitted signal $s(t)$, in complex baseband notation, equals

$$s(t) = \sum_{k=0}^{N_f-1} \sum_{n=0}^{N_t-1} a_{k,n} \jmath^{k+n} e^{\jmath(2\pi/T+\pi/2)k} h(t-nT/2), \tag{1}$$

where $\jmath$ is the imaginary unit and $a_{k,n}$ denotes the symbol at subcarrier k and time n. N_f is the number of subcarriers and N_t is the number of "time carriers". Since there are K symbols, we must have $K = N_f N_t$. $T/2$ is the time needed to transmit one symbol, although, the pulse $h(t)$ can have longer duration. Note that all symbols $a_{k,n}$ are real valued.

We are only interested in systems that are orthogonal, that is, there should be no interchannel interference (ICI) and no intersymbol interference (ISI). This can be mathematically stated as

$$\left[\mathcal{R}\left\{h(t-kT)e^{\jmath(\frac{2\pi}{T}t+\frac{\pi}{2})m} * g(t)\right\}\right]_{t=0} = \delta_{k,m} \quad (2)$$

$$\left[\mathcal{R}\left\{\jmath h\left(t-kT+\frac{T}{2}\right) e^{\jmath(\frac{2\pi}{T}t+\frac{\pi}{2})m} * g(t)\right\}\right]_{t=0} = 0 \quad (3)$$

$$\left[\mathcal{I}\left\{h(t-kT)e^{\jmath(\frac{2\pi}{T}t+\frac{\pi}{2})m} * g\left(t-\frac{T}{2}\right)\right\}\right]_{t=0} = 0 \quad (4)$$

$$\left[\mathcal{I}\left\{\jmath h\left(t-kT+\frac{T}{2}\right) e^{\jmath(\frac{2\pi}{T}t+\frac{\pi}{2})m} * g\left(t-\frac{T}{2}\right)\right\}\right]_{t=0} = \delta_{k,m} \quad (5)$$

where $\mathcal{R}\{\}$ and $\mathcal{I}\{\}$ denotes real and imaginary part and $\delta_{k,m}$ is the two dimensional Kronecker function. If the transmitter filters are real and symmetric, and the receiver filters are taken as

$$g(t) = h(-t), \quad (6)$$

then (3) and (4) are always satisfied. Moreover, (5) is now equivalent to (2) and is always satisfied when m is an odd number. Therefore the constraints (2)–(5) can be summarized into

$$G_{m,k} \triangleq \int_{-\infty}^{\infty} h(t)h(t-kT)\cos\left(\frac{2\pi}{T}2mt\right)dt = \delta_{m,k}. \quad (7)$$

While the equations (7) are present in the general case of an infinite blocklength, it is here not necessary to fullfill (7) for all m and k. Inspecting the signal generation form (1) and the constraints (7), it can be seen that for finite blocklength, the constraints that must be fullfilled are

$$G_{m,k} = \delta_{m,k},\ 0 \le k \le \left\lfloor \frac{N_t-1}{2} \right\rfloor,\ 0 \le m \le \left\lfloor \frac{N_f-1}{2} \right\rfloor. \quad (8)$$

The choice of symbol and carrier spacing in (1) needs a few words. In the asymptotic case $K = \infty$, it is well known that the product of the symbol and carrier spacing must be 0.5 Hz-s in order to have an ICI/ISI free system. In (1) the symbol spacing is $T/2$ seconds and the carrier spacing is $1/T$ Hz, thus, the product is indeed 0.5. For a *finite* number of symbols, there are only finitely many constraints in (8) to be fullfilled. Therefore the product can in principle be smaller than 0.5. However, in this paper we only investigate the case of a product equal to 0.5.

B. System Optimization

Assume that the pulse $h(t)$ is of duration LT. The total time consumed by (1) is

$$T_{\text{tot}} \triangleq N_t\frac{T}{2} + \left(L-\frac{1}{2}\right)T = N_t\frac{T}{2} + \epsilon_t T, \quad (9)$$

where we have introduced the *time overshoot*[1]

$$\epsilon_t \triangleq L - \frac{1}{2}. \quad (10)$$

In this paper we measure bandwidth with a FOBE constraint; by this we mean the frequency interval in which the pulse $h(t)$ holds, say, 99 % of the power. The choice of bandwidth measure will subsequently lead the paper to make use of derivations from [4] rather than those in [5]. For a given pulse $h(t)$, define W_C as

$$W_C \triangleq W : \frac{\int_{-W}^{W}|H(f)|^2 df}{\int_{-\infty}^{\infty}|H(f)|^2 df} = C. \quad (11)$$

Note that W_C is dependent on $h(t)$. The total consumed bandwidth becomes

$$W_{\text{tot}} \triangleq \frac{N_f-1}{T} + 2W_C = \frac{N_f}{T} + \frac{\epsilon_f}{T}, \quad (12)$$

where we have introduced the *frequency overshoot*

$$\epsilon_f \triangleq (2W_C - 1)T. \quad (13)$$

Note that we use perfectly time limited signals. A more general approach would be to use pulses that are allowed to take nonzero values at the entire time axis, and to measure time consumption by time out of band energy (TOBE).

The objective function to minimize becomes

$$\begin{aligned} W_{\text{tot}}T_{\text{tot}} &= \left(N_t\frac{T}{2} + \epsilon_t T\right)\left(\frac{N_f}{T} + \frac{\epsilon_f}{T}\right) \\ &= \frac{K}{2} + \frac{N_t}{2}\epsilon_f + \epsilon_t N_f + \epsilon_t\epsilon_f \end{aligned} \quad (14)$$

This is a constrained optimization over $h(t)$, N_t and N_f, the constraint being that $h(t)$ should be ICI/ISI free. For a fixed blocksize K, we should optimize the number of subcarriers such that $N_tN_f = K$.

To find the theoretical solution of optimization (14), take the derivative of $W_{\text{tot}}T_{\text{tot}}$ with respect to N_t.

$$\frac{\partial W_{\text{tot}}T_{\text{tot}}}{\partial N_t} = \frac{\epsilon_f}{2} - \frac{\epsilon_t K}{N_t^2} \quad (15)$$

Setting the derivative to zero gives $N_t = \sqrt{2K\epsilon_t/\epsilon_f}$. Inserting this into (14) gives

$$\begin{aligned} (W_{\text{tot}}T_{\text{tot}})_{\min} &= \frac{K}{2} + \sqrt{2\epsilon_t\epsilon_f K} + \sqrt{\frac{\epsilon_t\epsilon_f}{2K}} + \epsilon_t\epsilon_f \\ &= \frac{K}{2} + \sqrt{\epsilon 2K} + \sqrt{\frac{\epsilon}{2K}} + \epsilon \end{aligned} \quad (16)$$

where $\epsilon \triangleq \epsilon_t\epsilon_f$.

This shows that in order to minimize (14), we should minimize ϵ. This is closely related to time–frequency localization of pulses; the pulse with the smallest time-frequency occupancy is the Gaussian pulse which satisfies Heisenberg's uncertainty principle. But since the Gaussian

[1] To be technically correct, ϵ_t is a function of L, but this notation is omitted.

pulse is not ICI/ISI free, it is not the solution we are seeking in this paper.

The minimum (16) is in general achieved for non integer N_t and N_f, which is not possible in practice. The number of time- and subcarriers must be integers, so the optimization is over a combinatorial domain. The optimization problem can be formulated as

$$\begin{aligned} (W_{\rm tot}T_{\rm tot})_{\min} &= \min_{\epsilon_t,\epsilon_f,N_t,N_f} K + N_t\epsilon_f + \epsilon_t N_f + \epsilon_t\epsilon_f \\ \text{such that} &\quad N_tN_f = K,\ N_t, N_f \text{ integers} \\ &\quad h(t) \ \text{satisfies (8)}. \end{aligned} \tag{17}$$

It is obvious that for any combination ϵ_t, N_t, and N_f, the minimum of (17) is achieved by minimizing ϵ_f. This is in turn achieved by finding the pulse $h(t)$ with the smallest possible bandwidth W_C. Therefore we need to solve the optimization

$$\begin{aligned} \epsilon_{f,\rm opt} &\triangleq \min_{h(t)} (2W_C - 1)T \\ \text{such that} &\quad h(t) \text{ satisfies (8)} \\ &\quad h(t) \text{ has duration } \left(\epsilon_{\rm t} + \frac{1}{2}\right) \rm T. \end{aligned} \tag{18}$$

The objective of (18) can be reformulated as (assume $h(t)$ is unit energy)

$$\begin{aligned} W_{C,\rm opt} &\triangleq \frac{\epsilon_{f,\rm opt}}{2T} + \frac{1}{2} \\ &= \arg\min_{W} : \left[\max_{h(t)} \int_{-W}^{W} |H(f)|^2 df = C\right] \end{aligned} \tag{19}$$

The innermost optimization of (19) is exactly the optimization problem considered in [4]. For given N_t, N_f and L, the value $\epsilon_{f,\rm opt}$ is uniquely defined. This value is therefore denoted the *Vahlin-Holte* solution, and we write

$$\epsilon_{f,\rm opt} = \mathcal{VH}(\epsilon_t, N_t, N_f). \tag{20}$$

This implies that the optimization (17) can be expressed in the more compact notation

$$\begin{aligned} (W_{\rm tot}T_{\rm tot})_{\min} &= \min_{L,N_t} K + N_t\mathcal{VH}(\epsilon_t, N_t, K/N_t) \\ &\quad + \epsilon_t K/N_t + \epsilon_t\mathcal{VH}(\epsilon_t, N_t, K/N_t) \\ \text{such that} &\quad N_t, K/N_t \ \text{integers} \end{aligned} \tag{21}$$

If $N_f > N_f'$, then it follows that $\mathcal{VH}(\epsilon_t, N_t, N_f) > \mathcal{VH}(\epsilon_t, N_t, N_f')$ because there are more constraints present; the same is true for N_t and N_t'. When $\epsilon_t > \epsilon_t'$ it follows that $\mathcal{VH}(\epsilon_t, N_t, N_f) < \mathcal{VH}(\epsilon_t', N_t, N_f)$. This is true because the pulses of duration $L'T = (\epsilon_t' + 1/2)T$ are avaliable as solutions also for the longer duration $LT = (\epsilon_t + 1/2)T$.

A lemma on the optimal (N_t, N_f) combination closes this section.

Lemma 1: For finite duration pulses $LT \le L^*T$ and a finite blocklength K, there exists a value $C^* < 1$ such that if the power bandwidth is measured with $C \ge C^*$, then the optimal number of subcarriers is $N_f = K$.

Proof The optimal frequency overshoot is a function of ϵ_t, $\epsilon_{f,\rm opt} = \mathcal{VH}(\epsilon_t, N_t, N_f)$. To obtain $N_t = 1$ as optimal solution, for some L, it follows from (15) and (16) that $\epsilon_{f,\rm opt} > 2K\epsilon_t$. But since the pulse duration is finite, its Fourier transform must be infinite and it follows that $\epsilon_{f,\rm opt} \to \infty$, as $C \to 1$. Moreover, ϵ_t is finite as well. Therefore, the value C^* can be taken as $C^* = \arg\min_C : \mathcal{VH}(\epsilon_t, N_t, N_f)/2K\epsilon_t > 1,\ \forall L \le L^*$, with $N_t = N_f = 1$ because this give a lower bound on $\mathcal{VH}(\epsilon_t, N_t, N_f)$.■

From the proof of Lemma 1, the following corollary is obtained.

Corollary 1: Assume a finite duration pulse, $LT \le L^*T$. If $K \to \infty$ but $C \to 1$ so fast that $\epsilon_{f,\rm opt} > 2K\epsilon_t,\ \forall L \le L^*$, then the optimal number of subcarriers is $N_f = K$.

The implication of Lemma 1 and its corollary is that the OFDM symbol-lattice collapses into a single column of symbols when bandwidth is measured with a large C. Thus, OFDM, in the normal sense, is not the solution to the minimum time–frequency product in this case.

III. Numerical Results

We will only consider the case $C = 0.99$, i.e. the 99 % power bandwidth, in this paper.

The smallest pulse duration LT such that the constraints (7) can be fullfilled is $0.5T$. The unique solution is then a rectangular pulse. But with finite blocklength, there are only finitely many constraints in (8), and durations smaller than $0.5T$ can in principle be used.

In [4] it is shown that the solution $h(t)$ to the innermost optimization in (19) is of the form

$$h(t) = \sum_{m=0}^{\infty} c_m\psi_m(t), \tag{22}$$

where $\psi_m(t)$ is the mth even prolate spheroidal wave function truncated to the (time) interval $-(\epsilon_t + 1)/2 \le t/T \le (\epsilon_t + 1)/2$.

We follow [4] to actually find a numerical solution $\{c_m\}$. The problem is reduced to finite dimensionality; we have used 14 dimensions, that is, we allow 14 coefficients c_m. The optimization method we used is a MATLAB-built-in SQP method. As in [4] we have accepted a deviation $\sigma^2 = 10^{-6}$ from (8) where

$$\sigma^2 = \sum_{m,k} |G_{m,k} - \delta_{m,k}|^2. \tag{23}$$

In order to solve (21), complete knowledge of $\mathcal{VH}(\epsilon_t, N_t, K/N_t)$ must first be obtained. It turns out that this is a major difficulty when L is close to 0.5 or when N_f is large. This is why 14 dimensions are needed. It is possible to use branch and bound ideas prior to Lemma 1, to reduce the computational burden .

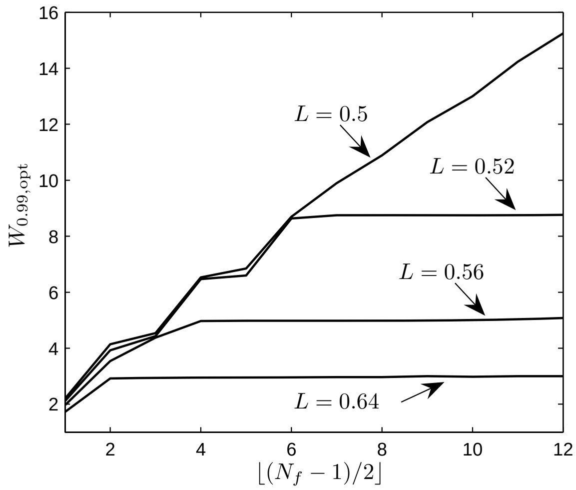

Fig. 1. Optimal 99 % bandwidths of the Vahlin-Holte ICI/ISI free pulses.

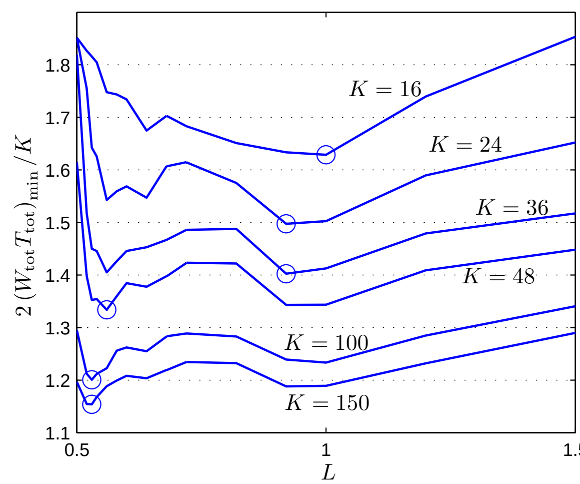

Fig. 3. Mimimum time–frequency product, relative to 0.5, as a function of pulse duration L. The circles mark the optimal durations.

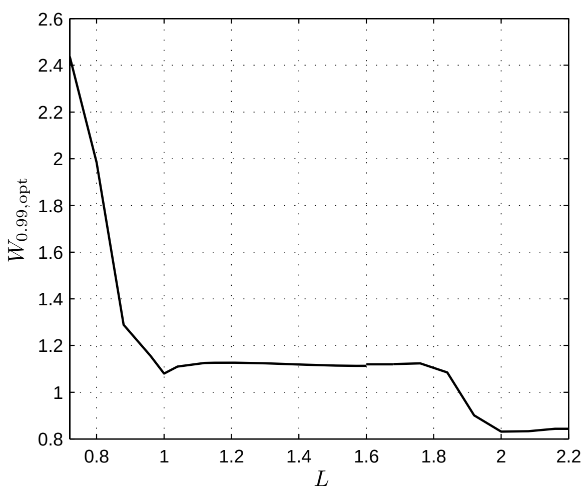

Fig. 2. Optimal 99 % bandwidth of ICI/ISI free pulses as a function of L for the case $N_t = 2\lceil L \rceil - 1$ and $N_f = 7$.

To illustrate the behaviour of the optimized pulses, we plot $W_{0.99,\mathrm{opt}}$ for $L = 0.50$, 0.52, 0.56 and 0.64 against $\lfloor (N_f - 1)/2 \rfloor$, which is the number of constraints in (8), in Figure 1. It is seen that the bandwidth $W_{0.99,\mathrm{opt}}$ essentially saturates for a certain number of subcarriers (this will eventually also occur for $L = 0.5$). This occurs because the optimized pulses are well localized in frequency, and are already almost orthogonal to pulses $N_f + M$ subcarriers away ($M > 0$) even if they were only optimized for N_f subcarriers.

While Figure 1 shows the optimal bandwidth as a function of N_f, Figure 2 shows the optimal bandwidth as a function of L, for the case $N_t = 2\lceil L \rceil - 1$ and $N_f = 7$. The choice for N_t implies that $h(t)$ must be orthogonal to all its T-shifts. Note that the curve in Figure 2 is not monotonically decreasing. At $L \approx 1.8$, this is a result of the finitely many dimensions in the pulse optimization. When L is an integer, new constraints are introduced in (8) which makes the outcome worse; this explaines the increase of $W_{0.99,\mathrm{opt}}$ at $L = 1$ and 2.

From Figure 2 it is seen that the optimal pulse shape cannot possibly be of duration $1 < L \lessapprox 1.8$. This is true since in this region, $\epsilon_{f,\mathrm{opt}}$ in (17) is almost constant, but ϵ_t is increasing. $W_{\mathrm{tot}}T_{\mathrm{tot}}$ will therefore take a smaller value at $L = 1$.

In Figure 3 the outcome of the optimization (21) is shown, for several blocklengths. We plot the minimum possible time-frequency product per symbol, relative to the asymptotic case 0.5, versus the pulse duration L. The pulses are here assumed to be exactly of duration LT; thus pulses of duration $L'T < LT$ are not available for duration LT. The optimal pulse durations are marked with circles. We have tested $L \leq 11$, but the optimal L is never above 1. The optimal pulse turns out to be of shorter duration when the blocklength increases. For a blocklength of 100 symbols, the excess time–frequency product is roughly 20 %.

It remains to discuss the optimal (N_t, N_r) combinations that corresponds to the optimal points in Figure 3. We give the combinations as well as the the actual time–frequency products in Table I. An interesting fact is that, for blocklengths from 48 up to at least 150 symbols, the optimal systems have $N_f = K$ and $N_t = 1$. Thus, the rectangular symbol lattice in OFDM collapses into a single column of symbols. Thus, it seems that the value $C = 0.99$ suffices as c^* in Lemma 1.

Finally we show the excess time–frequency product per symbol as a log–log plot in Figure 4. The asymptotic product is 0.5, and by 'excess' we mean $(W_{\mathrm{tot}}T_{\mathrm{tot}})_{\min}/K - 0.5$. It is interesting to observe that the excess product can be very well approximated with a straight line, for the

K	$(W_{\text{tot}}T_{\text{tot}})_{\min}/K$	Opt. L	(N_t, N_r)
16	1.629	1	(4,4)
24	1.497	0.92	(4,6)
36	1.396	0.92	(4,9)
48	1.334	0.56	(1,48)
90	1.216	0.53	(1,90)
96	1.206	0.53	(1,96)
100	1.201	0.53	(1,100)
112	1.185	0.53	(1,112)
120	1.177	0.53	(1,120)
150	1.154	0.53	(1,150)

TABLE I

OPTIMAL TIME–FREQUENCY PRODUCT PER SYMBOL AND SYSTEM PARAMETERS FOR VARIOUS BLOCKLENGTHS.

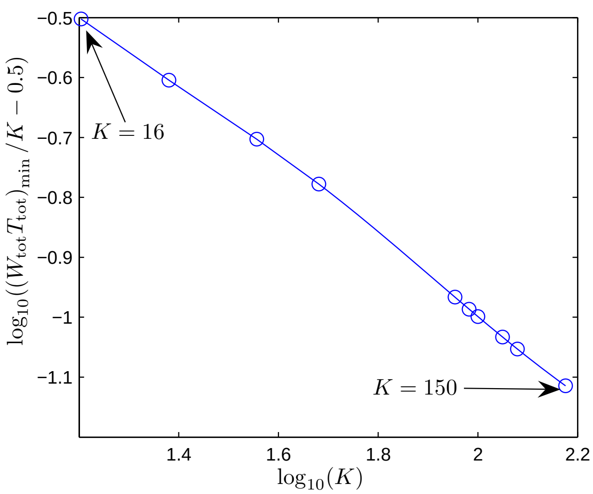

Fig. 4. Log-log plot of the minimum excess time-frequency product per symbol as a function of the blocklength K.

range of blocklengths presented in Figure 4. This leads to the empirical excess product law

$$\frac{(W_{\text{tot}}T_{\text{tot}})_{\min}}{K} - 0.5 \approx \alpha K^{\beta}, \quad \beta < 0 \tag{24}$$

IV. PROBLEM GENERALIZATIONS

We list some possible generalizations of the optimization problem considered in this paper.

Intentional Intersymbol Interference: Some of the constraints in (8) can be omitted, resulting in ICI or ISI. In this case the symbol and carrier spacing can have a product less than 0.5 Hz-s. Thus, the system has higher bandwidth efficiency. In the literature this is referred to as faster-than-Nyquist signaling. The ISI can be designed to be so mild that the BER of the system is unaffected. This can be achived by adding the constraint that the minimum distance of the system $d^2_{\min}$ must still equal the matched filter bound. In [10], $h(t)$ is taken as a root raised cosine pulse, the outcome is that the product can be as low as 0.25 Hz-s, without any performance degradation. In [11], the finite packet problem is considered.

Other time and bandwidth measures: As explained previously, the outcome of the paper is affected by the bandwidth measure. A straightforward approach is to measure bandwidth with other criteria, such as RMS etc. The time duration of a pulse can be measured in the same way as frequency is measured.

V. CONCLUSIONS

In this paper we have investigated the minimum time–frequency product needed to transmit K symbols over an AWGN channel by means of an OFDM/OQAM system. The bandwidth measure is the 99 % power bandwidth. The outcome is that for blocklengths from 48 up to, at least, 150 symbols, the optimal system has K subcarriers; only a single symbol interval in time is used. Thus, the rectangular symbol lattice in a normal OFDM system collapses. If the blocklength is 100 symbols, the time–bandiwdth penalty is roughly 20 %.

Moreover, the optimal pulse shape to use is very short, 0.51T–T seconds for the blocklengths considered here. The needed excess time–frequency product per symbol, seems to obey a power law.

VI. ACKNOWLEDGMENTS

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