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On the cost-vs-quality tradeoff in make-or-buy decisions

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Abstract

The make-or-buy decision is analyzed in a simple two-task principal-agent model. There is a cost-saving/quality tradeoff in effort provision. The principal faces a dichotomous choice between weak (“make”) and strong (“buy”) cost-saving incentives for the agent; the dichotomy is due to an incomplete-contracting limitation necessitating that one party be residual claimant. Choosing “buy” rather than “make” leads to higher cost-saving effort and – in a plausible “main case” – to lower quality effort; this in spite of stronger direct quality-provision incentives in the former case.

JEL Classification: D23, L24, L25

Keywords: make-or-buy decision, multitask principal-agent problem, quality

1 Introduction

In this note, we use a simple two-task principal-agent framework for addressing the effects of and the attractiveness of outsourcing. The aim is to provide a conceptually fruitful, yet simple, framework for assessing the tradeoffs encountered in practice by firms, organizations and government bodies.

The two tasks are geared towards saving costs and caring for quality. The prediction is that outsourcing will be unambiguously beneficial in terms of cost, and that this will be at the

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expense of quality in the “main case,” albeit not unambiguously; the reduction in quality will, moreover, result *in spite of stronger direct quality incentives*. We will also demonstrate, by a direct comparison of value functions, how the attractiveness of outsourcing is affected by the parameters of the model. In particular, a higher valuation of quality will make outsourcing less attractive precisely if the “main case” mentioned applies.

The key assumption made is that the cost-based performance measure is subject to an incomplete-contracting limitation that forces the principal to either pass on the full effect of cost-savings to the agent, or to make remuneration completely insensitive to cost-savings. This assumption can be justified in terms of costs being tied to an asset whose ownership can be shifted, while any sharing contract would be plagued by manipulation.¹ With this assumption, the principal faces a discrete choice between “make” – i.e. owning the asset and shielding the agent from monetary incentives – and “buy” – i.e. not owning the asset and exposing the agent to strong monetary incentives. Each of the options gives rise to an optimal remuneration policy in terms of the second performance measure indicating quality, and each of the resulting incentive schemes produces a distinct outcome in terms of the agent’s actual effort profile.

The point of departure for this note is thus the assumed dichotomy between “make” and “buy” in terms of direct monetary incentives. While the dichotomy is a direct consequence of contractual incompleteness, the association of the two resulting regimes with make and buy in practice is intended to reflect a stylized fact. Although some of the background for this stylized fact is “anecdotal” as argued by Gibbons (2005, p. 207), it has considerable backing by casual observation of remuneration of employees as compared with contractors.² There is also firm theoretical corroborating arguments; Acemoglu *et al.* (2008), for example, use a career-concerns framework to argue that firms, by design, “coarsify information” with weaker equilibrium effort incentives as a result. Tadelis (2002) makes a related point: The assets used by an employee are typically owned by someone else, making strong cost-based incentives hazardous due to multi-task effort-substitution incentives not to take due care of the asset.³

There is also evidence that incentives in other dimensions – e.g. direct incentives in terms of

¹An elaboration of this framework is made in Andersson (2009); also, Holmström and Milgrom (1991), Holmström (1999) and Gibbons (2005) make reference to an element of contractual incompleteness in a principal-agent model. For a basic discussion and justification for contractual incompleteness, see Hart (1995).

²An informal account corroborating this view is provided by Williamson (1985, Ch. 6).

³Also, Bajari and Tadelis (2001) provide an interesting foundation – in terms of complexity and adaptation costs – for the dichotomous choice between weak in-house cost-saving incentives, and strong cost-saving incentives in contracting.

quality measures – are stronger (and more explicit) in the context of contracting compared with in-house provision. This difference is likely to be more clear-cut in public sector contracting due to the limited potential for reputational mechanisms; it is discussed in e.g. Domberger and Jensen (1997). Marvel and Marvel (2007) argue – on the basis of survey-based evidence from municipal contracting – that overall monitoring does not really differ between in-house and contracted services; when monitoring is decomposed, however, contracting with for-profit entities (and other government entities) entails significantly stronger rewards of good performance and sanctions of poor performance (compared with in-house provision and contracting with non-profits).

One way of phrasing our main result is that outsourcing leads to lower costs but that the effect on quality is ambiguous absent further assumptions, although the effect is negative in the more plausible “main case” (defined by a genuine tradeoff between effort devoted to cost-saving and quality). The result is in line with those obtained by Hart, Shleifer and Vishny (1997) in their influential analysis of government contracting using a pure incomplete-contracting framework. While contracts are absent in their framework, our result reflects a *contractual response* to ownership being a dichotomous choice between weak and strong incentives (this dichotomy applies by construction in their framework as well). This contractual response is an empirically testable implication.⁴ Our basic approach is close Lindqvist (2008) who uses a related albeit somewhat different basic two-task framework and makes a similar assumption about regime choice with the important exception that he assumes that neither cost-savings, nor quality, can be contractually rewarded when the principal chooses a public agency rather than a private firm to perform a task; under these assumption, the attractiveness of a private solution is U-shaped as a function the valuation of quality.⁵

In the following we describe the basic model and the formal manifestation of the key assumption of regime choice. We then go on to the results; a brief concluding section follows.

⁴Also, our framework is not tailored to public-sector outsourcing, although – as we have noted – the relevance is more easily established there due to the limits of implicit contracts. Implicit contracts, clearly, can alleviate quality problems.

⁵Our result is also congruent with that obtained by Corneo and Rob (2003) who show that cost-based incentives are stronger in private compared with public firms, but that the ranking of overall productivity is ambiguous.

2 The Model

Basic framework We will employ a two-task specification of the Holmström-Milgrom (1991) multitask principal-agent model. A risk-neutral principal contracts with a risk-averse agent who exerts efforts, a_1 and a_2 , in two dimensions; following the Introduction, we sometimes refer to the dimensions in terms of cost-saving and quality. There are two output measures, x_1 and x_2 , that depend stochastically on effort according to

$$x_i = a_i + \varepsilon_i, \quad i = 1, 2,$$

where ε_i is noise; the outputs can thus also be thought of in terms of costs and quality. The noise terms are assumed to be jointly normally distributed with mean zero and a general covariance structure with $\text{Var}(\varepsilon_i) = v_i$, and $\text{Cov}(\varepsilon_1, \varepsilon_2) = \sigma$; some of the analysis will be done under the assumption that the errors are independent across i , i.e. $\sigma = 0$. In light of the interpretations discussed with a cost-saving dimension and a quality dimension, we assume that $v_2 > v_1$; i.e. that quality is measured less precisely than costs.⁶

The principal offers the agent a contract that specifies monetary compensation that is constrained to be linear in the performance measures,

$$y = F + m_1 x_1 + m_2 x_2.$$

The agent has preferences over monetary compensation and effort, $a = (a_1, a_2)$, according to a constant-absolute-risk-aversion utility function

$$u(y; a) = -\exp\{-r[y - c(a)]\}, \quad \text{where } c(a) = a_1^2 + 2\kappa a_1 a_2 + a_2^2,$$

where r is risk aversion and the parameter $\kappa \in [-1, 1]$ measures the degree of substitutability between a_1 and a_2 in the agent's disutility-of-effort function; $\kappa > 0$ means that the two tasks are substitutes and compete for effort in the sense that the marginal cost of a_1 is increasing in a_2 and vice versa. We will take the case $\kappa > 0$ as the *main case*; the complements case ($\kappa < 0$) gives the effort-extraction problem a “free-lunch flavor” that seems unnatural in most applications. The agent has reservation payoff u_0 .

The principal's problem We now consider the principal's problem when both m_1 and m_2 are chosen freely. The principal values the two dimensions of realized output at β_1 and β_2 per

⁶Note that this does not directly interfere with the assumption – to be made – that x_1 is non-contractible.

unit,⁷ and her problem is thus

$$\begin{aligned} & \max_{m_1, m_2, F} E [\beta_1 a_1 + \beta_2 a_2 - (F + m_1 x_1 + m_2 x_2)] \\ \text{s.t. } & -E \exp \left\{ -r [F + m_1(a_1 + \varepsilon_1) + m_2(a_2 + \varepsilon_2) - (a_1^2 + 2\kappa a_1 a_2 + a_2^2)] \right\} \geq u_0, \\ & \text{and } a \in \arg \max -E \exp \left\{ -r [F + m_1(a_1 + \varepsilon_1) + m_2(a_2 + \varepsilon_2) - (a_1^2 + 2\kappa a_1 a_2 + a_2^2)] \right\}. \end{aligned}$$

For the case of independent noise, the solution is (in the Appendix we state the solution for a general covariance structure)

$$m_1 = \frac{(1 + 2rv_2) \beta_1 - \kappa \beta_2}{4r^2(1 - \kappa^2)v_1v_2 + 2rv_1 + 2rv_2 + 1}, \quad (1)$$

and

$$m_2 = \frac{-\kappa \beta_1 + (1 + 2rv_1) \beta_2}{4r^2(1 - \kappa^2)v_1v_2 + 2rv_1 + 2rv_2 + 1}; \quad (2)$$

F is determined residually. The original key insight of Holmström and Milgrom (1991) is that there is, in general, an interdependence between the two output dimensions, (a_1, a_2) , in the sense that incentives provided for one component of the result affect inputs and results in both dimensions.⁸ This interdependence is a bit unwieldy even as we rule out stochastic dependence between the noise terms and assume that each output measure depends only on one input. Nevertheless, some general – and for our purposes important – properties can be demonstrated by considering some special cases.

- First, it may be worth noting that if noise (measured by v_i) or risk aversion vanishes, the incentive problem vanishes too, and the solution is $m_1 = \beta_1$ and $m_2 = \beta_2$.
- Second, consider the case where a_2 has no intrinsic value to the principal so that $\beta_2 = 0$. This gives

$$m_1 = \frac{\beta_1(2rv_2 + 1)}{4r^2(1 - \kappa^2)v_1v_2 + 2rv_1 + 2rv_2 + 1}; \quad m_2 = \frac{-2rv_1\kappa\beta_1}{4r^2(1 - \kappa^2)v_1v_2 + 2rv_1 + 2rv_2 + 1},$$

and we see that as long as the two inputs, (a_1, a_2) , are substitutes, the agent is punished for high output in the x_2 -dimension.

- Finally, consider the case where the informativeness about effort of one dimension of output, say 2, grows small, i.e. when $v_2 \rightarrow \infty$. In this case

$$m_1 = \frac{2r(\beta_1 - \beta_2\kappa)}{4r^2(1 - \kappa^2)v_1 + 2r}; \quad m_2 = 0,$$

⁷We could have assumed that $\beta_1 = 1$ with no loss of generality, but this would have made the expressions less transparent.

⁸Essentially the same insight was gained in a somewhat different framework by Baker (1992).

and we see that the incentives provided for x_1 must be used to control both dimensions of effort; from the expression one sees e.g. that if the uninformative dimension is important enough – more precisely if $\beta_1 < \beta_2\kappa$ – output in the other dimension is punished.

The last case highlights the point that there are important circumstances under which weak incentives are desirable for “second-best reasons.”

Manipulation and regime choice The key assumption of this paper is that the principal faces a dichotomous choice of cost-saving incentives, m_1 . Specifically, the principal is assumed to face the choice between giving up direct cost-saving incentives, setting $m_1^0 = 0$, or providing “full cost-saving incentives” – i.e. providing no insurance – setting $m_1^1 = \beta_1$. The origin of this restriction is the assumption that money counts are non-contracible whereas the control right of the revenue stream itself can be transferred. The underpinnings of this assumption have been discussed in the Introduction.

3 Results

This paper aims at addressing two questions: how do incentives and effort depend on the choice between make and buy, and how can we characterize the tradeoffs characterizing the choice of regime. Under the assumptions made about the dichotomous choice of cost-saving incentives, m_1 , the remaining objects – i.e. quality incentives measured by m_2 and the agent’s equilibrium effort (a_1, a_2) – can be solved for conditional on the regime choice and compared across regimes. The simplicity of the model, moreover, enables us to derive the principal’s value function for each case and thus characterize the choice of regime.

Comparing solutions While the comparisons made are simple in principle, the general expression of the model is a bit unwieldy and a general comparison is not particularly conclusive. We therefore proceed in the text by emphasizing comparisons based on the case with independent errors ($\sigma = 0$) and then making some comments about the case with dependent errors assuming that efforts are independent in the agent’s utility ($\kappa = 0$). The most immediate comparison is that between the direct quality incentives in the two regimes. The respective optimal solutions are

$$m_2^0 = \frac{\beta_2 - \beta_1\kappa}{(2r(1 - \kappa^2)v_2 + 1)}, \quad m_2^1 = \frac{\beta_2 - 2r(1 - \kappa^2)\sigma\beta_1}{(2r(1 - \kappa^2)v_2 + 1)}$$

in the case of “make” and “buy,” and the difference is

$$m_2^1 - m_2^0 = \frac{\kappa - 2r(1 - \kappa^2)\sigma}{2r(1 - \kappa^2)v_2 + 1}\beta_1. \quad (3)$$

We see that the difference has the sign of κ for $\sigma = 0$. In particular it is positive – i.e. incentives are stronger under “buy” – in the plausible reference case of the two efforts being substitutes (i.e. competing for attention) and independence between error components. We state a proposition:

PROPOSITION 1. *Suppose that the errors are independent and that the efforts are substitutes, $\kappa > 0$ (complements, $\kappa < 0$). Then the incentives provided for quality are stronger (weaker) when the principal chooses “buy.”*

In the case with $\sigma \neq 0$ and $\kappa = 0$, the difference has the opposite sign of σ ; as we will see and comment on shortly, this pattern is the sole exception from the the pattern that κ and σ have broadly similar effects.

Going on to the equilibrium effort in each regime, the differences are

$$a_1^{1*} - a_1^{0*} = \frac{(2r(v_2 + \kappa\sigma) + 1)\beta_1}{2(2r(1 - \kappa^2)v_2 + 1)}; \quad a_2^{1*} - a_2^{0*} = \frac{-r(\kappa v_2 + \sigma)\beta_1}{2r(1 - \kappa^2)v_2 + 1}. \quad (4)$$

The first difference is guaranteed to be positive whenever quality is measured less precisely than cost.⁹ The second difference is negative when both κ and σ are non-negative and it has the sign of $-\kappa$ for $\sigma = 0$; in particular, it is negative when the efforts are substitutes ($\kappa > 0$) and the errors are independent. We state the following proposition:

PROPOSITION 2. *Equilibrium effort devoted to cost-savings is higher when the principal chooses buy. Assuming that the errors are independent, $\sigma = 0$, and that the efforts are substitutes, $\kappa > 0$ (complements, $\kappa < 0$), equilibrium effort devoted to quality is lower (higher) when the principal chooses “buy.”*

In the case with $\sigma \neq 0$ and $\kappa = 0$, the result is, as a matter of fact, perfectly similar – the difference in quality effort has the same sign as $-\sigma$.¹⁰

Regime choice We now turn to the choice between make and buy. In formal terms we are interested in the comparative statics of the difference between the principal’s value function from choosing “buy” and “make” with respect to the parameters of the model. We have:

⁹This follows from $|\kappa| \leq 1$, the inequality limiting a covariance $|\sigma| \leq \sqrt{v_1 v_2} \leq (v_1 + v_2)/2$, and the assumption $v_1 \leq v_2$; note that the non-contractibility of cost-savings does not reflect a lack of measurability.

¹⁰It is a rather subtle matter to note that while the difference between m_2 ’s in (3) has the sign of $-\sigma$ for $\kappa = 0$ (compared with having the sign of κ for $\sigma = 0$), Proposition 2 shows that κ and σ have similar effects on equilibrium effort. The latter property is more fundamental, and by inspection of the general solution of the “unconstrained” problem with both m_1 and m_2 endogenous one can see that κ and σ have similar qualitative effects on that solution.

PROPOSITION 3. *Suppose that the errors are independent, and that the efforts are substitutes, $\kappa > 0$, (complements, $\kappa < 0$). The effect of an increase in each of the parameters on the attractiveness of “buy” is detailed by the following list:*

- β_2 (the principal’s relative valuation of quality): *negative (positive)*;
- v_1 (the measurement error in cost-savings): *negative*;
- v_2 (the measurement error in quality): *negative (positive)*;
- κ (the degree to which effort spent on quality competes with effort spent on cost savings): *negative*;
- r (the agent’s risk aversion): *negative (ambiguous)*.

In terms of intuition, all parameters would probably be expected to make outsourcing decisions less attractive given the conflict between cost-saving and quality that prevails when $\kappa > 0$. The reversal when $\kappa < 0$ is natural in the case of β_2 (quality is higher under “buy”); for v_2 it reflects the fact that m_2 is higher under “make” in this case which is costlier the higher is v_2 ; the ambiguous impact of risk aversion in this case reflects a combination of a direct effect and an indirect effect similar to that of v_2 .

In terms of implications, recalling that the sign of κ is decisive for the prediction, the key observations are that:

1. the comparative statics with respect to the valuation of quality, β_2 , imply that a higher valuation of quality leads to higher equilibrium quality; and,
2. that the effect of quality being less well measurable, increasing v_2 , is similar to the effect of it being more valuable (increasing β_2).

The case with dependent signals ($\sigma \neq 0$) and separable effort ($\kappa = 0$) gives somewhat less clear-cut, and less clearly interpretable results. The key conclusion with respect to β_2 , however, is perfectly consonant with the above: “buy” grows less attractive with β_2 when $\sigma > 0$, and this is reversed when $\sigma < 0$; that is, conclusion 1. above remains. Conclusion 2. does not generally go through in this case.

4 Conclusions

The main conclusion from our analysis is that outsourcing leads to lower costs, while the effects on quality are likely to be negative, although this depends on a precise condition in terms of

effort-substitution possibilities. While this conclusion is in line with previous work – in particular with Hart, Shleifer and Vishny (1997) – a distinguishing feature of this paper is that we establish this in a contracting framework where the effects are permeated by tangible incentive contracts. Apart from this being “realistic” in many applications, this has the benefit of producing a richer set of empirical implications.¹¹ In particular, in the “main case” singled out in this paper – whose relevance can likely be directly or indirectly established in many contexts – the prediction is that outsourcing will be accompanied by stronger rewards for quality, while still producing lower quality than comparable in-house arrangements.

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¹¹This said, the empirical distinction between the role of quality in our agency framework and the role of quality in pure incomplete contracting models is not razor sharp. The empirical account of prison privatization given by Hart, Shleifer and Vishny (1997, Sec. IV), for example, is broadly consistent both with an incomplete contracting view and a contractual view; in particular, it seems clear from that account that the explicit contractual regulation of quality is more stringent for private prisons, while in many instances quality in the end is lower.

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Appendix

A.1 General solution

Optimal contracts in the basic two-task model The problem – with general covariance matrix and $\text{Cov}(\varepsilon_1, \varepsilon_2) = \sigma$ – can be written

$$\begin{aligned} & \max_{m, F} [(\beta_1 - m_1)a_1 + (\beta_2 - m_2)a_2 - F] \\ \text{s.t. } & -\exp(-r(F + m_1a_1 + m_2a_2 - \frac{r}{2}m_1^2v_1 - \frac{r}{2}m_2^2v_2 - rm_1m_2\sigma - [a_1^2 + 2\kappa a_1a_2 + a_2^2])) \geq u_0 \end{aligned}$$

and optimality for the agent, the first-order conditions for which are

$$m_1 - 2(a_1 + \kappa a_2) = 0; \quad m_2 - 2(\kappa a_1 + a_2) = 0.$$

Maximization by the agent yields

$$a_1^* = \frac{m_1 - \kappa m_2}{2(1 - \kappa^2)}; \quad a_2^* = \frac{m_2 - \kappa m_1}{2(1 - \kappa^2)} \quad (\text{A.1})$$

and the principal’s objective function is (with a_1^* and a_2^* inserted, $\tilde{u} = -\ln(-u_0)/r$, and after substituting the constraint)

$$\begin{aligned} \phi(\beta_1, \beta_2) &= \beta_1 \frac{m_1 - \kappa m_2}{2(1 - \kappa^2)} + \beta_2 \frac{m_2 - \kappa m_1}{2(1 - \kappa^2)} - \frac{r}{2}m_1^2v_1 - \frac{r}{2}m_2^2v_2 - rm_1m_2\sigma \\ &\quad - \frac{1}{4(1 - \kappa^2)^2}[(m_1 - \kappa m_2)^2 + 2\kappa(m_1 - \kappa m_2)(m_2 - \kappa m_1) + (m_2 - \kappa m_1)^2] - \tilde{u}, \end{aligned}$$

simplifying and multiplying by $2(1 - \kappa^2)$, we have

$$\begin{aligned} \phi(\beta_1, \beta_2) &= \beta_1(m_1 - \kappa m_2) + \beta_2(m_2 - \kappa m_1) - r(1 - \kappa^2)m_1^2v_1 - r(1 - \kappa^2)m_2^2v_2 - 2r(1 - \kappa^2)m_1m_2\sigma \\ &\quad - \frac{1}{1 - \kappa^2}[\frac{1}{2}(m_1 - \kappa m_2)^2 + \kappa(m_1 - \kappa m_2)(m_2 - \kappa m_1) + \frac{1}{2}(m_2 - \kappa m_1)^2] - \hat{\tilde{u}}. \end{aligned}$$

The first-order conditions w.r.t. (m_1, m_2) are:

$$\begin{aligned} & \beta_1 - \beta_2 \kappa - 2r(1 - \kappa^2)(v_1 m_1 + \sigma m_2) - \\ & \frac{1}{1 - \kappa^2} [(m_1 - \kappa m_2) + \kappa [(m_2 - \kappa m_1) - \kappa(m_1 - \kappa m_2) - \kappa(m_2 - \kappa m_1)]] = 0, \\ & \beta_2 - \beta_1 \kappa - 2r(1 - \kappa^2)(v_2 m_2 + \sigma m_1) - \\ & \frac{1}{1 - \kappa^2} [-\kappa(m_1 - \kappa m_2) + \kappa [(m_1 - \kappa m_2) - \kappa(m_2 - \kappa m_1) + (m_2 - \kappa m_1)]] = 0. \end{aligned}$$

Simplifying,

$$\begin{aligned} \beta_1 - \beta_2 \kappa &= \left(2r(1 - \kappa^2)v_1 + \frac{1 - \kappa^2}{1 - \kappa^2} \right) m_1 + \left(\frac{\kappa^3 - \kappa}{1 - \kappa^2} + 2r(1 - \kappa^2)\sigma \right) m_2, \\ \beta_2 - \beta_1 \kappa &= \left(\frac{\kappa^3 - \kappa}{1 - \kappa^2} + 2r(1 - \kappa^2)\sigma \right) m_1 + \left(2r(1 - \kappa^2)v_2 + \frac{1 - \kappa^2}{1 - \kappa^2} \right) m_2; \end{aligned}$$

simplifying further

$$\beta_1 - \beta_2 \kappa = (2r(1 - \kappa^2)v_1 + 1) m_1 + (2r(1 - \kappa^2)\sigma - \kappa) m_2, \quad (\text{A.2})$$

$$\beta_2 - \beta_1 \kappa = (2r(1 - \kappa^2)\sigma - \kappa) m_1 + (2r(1 - \kappa^2)v_2 + 1) m_2. \quad (\text{A.3})$$

The full system can be written

$$\begin{pmatrix} 2r(1 - \kappa^2)v_1 + 1 & 2r(1 - \kappa^2)\sigma - \kappa \\ 2r(1 - \kappa^2)\sigma - \kappa & 2r(1 - \kappa^2)v_2 + 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \beta_1 - \beta_2 \kappa \\ \beta_2 - \beta_1 \kappa \end{pmatrix},$$

or,

$$\begin{pmatrix} 2r(1 - \kappa^2)v_1 + 1 & 2r(1 - \kappa^2)\sigma - \kappa \\ 2r(1 - \kappa^2)\sigma - \kappa & 2r(1 - \kappa^2)v_2 + 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 & -\kappa \\ -\kappa & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

or, invertring the RHS matrix,

$$\frac{1}{1 - \kappa^2} \begin{pmatrix} 1 & \kappa \\ \kappa & 1 \end{pmatrix} \begin{pmatrix} 2r(1 - \kappa^2)v_1 + 1 & 2r(1 - \kappa^2)\sigma - \kappa \\ 2r(1 - \kappa^2)\sigma - \kappa & 2r(1 - \kappa^2)v_2 + 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

The LHS can be written

$$\frac{1}{1 - \kappa^2} \begin{pmatrix} 1 & \kappa \\ \kappa & 1 \end{pmatrix} \begin{pmatrix} 2r(1 - \kappa^2)v_1 + 1 & 2r(1 - \kappa^2)\sigma - \kappa \\ 2r(1 - \kappa^2)\sigma - \kappa & 2r(1 - \kappa^2)v_2 + 1 \end{pmatrix} = \begin{pmatrix} 2r(v_1 + \kappa\sigma) + 1 & 2r(\kappa v_2 + \sigma) \\ 2r(\kappa v_1 + \sigma) & 2r(v_2 + \kappa\sigma) + 1 \end{pmatrix},$$

and the determinant is

$$D = 4r^2 (v_1 v_2 + v_1 \sigma \kappa + v_2 \sigma \kappa + \sigma^2 \kappa^2) + 2r(v_1 + \kappa\sigma) + 2r(v_2 + \kappa\sigma) + 1 - 4r^2 (\kappa^2 v_1 v_2 + v_1 \sigma \kappa + v_2 \sigma \kappa + \sigma^2),$$

or

$$D = 4r^2 (1 - \kappa^2) (v_1 v_2 - \sigma^2) + 2r(v_1 + \kappa\sigma) + 2r(v_2 + \kappa\sigma) + 1.$$

The solution in terms of m is thus:

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} 2r(v_2 + \kappa\sigma) + 1 & -2r(\kappa v_2 + \sigma) \\ -2r(\kappa v_1 + \sigma) & 2r(v_1 + \kappa\sigma) + 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

or

$$m_1 = \frac{(2r(v_2 + \kappa\sigma) + 1)\beta_1 - 2r(\kappa v_2 + \sigma)\beta_2}{D},$$

and

$$m_2 = \frac{-2r(\kappa v_1 + \sigma)\beta_1 + (2r(v_1 + \kappa\sigma) + 1)\beta_2}{D}.$$

A.2 Characterizing the regimes

Optimal contracts for each regime Now, from the first-order condition with respect to m_2 , we get the optimal m_2 conditional on $\bar{m}_1 \in \{0, \beta_1\}$,

$$m_2(\bar{m}_1) = \frac{\beta_2 - \beta_1\kappa - (2r(1 - \kappa^2)\sigma - \kappa)\bar{m}_1}{(2r(1 - \kappa^2)v_2 + 1)},$$

specifically,

$$m_2^0 = \frac{\beta_2 - \beta_1\kappa}{(2r(1 - \kappa^2)v_2 + 1)}, \quad m_2^1 = \frac{\beta_2 - 2r(1 - \kappa^2)\sigma\beta_1}{(2r(1 - \kappa^2)v_2 + 1)},$$

and we see that the difference is

$$m_2^1 - m_2^0 = \frac{(\kappa - 2r(1 - \kappa^2)\sigma)\beta_1}{(2r(1 - \kappa^2)v_2 + 1)}$$

which has the sign of κ for $\sigma = 0$, and the opposite sign of σ for $\kappa = 0$.

Equilibrium effort: We can now simply calculate equilibrium effort by plugging in the m 's in (A.1); for the case of $\bar{m}_1 = m_1^0 = 0$ it is

$$a_1^{0*} = \frac{-\kappa(\beta_2 - \beta_1\kappa)}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)}; \quad a_2^{0*} = \frac{\beta_2 - \beta_1\kappa}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)},$$

and for the case of $\bar{m}_1 = m_1^1 = \beta_1$ it is

$$a_1^{1*} = \frac{1}{2(1 - \kappa^2)} \left(\beta_1 - \frac{\kappa(\beta_2 - 2r(1 - \kappa^2)\sigma\beta_1)}{(2r(1 - \kappa^2)v_2 + 1)} \right); \quad a_2^{1*} = \frac{1}{2(1 - \kappa^2)} \left(\frac{\beta_2 - 2r(1 - \kappa^2)\sigma\beta_1}{(2r(1 - \kappa^2)v_2 + 1)} - \kappa\beta_1 \right),$$

or, developing,

$$\begin{aligned} a_1^{1*} &= \frac{(2r(1 - \kappa^2)v_2 + 1)\beta_1 - \kappa(\beta_2 - 2r(1 - \kappa^2)\sigma\beta_1)}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)} = \frac{(2r(1 - \kappa^2)(v_2 + \kappa\sigma) + 1)\beta_1 - \kappa\beta_2}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)}; \\ a_2^{1*} &= \frac{\beta_2 - 2r(1 - \kappa^2)\sigma\beta_1 - \kappa\beta_1(2r(1 - \kappa^2)v_2 + 1)}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)} = \frac{-(2r(1 - \kappa^2)(\kappa v_2 + \sigma) + \kappa)\beta_1 + \beta_2}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)}. \end{aligned}$$

The differences are

$$a_1^{1*} - a_1^{0*} = \frac{(2r(1 - \kappa^2)(v_2 + \kappa\sigma) + 1)\beta_1 - \kappa\beta_2}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)} - \frac{-\kappa(\beta_2 - \beta_1\kappa)}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)},$$

$$a_1^{1*} - a_1^{0*} = \frac{(2r(1 - \kappa^2)(v_2 + \kappa\sigma) + (1 - \kappa^2))\beta_1}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)} = \frac{(2r(v_2 + \kappa\sigma) + 1)\beta_1}{2(2r(1 - \kappa^2)v_2 + 1)},$$

and

$$a_2^{1*} - a_2^{0*} = \frac{-(2r(1 - \kappa^2)(\kappa v_2 + \sigma) + \kappa)\beta_1 + \beta_2}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)} - \frac{\beta_2 - \beta_1\kappa}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)},$$

or, simplifying,

$$a_2^{1*} - a_2^{0*} = \frac{-2r(1 - \kappa^2)(\kappa v_2 + \sigma)\beta_1}{2(1 - \kappa^2)(2r(1 - \kappa^2)v_2 + 1)} = \frac{-r(\kappa v_2 + \sigma)\beta_1}{(2r(1 - \kappa^2)v_2 + 1)}.$$

In short,

$$a_1^{1*} - a_1^{0*} = \frac{(2r(v_2 + \kappa\sigma) + 1)\beta_1}{2(2r(1 - \kappa^2)v_2 + 1)}; \quad a_2^{1*} - a_2^{0*} = \frac{-r(\kappa v_2 + \sigma)\beta_1}{(2r(1 - \kappa^2)v_2 + 1)}.$$

The specializations where either κ or σ are zero are useful:

- $\sigma = 0$:

$$a_1^{1*} - a_1^{0*} = \frac{(2rv_2 + 1)\beta_1}{2(2r(1 - \kappa^2)v_2 + 1)} > 0$$

$$a_2^{1*} - a_2^{0*} = \frac{-r\kappa v_2\beta_1}{(2r(1 - \kappa^2)v_2 + 1)} < 0^*$$

where * indicates reversal when $\kappa < 0$;

- $\kappa = 0$:

$$a_1^{1*} - a_1^{0*} = \frac{\beta_1}{2} > 0$$

$$a_2^{1*} - a_2^{0*} = \frac{-r\sigma\beta_1}{2rv_2 + 1} < 0^*$$

where * indicates reversal when $\sigma < 0$.

A.3 Regime choice

In order to sort out the forces at work we make explicit comparisons for the two cases with $\kappa \neq 0$, $\sigma = 0$ and $\sigma \neq 0$, $\kappa = 0$; the first is the more interesting one, and a general comparison becomes too hairy to be useful.

Comparing value functions, $\kappa \neq 0, \sigma = 0$ The value function is

$$\phi(\beta_1, \beta_2) = \beta_1 \frac{m_1 - \kappa m_2}{2(1 - \kappa^2)} + \beta_2 \frac{m_2 - \kappa m_1}{2(1 - \kappa^2)} - \frac{r}{2} m_1^2 v_1 - \frac{r}{2} m_2^2 v_2 - r m_1 m_2 \sigma$$

$$- \frac{1}{4(1 - \kappa^2)^2} [(m_1 - \kappa m_2)^2 + 2\kappa(m_1 - \kappa m_2)(m_2 - \kappa m_1) + (m_2 - \kappa m_1)^2] - \tilde{u};$$

without loss of generality (for the comparison as well as the solution, by the way) we assume $\tilde{u} = 0$. We have

$$m_2 = \frac{\beta_2 - \beta_1\kappa - (2r(1 - \kappa^2)\sigma - \kappa)\bar{m}_1}{(2r(1 - \kappa^2)v_2 + 1)},$$

and the relevant comparison is between $\overline{m}_1 = 0$ and $\overline{m}_1 = \beta_1$ with respectively

$$m_2^0 = \frac{\beta_2 - \beta_1 \kappa}{(2r(1 - \kappa^2)v_2 + 1)} \text{ and } m_2^1 = \frac{\beta_2}{(2r(1 - \kappa^2)v_2 + 1)}.$$

The case $\overline{m}_1 = 0$ The value function can be expressed

$$\phi^0 = \beta_1 \frac{-\kappa m_2}{2(1 - \kappa^2)} + \beta_2 \frac{m_2}{2(1 - \kappa^2)} - \frac{r}{2} m_2^2 v_2 - \frac{1}{4(1 - \kappa^2)^2} [(-\kappa m_2)^2 + 2\kappa(-\kappa m_2)(m_2) + (m_2)^2],$$

and multiplying by the denominator,

$$2(1 - \kappa^2) \phi^0 = \beta_1 (-\kappa m_2) + \beta_2 m_2 - \frac{r}{2} 2(1 - \kappa^2) m_2^2 v_2 - \frac{m_2^2}{2},$$

or

$$2(1 - \kappa^2) \phi^0 = (\beta_2 - \kappa \beta_1) m_2 - \frac{1}{2} (2r(1 - \kappa^2) v_2 + 1) m_2^2.$$

Inserting m_2^0 , we get

$$2(1 - \kappa^2) \phi^0 = (\beta_2 - \kappa \beta_1) \frac{\beta_2 - \beta_1 \kappa}{(2r(1 - \kappa^2)v_2 + 1)} - \frac{1}{2} (2r(1 - \kappa^2) v_2 + 1) \left(\frac{\beta_2 - \beta_1 \kappa}{(2r(1 - \kappa^2)v_2 + 1)} \right)^2,$$

or

$$(2r(1 - \kappa^2)v_2 + 1) 2(1 - \kappa^2) \phi^0 = \frac{1}{2} (\beta_2 - \beta_1 \kappa)^2,$$

or

$$\phi^0 = \frac{1}{(2r(1 - \kappa^2)v_2 + 1) 2(1 - \kappa^2)} \frac{1}{2} (\beta_2 - \beta_1 \kappa)^2.$$

The case $\overline{m}_1 = \beta_1$ The value function can be expressed

$$\begin{aligned} \phi^1 &= \beta_1 \frac{\beta_1 - \kappa m_2}{2(1 - \kappa^2)} + \beta_2 \frac{m_2 - \kappa \beta_1}{2(1 - \kappa^2)} - \frac{r}{2} \beta_1^2 v_1 - \frac{r}{2} m_2^2 v_2 - \\ &\quad \frac{1}{4(1 - \kappa^2)^2} [(\beta_1 - \kappa m_2)^2 + 2\kappa(\beta_1 - \kappa m_2)(m_2 - \kappa \beta_1) + (m_2 - \kappa \beta_1)^2], \end{aligned}$$

or, multiplying,

$$\begin{aligned} 2(1 - \kappa^2) \phi^1 &= \beta_1 (\beta_1 - \kappa m_2) + \beta_2 (m_2 - \kappa \beta_1) - \frac{r}{2} 2(1 - \kappa^2) \beta_1^2 v_1 - \frac{r}{2} 2(1 - \kappa^2) m_2^2 v_2 - \\ &\quad \frac{1}{2(1 - \kappa^2)} [(\beta_1 - \kappa m_2)^2 + 2\kappa(\beta_1 - \kappa m_2)(m_2 - \kappa \beta_1) + (m_2 - \kappa \beta_1)^2]; \end{aligned}$$

inserting m_2^1 , we get

$$\begin{aligned} 2(1 - \kappa^2) \phi^1 &= \beta_1 \left(\frac{\beta_1 (2r(1 - \kappa^2) v_2 + 1) - \kappa \beta_2}{(2r(1 - \kappa^2) v_2 + 1)} \right) + \beta_2 \left(\frac{\beta_2 - \kappa \beta_1 (2r(1 - \kappa^2) v_2 + 1)}{(2r(1 - \kappa^2) v_2 + 1)} \right) - \\ &\quad \frac{r}{2} 2(1 - \kappa^2) \beta_1^2 v_1 - \frac{r}{2} 2(1 - \kappa^2) \left(\frac{\beta_2}{(2r(1 - \kappa^2) v_2 + 1)} \right)^2 v_2 - \\ &\quad \frac{1}{2(1 - \kappa^2)} \left[\left(\frac{\beta_1 (2r(1 - \kappa^2) v_2 + 1) - \kappa \beta_2}{(2r(1 - \kappa^2) v_2 + 1)} \right)^2 + \right. \\ &\quad \left. 2\kappa \left(\frac{\beta_1 (2r(1 - \kappa^2) v_2 + 1) - \kappa \beta_2}{(2r(1 - \kappa^2) v_2 + 1)} \right) \left(\frac{\beta_2 - \kappa \beta_1 (2r(1 - \kappa^2) v_2 + 1)}{(2r(1 - \kappa^2) v_2 + 1)} \right) + \left(\frac{\beta_2 - \kappa \beta_1 (2r(1 - \kappa^2) v_2 + 1)}{(2r(1 - \kappa^2) v_2 + 1)} \right)^2 \right], \end{aligned}$$

and letting $\Omega = 2r(1 - \kappa^2)v_2 + 1$, we get

$$2(1 - \kappa^2)\Omega\phi^1 = \beta_1(\beta_1\Omega - \kappa\beta_2) + \beta_2(\beta_2 - \kappa\beta_1\Omega) - r(1 - \kappa^2)\beta_1^2\Omega v_1 - \frac{1}{\Omega}r(1 - \kappa^2)\beta_2^2v_2 \\ - \frac{1}{2(1 - \kappa^2)\Omega} \left[((\beta_1\Omega - \kappa\beta_2))^2 + 2\kappa((\beta_1\Omega - \kappa\beta_2))(\beta_2 - \kappa\beta_1\Omega) + (\beta_2 - \kappa\beta_1\Omega)^2 \right],$$

or,

$$2(1 - \kappa^2)\Omega\phi^1 = \beta_2^2 - \kappa\beta_1\beta_2(1 + \Omega) + \beta_1^2\Omega - r(1 - \kappa^2)\beta_1^2v_1 \\ - \frac{1}{2(1 - \kappa^2)\Omega} \left[(\beta_1\Omega - \kappa\beta_2)^2 + 2\kappa(\beta_1\Omega - \kappa\beta_2)(\beta_2 - \kappa\beta_1\Omega) + (\beta_2 - \kappa\beta_1\Omega)^2 + 2r(1 - \kappa^2)^2\beta_2^2v_2 \right].$$

As an intermediate calculation, we simplify the bracketed expression:

$$(\beta_1\Omega - \kappa\beta_2)^2 + 2\kappa(\beta_1\Omega - \kappa\beta_2)(\beta_2 - \kappa\beta_1\Omega) + (\beta_2 - \kappa\beta_1\Omega)^2, \\ \beta_1^2\Omega^2 - 2\kappa\beta_1\beta_2\Omega + \kappa^2\beta_2^2 - 2\kappa^2\beta_1^2\Omega^2 + 2\kappa(1 + \kappa^2)\beta_1\beta_2\Omega - 2\kappa^2\beta_2^2 + \kappa^2\beta_1^2\Omega^2 - 2\kappa\beta_1\beta_2\Omega + \beta_2^2, \\ (1 - \kappa^2)\beta_1^2\Omega^2 + (1 - \kappa^2)\beta_2^2 - 2\kappa(1 - \kappa^2)\beta_1\beta_2\Omega.$$

Let us next resume calculating the value function after the intermediate calculation,

$$2(1 - \kappa^2)\Omega\phi^1 = \beta_2^2 - \kappa\beta_1\beta_2(1 + \Omega) + \beta_1^2\Omega - r(1 - \kappa^2)\beta_1^2\Omega v_1 \\ - \frac{1}{2(1 - \kappa^2)\Omega} \left[(1 - \kappa^2)\beta_1^2\Omega^2 + (1 - \kappa^2)\beta_2^2 - 2\kappa(1 - \kappa^2)\beta_1\beta_2\Omega + 2r(1 - \kappa^2)^2\beta_2^2v_2 \right],$$

or,

$$2(1 - \kappa^2)\Omega\phi^1 = \beta_2^2 - \kappa\beta_1\beta_2(1 + \Omega) + \beta_1^2\Omega - r(1 - \kappa^2)\beta_1^2\Omega v_1 \\ - \frac{1}{2\Omega} [\beta_1^2\Omega^2 + \beta_2^2 - 2\kappa\beta_1\beta_2\Omega + 2r(1 - \kappa^2)\beta_2^2v_2],$$

or,

$$2(1 - \kappa^2)\Omega\phi^1 = \beta_2^2 - \kappa\beta_1\beta_2(1 + \Omega) + \beta_1^2\Omega - r(1 - \kappa^2)\beta_1^2\Omega v_1 \\ - \frac{1}{2\Omega} [\beta_1^2\Omega^2 - 2\kappa\beta_1\beta_2\Omega + \beta_2^2\Omega],$$

or,

$$2(1 - \kappa^2)\Omega\phi^1 = \beta_2^2 - \kappa\beta_1\beta_2(1 + \Omega) + \beta_1^2\Omega - r(1 - \kappa^2)\beta_1^2\Omega v_1 - \frac{1}{2} [\beta_1^2\Omega - 2\kappa\beta_1\beta_2 + \beta_2^2],$$

or,

$$2(1 - \kappa^2)\Omega\phi^1 = \frac{1}{2}\beta_2^2 + \beta_1^2\Omega \left(1 - r(1 - \kappa^2)v_1 - \frac{1}{2} \right) - \kappa\beta_1\beta_2\Omega,$$

or,

$$2(1 - \kappa^2)\Omega\phi^1 = \frac{1}{2}\beta_2^2 + \frac{1}{2}\beta_1^2\Omega(1 - 2r(1 - \kappa^2)v_1) - \kappa\beta_1\beta_2\Omega.$$

Comparison To compare, we recapitulate the value functions,

$$2(1 - \kappa^2) \Omega \phi^0 = \frac{1}{2} (\beta_2 - \beta_1 \kappa)^2 = \frac{1}{2} (\beta_2^2 + \kappa^2 \beta_1^2 - 2\kappa \beta_1 \beta_2),$$

and

$$2(1 - \kappa^2) \Omega \phi^1 = \frac{1}{2} \beta_2^2 + \frac{1}{2} \beta_1^2 \Omega (1 - 2r(1 - \kappa^2) v_1) - \kappa \beta_1 \beta_2 \Omega.$$

The difference (the prime denoting that we still have a multiplying factor for simplification) is

$$\Delta' = \frac{1}{2} \beta_1^2 \Omega (1 - \kappa^2 - 2r(1 - \kappa^2) v_1) + \kappa \beta_1 \beta_2 (1 - \Omega),$$

or,

$$\Delta' = \frac{1}{2} \beta_1^2 \Omega (1 - \kappa^2 - 2r(1 - \kappa^2) v_1) - 2r(1 - \kappa^2) v_2 \kappa \beta_1 \beta_2,$$

or,

$$2(1 - \kappa^2) \Omega \Delta \phi = 2(1 - \kappa^2) \left[\frac{1}{2} \beta_1^2 \Omega \frac{(1 - 2rv_1)}{2} - rv_2 \kappa \beta_1 \beta_2 \right].$$

Returning to the original value function, we have

$$\Delta \phi = \left[\frac{1}{2} \beta_1^2 \frac{(1 - 2rv_1)}{2} - \frac{rv_2 \kappa \beta_1 \beta_2}{\Omega} \right],$$

or with $\beta_1 = 1$, $\Omega = 2r(1 - \kappa^2) v_2 + 1$,

$$\Delta \phi = \left[\frac{(1 - 2rv_1)}{4} - \frac{rv_2 \kappa \beta_2}{2r(1 - \kappa^2) v_2 + 1} \right],$$

or,

$$\Delta \phi = \left[\frac{1}{4} - \frac{rv_1}{2} - \frac{\kappa}{2(1 - \kappa^2) + 1/rv_2} \cdot \beta_2 \right].$$

The comparative statics can be read straightforwardly from this; the effects of an increase in the respective parameters are stated below, with * meaning that the direction is switched if $\kappa < 0$:

- β_2 : negative*,
- v_1 : negative,
- κ : negative,
- v_2 : negative*,
- r : ambiguous (it is negative, for $\kappa > 0$, and ambiguous if $\kappa < 0$).

Comparing value functions, $\sigma \neq 0, \kappa = 0$ The value function is (assuming w.l.g. that $\tilde{u} = 0$)

$$\begin{aligned} \phi(\beta_1\beta_2) &= \beta_1 \frac{m_1 - \kappa m_2}{2(1 - \kappa^2)} + \beta_2 \frac{m_2 - \kappa m_1}{2(1 - \kappa^2)} - \frac{r}{2} m_1^2 v_1 - \frac{r}{2} m_2^2 v_2 - r m_1 m_2 \sigma \\ &\quad - \frac{1}{4(1 - \kappa^2)^2} [(m_1 - \kappa m_2)^2 + 2\kappa(m_1 - \kappa m_2)(m_2 - \kappa m_1) + (m_2 - \kappa m_1)^2], \end{aligned}$$

or

$$\phi(\beta_1\beta_2) = \beta_1 \frac{m_1}{2} + \beta_2 \frac{m_2}{2} - \frac{r}{2} m_1^2 v_1 - \frac{r}{2} m_2^2 v_2 - r m_1 m_2 \sigma - \frac{1}{4} [m_1^2 + m_2^2].$$

We have

$$m_2 = \frac{\beta_2 - \beta_1 \kappa - (2r(1 - \kappa^2)\sigma - \kappa) \bar{m}_1}{(2r(1 - \kappa^2)v_2 + 1)},$$

and the obvious comparison is between $\bar{m}_1 = 0$ and $\bar{m}_1 = \beta_1$ with

$$m_2^0 = \frac{\beta_2}{(2rv_2 + 1)} \text{ and } m_2^1 = \frac{\beta_2 - \beta_1 2r\sigma}{(2rv_2 + 1)}.$$

The case $\bar{m}_1 = 0$ The value function can be expressed

$$\phi^0 = \beta_2 \frac{m_2}{2} - \frac{r}{2} m_2^2 v_2 - \frac{1}{4} [(m_2)^2],$$

or

$$2\phi^0 = \beta_2 m_2 - r m_2^2 v_2 - \frac{m_2^2}{2},$$

or

$$2\phi^0 = \beta_2 m_2 - \frac{1}{2} (2rv_2 + 1) m_2^2;$$

inserting m_2^0 ,

$$2\phi^0 = \beta_2 \frac{\beta_2}{(2rv_2 + 1)} - \frac{1}{2} (2rv_2 + 1) \left(\frac{\beta_2}{(2rv_2 + 1)} \right)^2,$$

or

$$(2rv_2 + 1) 2\phi^0 = \frac{1}{2} \beta_2^2.$$

The case $\bar{m}_1 = \beta_1$ The value function can be expressed

$$\phi^1 = \beta_1 \frac{\beta_1}{2} + \beta_2 \frac{m_2}{2} - \frac{r}{2} \beta_1^2 v_1 - \frac{r}{2} m_2^2 v_2 - r \beta_1 m_2 \sigma - \frac{1}{4} [\beta_1^2 + m_2^2],$$

or

$$2\phi^1 = \beta_1^2 + \beta_2 m_2 - r \beta_1^2 v_1 - r m_2^2 v_2 - r 2\beta_1 m_2 \sigma - \frac{1}{2} [\beta_1^2 + m_2^2];$$

inserting m_2^1 , we get

$$2\phi^1 = \beta_1^2 + \beta_2 \left(\frac{\beta_2 - \beta_1 2r\sigma}{(2rv_2 + 1)} \right) - r\beta_1^2 v_1 - r \left(\frac{\beta_2 - \beta_1 2r\sigma}{(2rv_2 + 1)} \right)^2 v_2 - 2r\beta_1 \left(\frac{\beta_2 - \beta_1 2r\sigma}{(2rv_2 + 1)} \right) \sigma - \frac{1}{2} \left[\beta_1^2 + \left(\frac{\beta_2 - \beta_1 2r\sigma}{(2rv_2 + 1)} \right)^2 \right],$$

or, letting $\Omega = 2rv_2 + 1$,

$$2\Omega\phi^1 = \beta_1^2\Omega + \beta_2(\beta_2 - \beta_1 2r\sigma) - r\beta_1^2\Omega v_1 - \frac{1}{\Omega}r(\beta_2 - \beta_1 2r\sigma)^2 v_2 - 2r\beta_1(\beta_2 - \beta_1 2r\sigma)\sigma - \frac{1}{2\Omega} \left[\beta_1^2\Omega^2 + (\beta_2 - \beta_1 2r\sigma)^2 \right],$$

or

$$2\Omega\phi^1 = \beta_1^2\Omega + \beta_2^2 - 2r\beta_1\beta_2\sigma - r\beta_1^2\Omega v_1 - \frac{1}{\Omega}r(\beta_2^2 + 4\beta_1^2 r^2 \sigma^2 - 4r\beta_1\beta_2\sigma) v_2 + 4\beta_1^2 r^2 \sigma^2 - 2r\beta_1\beta_2\sigma - \frac{1}{2\Omega} \left[\beta_1^2\Omega^2 + (\beta_2^2 + 4\beta_1^2 r^2 \sigma^2 - 4r\beta_1\beta_2\sigma) \right],$$

or

$$2\Omega\phi^1 = \beta_1^2\Omega + \beta_2^2 - 2r\beta_1\beta_2\sigma - r\beta_1^2\Omega v_1 + 4\beta_1^2 r^2 \sigma^2 - 2r\beta_1\beta_2\sigma - \frac{1}{2\Omega} \left[\beta_1^2\Omega^2 + (\beta_2^2 + 4\beta_1^2 r^2 \sigma^2 - 4r\beta_1\beta_2\sigma) + 2r(\beta_2^2 + 4\beta_1^2 r^2 \sigma^2 - 4r\beta_1\beta_2\sigma) v_2 \right],$$

or,

$$2\Omega\phi^1 = \beta_1^2\Omega + \beta_2^2 - 2r\beta_1\beta_2\sigma - r\beta_1^2\Omega v_1 + 4\beta_1^2 r^2 \sigma^2 - 2r\beta_1\beta_2\sigma - \frac{1}{2\Omega} \left[\beta_1^2\Omega^2 + \beta_2^2\Omega + 4\beta_1^2 r^2 \sigma^2\Omega - 4r\beta_1\beta_2\sigma\Omega \right],$$

or

$$2\Omega\phi^1 = \frac{1}{2}\beta_1^2\Omega + \frac{1}{2}\beta_2^2 - 2r\beta_1\beta_2\sigma - r\beta_1^2\Omega v_1 + 2\beta_1^2 r^2 \sigma^2 - 2r\beta_1\beta_2\sigma + 2r\beta_1\beta_2\sigma,$$

or

$$2\Omega\phi^1 = \frac{1}{2}\beta_1^2 (\Omega(1 - 2rv_1) + 4r^2\sigma^2) + \frac{1}{2}\beta_2^2 - 2r\beta_1\beta_2\sigma.$$

Comparison We first consider the difference between the re-normalized value functions, recalling

$$2\Omega\phi^0 = \frac{1}{2}\beta_2^2,$$

the difference is

$$\Delta' = \frac{1}{2}\beta_1^2 (\Omega(1 - 2rv_1) + 4r^2\sigma^2) - 2r\beta_1\beta_2\sigma;$$

dividing by 2Ω and setting $\beta_1 = 1$,

$$\Delta = \frac{1}{2} \left(\frac{1}{2} - rv_1 + \frac{2r^2\sigma^2}{\Omega} \right) - \frac{r\beta_2\sigma}{\Omega},$$

or

$$\Delta = \frac{1}{2} \left(\frac{1}{2} - rv_1 + \frac{2r^2\sigma^2}{2rv_2 + 1} \right) - \frac{r\beta_2\sigma}{2rv_2 + 1},$$

or

$$\Delta = \frac{1}{2} \left(\frac{1}{2} - rv_1 \right) + \frac{r\sigma}{2rv_2 + 1} (r\sigma - \beta_2).$$

Noting that

$$\frac{\partial \Delta}{\partial \sigma} = \frac{1}{2rv_2 + 1} r (2r\sigma - \beta_2),$$

we have the following signs:

- β_2 : negative*
- v_1 : negative
- σ : negative (for $\sigma < \beta_2/2r$)
- v_2 : positive (for $\sigma < \beta_2/r$)
- r : ambiguous.