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A Minimum Distance Analysis of a Certain Class of Two Dimensional ISI Channels

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***Abstract*— In this paper we perform a minimum distance analysis of a class of two dimensional intersymbol interference channels. In particular, some important cases of multitrack multihead magnetic recording systems fall into the studied class. Previously, Soljanin and Georghiades have studied the same problem as we do. The results derived in this paper are more conclusive and they improve upon theirs. The fundamental proof technique that we will use is to transform the channel into an equivalent minimum phase channel.**

I. Introduction

In this paper we perform a minimum distance analysis of a certain class of two dimensional intersymbol interference (ISI) channels. The class of channels is broad and includes non-linear as well as linear channels. However the most important special case is two dimensional linear channels for magnetic recording systems. For that special case, our analysis significantly extends the state of the art.

Consider the N information symbol sequences $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$, where $\boldsymbol{a}_k = [a_{k0}, a_{k1}, \ldots]$ for $1 \leq k \leq N$. Each symbol belongs to a modulation alphabet $\mathcal{M}$. Let $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N]'$, where $(\cdot)'$ is the transpose operator. These N information symbol sequences generate N signals according to some signal generation rule F:

$$\boldsymbol{s}_k = F(\boldsymbol{a}_k), \quad 1 \leq k \leq N. \tag{1}$$

For the ease of description, $F(\boldsymbol{a}_k)$ is assumed to be a vector over the reals throughout the paper, but it can be easily extended to the other forms. Note that this rule can be due to the channel, or intentionally designed in the transmitter, or a combination thereof. Upon the rule F we impose no requirements; e.g., we can allow non-linear F.

For the alphabet $\mathcal{M}$ of interest an important parameter of any F is its *single-track minimum distance* d_0^2:

$$d_0^2 \triangleq \inf_{\boldsymbol{a}, \boldsymbol{e} \neq \boldsymbol{0}} \|F(\boldsymbol{a}) - F(\boldsymbol{a}+\boldsymbol{e})\|^2,$$

where $\boldsymbol{e}$ is any error event such that $\boldsymbol{a}+\boldsymbol{e}$ is a valid information sequence and the Euclidean norm is taken, i.e., $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}\boldsymbol{x}'}$.

The signals at the output of the channel are given by

$$\boldsymbol{y}_k = \alpha \boldsymbol{s}_{k-1} + \boldsymbol{s}_k + \alpha \boldsymbol{s}_{k+1} + \boldsymbol{w}_k, \quad 0 \leq k \leq N+1 \tag{2}$$

where $0 \leq \alpha \leq 1/2$, $\boldsymbol{w}_k$ are vectors of white Gaussian noise and $\boldsymbol{s}_k \triangleq \boldsymbol{0}$ for $k \leq 0$ and $k \geq N+1$. The presence of signals $\boldsymbol{s}_{k-1}$ and $\boldsymbol{s}_{k+1}$ in the signal $\boldsymbol{y}_k$ is commonly referred to as interchannel interference or, in the magnetic literature, as *intertrack interference* (ITI) which will be the notation in this paper. Note that (1) and (2) describe a two dimensional interference channel. In (2), α is a weighting factor of the ITI that represents the influence of signals $\boldsymbol{s}_{k-1}$ and $\boldsymbol{s}_{k+1}$ on $\boldsymbol{s}_k$. Actually, (2) is a simplified model because it assumes that both $\boldsymbol{s}_{k-1}$ and $\boldsymbol{s}_{k+1}$ influence $\boldsymbol{s}_k$ with the same amount. In the case of a linear F, the orders of (1) and (2) can be interchanged[1] and the interference is said to be separable. The z-transform of the ITI response is $t(z) \triangleq \alpha + z^{-1} + \alpha z^{-2}$.

Note that there are $N+2$ output signals for N input signals. Due to this, (2) is referred to as the *non-square channel model*.

By defining the $(N+2) \times N$ matrix

$$\boldsymbol{T} \triangleq \begin{bmatrix} \alpha & 0 & \cdots & \cdots & 0 \\ 1 & \alpha & 0 & \cdots & \vdots \\ \alpha & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \alpha \\ \vdots & \cdots & 0 & \alpha & 1 \\ 0 & \cdots & \cdots & 0 & \alpha \end{bmatrix}, \tag{3}$$

we can express (2) as

$$\begin{bmatrix} \boldsymbol{y}_0 \\ \vdots \\ \boldsymbol{y}_{N+1} \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \boldsymbol{s}_1 \\ \vdots \\ \boldsymbol{s}_N \end{bmatrix} + \begin{bmatrix} \boldsymbol{w}_0 \\ \vdots \\ \boldsymbol{w}_{N+1} \end{bmatrix}. \tag{4}$$

Another model of interest, the *square channel model*, is given by

$$\boldsymbol{y}_k = \alpha \boldsymbol{s}_{k-1} + \boldsymbol{s}_k + \alpha \boldsymbol{s}_{k+1} + \boldsymbol{w}_k, \quad 1 \leq k \leq N. \tag{5}$$

The analysis of this model is not given due to space limitation, but will be reported in the near future.

Both the non-square and the square channels appear frequently in magnetic recording research, with a linear F, under the name *multitrack, multihead recording systems* [1]. The non-square case appears if there are more (reading) heads than tracks; the square case appears if the numbers are equal. In magnetic recording, F takes the form

$$F(\boldsymbol{a}) = \boldsymbol{a} \star \boldsymbol{h}, \tag{6}$$

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[1] A slight change of notation is needed though.

where $\boldsymbol{h}$ is an ISI response and $\star$ denotes convolution. Moreover, the most important case for magnetic recording systems is an ISI response of the form

$$\boldsymbol{h} = [1\ \ {-1}] \star \underbrace{[1\ \ 1] \star [1\ \ 1] \star \cdots \star [1\ \ 1]}_{L \text{ terms}}, \tag{7}$$

for some integer $L \geq 1$.

We consider the minimum Euclidean distance, which is defined by

$$d^2_{\min} \triangleq \min_{\boldsymbol{E} \neq \boldsymbol{0}} d^2(\boldsymbol{E}),$$

with $\boldsymbol{E} \triangleq [\boldsymbol{e}_1 \ldots, \boldsymbol{e}_N]'$. The fundamental importance of $d^2_{\min}$ is that the error probability at high signal-to-noise ratio is well approximated by $Q(\sqrt{d^2_{\min}/2N_0})$, where N_0 is the power spectral density of the additive white Gaussian noise, and $Q(\cdot)$ is the standard Q-function. It should be pointed out that no energy normalization of the Euclidean distance is done in this paper; this follows the approach of [2].

The aim of this paper is to analytically derive the minimum distance in terms of the single-track minimum distance d_0^2 for any α in the range $0 \leq \alpha \leq 1/2$. A number of results have been derived previously by Soljanin and Georghiades [2]. In order to compare the contribution of this work with theirs, we list their results for the non-square case as follows.

Results of [2]

1) If $\alpha \leq 1/4$, then $d^2_{\min} = (1 + 2\alpha^2)d_0^2$ for all N and F.
2) If $N \leq 3$, $\mathcal{M} = \{\pm 1\}$ and $F(\boldsymbol{a}) = \boldsymbol{a} \star \boldsymbol{h}$ with $\boldsymbol{h}$ as in (7), then

$$d^2_{\min} = \begin{cases} (1+2\alpha^2)d_0^2, & \alpha \leq 1 - 1/\sqrt{2} \\ (2 + 4\alpha^2 - 4\alpha)d_0^2, & 1 - 1/\sqrt{2} \leq \alpha \leq 1/2, \end{cases} \tag{8}$$

provided that $d_0^2 \leq 6$. For a $\boldsymbol{h}$ of the form (7), $d_0^2 \leq 6$ for $L = 1$, 2, 3.

An obvious weakness of [2] is that the result 2) only considers $N \leq 3$. This work will relax that condition to an arbitrary N. Moreover, the result 1) will be improved. A minimum distance analysis for some other two dimensional interference channels can be found in [3].

A. Euclidean Distance and Minimum Euclidean Distance

In this section we give the basics of the Euclidean distance computation between two data signals. For notational convenience, the presentation is based on a linear F, but it can be easily extended to a non-linear F.

The distance between the signals generated from the information blocks $\boldsymbol{A}$ and $\boldsymbol{A} + \boldsymbol{E}$ is

$$\begin{aligned} d^2(\boldsymbol{A}, \boldsymbol{A}+\boldsymbol{E}) &= \sum_{k=0}^{N+1} \|\alpha(F(\boldsymbol{a}_k + \boldsymbol{e}_k) - F(\boldsymbol{a}_k)) \\ &\quad + (F(\boldsymbol{a}_{k-1} + \boldsymbol{e}_{k-1}) - F(\boldsymbol{a}_{k-1})) \\ &\quad + \alpha(F(\boldsymbol{a}_{k-2} + \boldsymbol{e}_{k-2}) - F(\boldsymbol{a}_{k-2}))\|^2 \\ &= \sum_{k=0}^{N+1} \|\alpha F(\boldsymbol{e}_k) + F(\boldsymbol{e}_{k-1}) + \alpha F(\boldsymbol{e}_{k-2})\|^2 \\ &= d^2(\boldsymbol{E}), \end{aligned} \tag{9}$$

where $F(\boldsymbol{e}_k) \triangleq \boldsymbol{0}$, for $k \leq 0$ and $k \geq N+1$. Thus, for a linear F, the Euclidean distance between two signals is a function only of the error event $\boldsymbol{E}$. If we define $g(z)$ as the z-transform of the autocorrelation of the ITI response $t(z)$:

$$\begin{aligned} g(z) &= t(z)t(z^{-1}) \\ &= \alpha^2 z^{-2} + 2\alpha z^{-1} + (1+2\alpha^2) + 2\alpha z + \alpha^2 z^2, \end{aligned}$$

the distance $d^2(\boldsymbol{E})$ can be written as

$$d^2(\boldsymbol{E}) = \sum_{k=1}^{N}\sum_{\ell=1}^{N} g_{k-\ell} F(\boldsymbol{e}_k) F(\boldsymbol{e}_\ell)'. \tag{10}$$

We can also write (10) as

$$d^2(\boldsymbol{E}) = [F(\boldsymbol{e}_1), \ldots, F(\boldsymbol{e}_N)]\, \boldsymbol{T}'\boldsymbol{T} \begin{bmatrix} F(\boldsymbol{e}_1) \\ \vdots \\ F(\boldsymbol{e}_N) \end{bmatrix},$$

where $\boldsymbol{T}'\boldsymbol{T}$ becomes the Toeplitz matrix

$$\begin{bmatrix} 1+2\alpha^2 & 2\alpha & \alpha^2 & & & \\ 2\alpha & 1+2\alpha^2 & 2\alpha & \ddots & & \\ \alpha^2 & 2\alpha & 1+2\alpha^2 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \alpha^2 \\ & & \alpha^2 & 2\alpha & 1+2\alpha^2 & 2\alpha \\ & & & \alpha^2 & 2\alpha & 1+2\alpha^2 \end{bmatrix}.$$

The results in forthcoming sections are based on the fact that $\boldsymbol{T}'\boldsymbol{T}$ determines the Euclidean distance, not $\boldsymbol{T}$ itself. Thus, we can replace $\boldsymbol{T}$ in (4) with any $\boldsymbol{V}$ such that $\boldsymbol{V}'\boldsymbol{V} = \boldsymbol{T}'\boldsymbol{T}$.

In forthcoming sections we will frequently use the term K-track error event, where $1 \leq K \leq N$. By this we mean, without any loss of generality, that $\boldsymbol{e}_1 \neq \boldsymbol{0}, \boldsymbol{e}_K \neq \boldsymbol{0}$ and $\boldsymbol{e}_k = \boldsymbol{0}$, for $k > K$. Moreover, it is simple to prove that if $\boldsymbol{e}_k = \boldsymbol{0}$ for $k = 2$ or $k = K-1$, then such an event cannot possibly achieve $d^2_{\min}$. The reasons for this are as follows. Assume $\boldsymbol{e}_2 = \boldsymbol{0}$. At the tracks 0 and $K+1$ we will always pile up at least an amount $2\alpha^2 d_0^2$ of Euclidean distance. At track 1, since $\boldsymbol{e}_2 = \boldsymbol{0}$ at least a distance d_0^2 is built up. In total this will generate at least a distance $(1+2\alpha^2)d_0^2$. Subsequently, $(1+2\alpha^2)d_0^2$ will appear as a lower bound to $d^2_{\min}$ for all α; thus, we can safely assume that $\boldsymbol{e}_k \neq \boldsymbol{0}$, for $k = 1, 2, K-1, K$ and that $\boldsymbol{e}_k = \boldsymbol{0}$, for $k > K$.

The rest of this paper is organized as follows. In Section II, the main results are stated. In order to not destroy the fluency of the paper, the proof-techniques and the proofs are deferred to Section III. Finally, conclusions are presented in Section IV.

II. Main Results

We start with exact results on $d^2_{\min}$.

Theorem 1: For an arbitrary $N \geq 2$ and any $\mathcal{M}$ as well as F,

$$d^2_{\min} = \begin{cases} (1+2\alpha^2)d_0^2, & \alpha \leq 1 - 1/\sqrt{2} \\ (2 + 4\alpha^2 - 4\alpha)d_0^2, & 1 - 1/\sqrt{2} \leq \alpha \leq \xi, \end{cases} \tag{11}$$

where ξ is the root of $2 - 7\alpha + 4\alpha^2 + 2\alpha^3$ satisfying $0 < \xi < 1/2$ ($\xi \approx 0.389$).

For arbitrary N, any modulation alphabet $\mathcal{M}$ and signal generation rule F, the minimum distance is still achieved by a single-track error event for $\alpha \leq 1-1/\sqrt{2}$. This improves upon the result from [2] that suggests that it is true for $\alpha \leq 1/4$. The second part of Theorem 1 states that the worst event in the range $1 - 1/\sqrt{2} \leq \alpha \leq 0.389$ is a two-track error event for any N, $\mathcal{M}$ and F.

The next result treats the range $0.389 \leq \alpha \leq 0.4082$.

Theorem 2: For $\xi \leq \alpha \leq 1/\sqrt{6} \approx 0.4082$, $N \geq 3$ and arbitrary $\mathcal{M}$ and F, the $d^2_{\min}$-achieving event is either a two-track or a three-track event and

$$2\frac{1 - 3\alpha^2 + 6\alpha^4}{1 + 2\alpha^2}d_0^2 \leq d^2_{\min} \leq (2 + 4\alpha^2 - 4\alpha)d_0^2. \tag{12}$$

Moreover, there exist $\mathcal{M}$ and F such that $d^2_{\min}$ achieves its boundary values.

The implication of Theorem 2 is that for such an α, there is no need to consider any four or more track error events when searching for $d^2_{\min}$. This significantly eases the computational burden.

In the range $1/\sqrt{6} \leq \alpha \leq 1/2$, we can lower-bound $d^2_{\min}$ as follows.

Theorem 3: For $1/\sqrt{6} \leq \alpha \leq 1/2$, the worst error event can involve more than three tracks and $d^2_{\min}$ is bounded by

$$2\frac{1 - 18\alpha^4 + 72\alpha^6 - 216\alpha^8}{(1 + 6\alpha^2)(1 - 12\alpha^4)}d_0^2 \leq d^2_{\min} \leq (2 + 4\alpha^2 - 4\alpha)d_0^2. \tag{13}$$

Although the above theorems are conclusive and improve significantly upon [2], there is still some ambiguity in the region $0.389 \leq \alpha \leq 1/2$. This ambiguity results from our very general assumptions on F. If more assumptions on F are made, we can obtain more precise result in the entire range $0 \leq \alpha \leq 1/2$.

Theorem 4: Let $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$ be two valid non-zero error events. For $N \geq 2$, if

$$\min_{\boldsymbol{e},\tilde{\boldsymbol{e}} \neq \boldsymbol{0}} \left\| F(\boldsymbol{a}) - F(\boldsymbol{a}+\boldsymbol{e}) - \frac{1}{2}(F(\tilde{\boldsymbol{a}}) - F(\tilde{\boldsymbol{a}}+\tilde{\boldsymbol{e}}))\right\|^2 \geq \frac{1}{4}d_0^2$$

then

$$d^2_{\min} = (2 + 4\alpha^2 - 4\alpha)d_0^2, \quad 1 - 1/\sqrt{2} \leq \alpha \leq 1/2.$$

For our most important special case, a binary modulation alphabet $\mathcal{M} = \{\pm 1\}$ (i.e., 2PAM) and a linear F of the form (6), the condition in Theorem 4 can be written as

$$\min \left\| \left(\boldsymbol{e} + \frac{1}{2}\tilde{\boldsymbol{e}}\right) \star \boldsymbol{h}\right\|^2 \geq \frac{1}{4}d_0^2$$

which is equivalent to

$$\min \|(2\boldsymbol{e} + \tilde{\boldsymbol{e}}) \star \boldsymbol{h}\|^2 \geq d_0^2.$$

For the binary modulation alphabet, the error events consist of symbols from the ternary error alphabet $\{-2, 0, 2\}$, which implies that the event $2\boldsymbol{e} + \tilde{\boldsymbol{e}}$ consists of symbols from the error alphabet $\{0, \pm 2, \pm 4, \pm 6\}$. But this is precisely the error alphabet that would arise if the modulation alphabet was the quaternary alphabet $\{\pm 1, \pm 3\}$ (i.e., 4PAM). We have therefore shown the following.

Corollary 1: For a 2PAM modulation alphabet, $N \geq 2$ and $F(\boldsymbol{a}) = \boldsymbol{a} \star \boldsymbol{h}$, if the minimum distance of $F(\boldsymbol{a})$ under a 4PAM alphabet satisfies

$$d^2_{0,\text{4PAM}} = d_0^2 \tag{14}$$

we have

$$d^2_{\min} = \begin{cases} (1 + 2\alpha^2)d_0^2, & \alpha \leq 1 - 1/\sqrt{2} \\ (2 + 4\alpha^2 - 4\alpha)d_0^2, & 1 - 1/\sqrt{2} \leq \alpha \leq 1/2 \end{cases} \tag{15}$$

for the 2PAM alphabet under investigation.

The corollary surprisingly states that if $F(\boldsymbol{a})$ has the same minimum distance for 2PAM and 4PAM, then $d^2_{\min}$ of the two dimensional channel is completely determined. If this is not the case, $d^2_{\min}$ is still determined for $\alpha < 0.389$, but a full search has to be performed for $\alpha \geq 0.389$. There are several complexity reduction techniques for such a search, but those are not the target of this paper. Note that (14) is fulfilled for all ISI responses with exactly two non-zero taps.

We now assume a $\boldsymbol{h}$ of the form (7) and check for which L (14) is satisfied. The outcome is that it is satisfied for (at least) $1 \leq L \leq 3$ which implies that the minimum distance is exactly given by (15) for those L regardless of N. Compared with [2], this is a major improvement.

III. Proofs

Before proceeding to the proofs, we make some preparations. Denote by $\boldsymbol{F}(\boldsymbol{e})$ the matrix $[F(\boldsymbol{e}_1), \ldots, F(\boldsymbol{e}_N)]$ and by $\boldsymbol{F}_{x_1}^{x_2}(\boldsymbol{e})$ the vector $[F(\boldsymbol{e}_{x_1}), \ldots, F(\boldsymbol{e}_{x_2})]$. We will make use of the following three facts.

Fact 1: As stated in Section I, it is only the matrix $\boldsymbol{T}'\boldsymbol{T}$ that is important for a distance computation, and not $\boldsymbol{T}$ itself. Thus, we are free to choose any factorization of $\boldsymbol{T}'\boldsymbol{T}$, and the one we will use is the so-called *minimum phase factorization*. Since it is well known that $g(z)$ is the autocorrelation of the real-valued polynomial $t(z) = \alpha + z^{-1} + \alpha z^{-2}$, it follows that the roots of $g(z)$ appear in complex conjugated pairs and that if r is a root, then is also $1/r^*$, where $(\cdot)^*$ is the complex conjugate operator. The minimum phase version of $g(z)$ results if $h(z)$ is created from the roots that lie inside of the unit circle. The four roots of $g(z)$ are

$$\begin{aligned} r_1 = r_2 &= -\frac{-1 + \sqrt{1 - 4\alpha^2}}{2\alpha} \\ r_3 = r_4 &= -\frac{-1 - \sqrt{1 - 4\alpha^2}}{2\alpha}. \end{aligned} \tag{16}$$

It is easy to see that r_1 and r_2 lie inside the unit circle. Thus, we can express the minimum phase version as

$$v(z) \triangleq v_0 + v_1 z^{-1} + v_2 z^{-2} = c\left(1 + z^{-1}\frac{\sqrt{1 - 4\alpha^2} - 1}{2\alpha}\right)^2,$$

where c is a constant that normalizes the energy of $v(z)$ to $1+2\alpha^2$. In what follows, we will only need the middle coefficient of $v(z)$, namely v_1, whose value is equal to 2α after some lines of manipulations.

Fact 2: It is true that

$$\begin{aligned}&[F(\boldsymbol{e}_1),\ldots,F(\boldsymbol{e}_N)]\,\boldsymbol{T}'\boldsymbol{T}\,[F(\boldsymbol{e}_1),\ldots,F(\boldsymbol{e}_N)]' \\ &= [F(\boldsymbol{e}_N),\ldots,F(\boldsymbol{e}_1)]\,\boldsymbol{T}'\boldsymbol{T}\,[F(\boldsymbol{e}_N),\ldots,F(\boldsymbol{e}_1)]'. \quad (17)\end{aligned}$$

Fact 3: For any symmetric positive definite matrix $\boldsymbol{G}$ of size 3 by 3, the optimization problem

$$\begin{array}{ll}\min & [\boldsymbol{x}_1\,\boldsymbol{x}_2\,\boldsymbol{x}_3]\,G\,[\boldsymbol{x}_1\,\boldsymbol{x}_2\,\boldsymbol{x}_3]' \\ \text{subject to} & \|\boldsymbol{x}_1\|^2, \|\boldsymbol{x}_2\|^2, \|\boldsymbol{x}_3\|^2 \geq d_0^2\end{array}$$

is solved by three vectors that lie in the same plane.

We are now well prepared to prove Theorems 1–3.

Proof of Theorems 1–3 Denote by d_ℓ^2 the minimum distance of any ℓ-track error event. Clearly, we have that $d_1^2 = (1+2\alpha^2)d_0^2$ and $d_{\min}^2 = \min\{d_1^2,\ldots,d_N^2\}$. This proof will compute d_2^2 and d_3^2 explicitly and a lower bound for d_ℓ^2, $\ell \geq 4$. The lower bound will then be shown to be larger than d_1^2, d_2^2 and d_3^2 in a certain range of α.

Consider a two-track error event $\boldsymbol{E} = [\boldsymbol{e}_1\,\boldsymbol{e}_2]'$. Because $d^2(c\boldsymbol{E}) = c^2 d^2(\boldsymbol{E})$, we can without loss of generality assume that $\|F(\boldsymbol{e}_1)\|^2 = d_0^2$. We then have the optimization problem

$$\min_{\boldsymbol{e}_2} \quad \boldsymbol{F}(\boldsymbol{e}) \begin{bmatrix} 1+2\alpha^2 & 2\alpha \\ 2\alpha & 1+2\alpha^2 \end{bmatrix} \boldsymbol{F}(\boldsymbol{e})'$$

under the constraint $\|F(\boldsymbol{e}_2)\|^2 \geq d_0^2$. The solution to this optimization is of the form $F(\boldsymbol{e}_2) = -kF(\boldsymbol{e}_1),\ k \geq 1$. Let $f(k)$ denote the distance resulting from $F(\boldsymbol{e}_2) = -kF(\boldsymbol{e}_1)$. To find the minimum distance of two-track error events, we take the derivative of $f(k)$: $f'(k) = 2k(1+2\alpha^2) - 4\alpha = 0$ which has the solution $k = 2\alpha/(1+2\alpha^2)$. But this solution violates the constraint $\|F(\boldsymbol{e}_2)\|^2 \geq d_0^2$ and $k=1$ becomes the minimizer. This results in the minimum distance

$$d_2^2 = (2+4\alpha^2-4\alpha)d_0^2.$$

For d_3^2, the optimization problem to be solved is

$$d_3^2 = \min_{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3} \quad \boldsymbol{F}(\boldsymbol{e}) \begin{bmatrix} 1+2\alpha^2 & 2\alpha & \alpha^2 \\ 2\alpha & 1+2\alpha^2 & 2\alpha \\ \alpha^2 & 2\alpha & 1+2\alpha^2 \end{bmatrix} \boldsymbol{F}(\boldsymbol{e})' \quad (18)$$

under the constraint $\|F(\boldsymbol{e}_\ell)\|^2 \geq d_0^2,\ \ell = 1,2,3$. Due to Fact 3, we can safely assume that $F(\boldsymbol{e}_1), F(\boldsymbol{e}_2)$ and $F(\boldsymbol{e}_3)$ lie in the same plane, and we can therefore treat them as complex numbers. It is simple to show that the minimum of (18) occurs for $\boldsymbol{e}_1 = \boldsymbol{e}_3 = \boldsymbol{e}_{\min}$, where $\boldsymbol{e}_{\min}$ is any error event with $\|F(\boldsymbol{e}_{\min})\|^2 = d_0^2$, and

$$F(\boldsymbol{e}_2) = \begin{cases} -F(\boldsymbol{e}_1), & 0 \leq \alpha \leq 1-1/\sqrt{2} \\ -\frac{4\alpha}{1+2\alpha^2}F(\boldsymbol{e}_1), & 1-1/\sqrt{2} \leq \alpha \leq 1/2. \end{cases} \quad (19)$$

Consequently,

$$d_3^2 = \begin{cases} (3-8\alpha+8\alpha^2)d_0^2, & 0 \leq \alpha \leq 1-1/\sqrt{2} \\ \frac{2}{1+2\alpha^2}(1-3\alpha^2+6\alpha^4)d_0^2, & 1-1/\sqrt{2} \leq \alpha \leq 1/2. \end{cases}$$

We now turn to error event that involves four or more tracks. For a K-track error event $\boldsymbol{E}$, by using the minimum phase factorization from Fact 1, we have the lower bound

$$\begin{aligned} d^2(\boldsymbol{E}) &\geq \sum_{k=0}^{2} \|v_0F(\boldsymbol{e}_k) + v_1F(\boldsymbol{e}_{k-1}) + v_2F(\boldsymbol{e}_{k-2})\|^2 \\ &+ \sum_{k=K-1}^{K+1} \|v_0F(\boldsymbol{e}_k) + v_1F(\boldsymbol{e}_{k-1}) + v_2F(\boldsymbol{e}_{k-2})\|^2. \\ &= v_0^2\|F(\boldsymbol{e}_1)\|^2 + \|v_0F(\boldsymbol{e}_2) + v_1F(\boldsymbol{e}_1)\|^2 \\ &+ \|v_0F(\boldsymbol{e}_3) + v_1F(\boldsymbol{e}_2) + v_2F(\boldsymbol{e}_1)\|^2 \\ &+ \|v_0F(\boldsymbol{e}_N) + v_1F(\boldsymbol{e}_{N-1}) + v_2F(\boldsymbol{e}_{N-2})\|^2 \\ &+ \|v_1F(\boldsymbol{e}_N) + v_2F(\boldsymbol{e}_{N-1})\|^2 + v_2^2\|F(\boldsymbol{e}_N)\|^2. \end{aligned}$$

Note that $\boldsymbol{e}_{N-2}$ equals $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ in the case of $N=4$ or 5, respectively. The above lower bound is unfortunately too loose. In order to strengthen it we invoke Fact 2:

$$\begin{aligned} 2d^2(\boldsymbol{E}) &\geq v_0^2\|F(\boldsymbol{e}_1)\|^2 + \|v_0F(\boldsymbol{e}_2) + v_1F(\boldsymbol{e}_1)\|^2 \\ &+ \|v_0F(\boldsymbol{e}_3) + v_1F(\boldsymbol{e}_2) + v_2F(\boldsymbol{e}_1)\|^2 \\ &+ \|v_0F(\boldsymbol{e}_N) + v_1F(\boldsymbol{e}_{N-1}) + v_2F(\boldsymbol{e}_{N-2})\|^2 \\ &+ \|v_1F(\boldsymbol{e}_N) + v_2F(\boldsymbol{e}_{N-1})\|^2 + v_2^2\|F(\boldsymbol{e}_N)\|^2. \\ &+ v_0^2\|F(\boldsymbol{e}_N)\|^2 + \|v_0F(\boldsymbol{e}_{N-1}) + v_1F(\boldsymbol{e}_N)\|^2 \\ &+ \|v_0F(\boldsymbol{e}_{N-2}) + v_1F(\boldsymbol{e}_{N-1}) + v_2F(\boldsymbol{e}_N)\|^2 \\ &+ \|v_0F(\boldsymbol{e}_1) + v_1F(\boldsymbol{e}_2) + v_2F(\boldsymbol{e}_3)\|^2 \\ &+ \|v_1F(\boldsymbol{e}_1) + v_2F(\boldsymbol{e}_2)\|^2 + v_2^2\|F(\boldsymbol{e}_1)\|^2. \end{aligned}$$

Rearranging the terms by using the afore-mentioned notation $\boldsymbol{F}_{x_1}^{x_2}(\boldsymbol{e})$ and the facts that $v_0^2+v_1^2+v_2^2 = 1+2\alpha^2$, $v_0v_1+v_1v_2 = 2\alpha$ and $v_0v_2 = \alpha^2$, we get

$$2d^2(\boldsymbol{E}) \geq \boldsymbol{F}_1^3(\boldsymbol{e})\,\boldsymbol{G}_3\,\boldsymbol{F}_1^3(\boldsymbol{e})' + \boldsymbol{F}_{N-2}^N(\boldsymbol{e})\,\tilde{\boldsymbol{G}}_3\,\boldsymbol{F}_{N-2}^N(\boldsymbol{e})'$$

where

$$\boldsymbol{G}_3 = \begin{bmatrix} 2+4\alpha^2 & 4\alpha & 2\alpha^2 \\ 4\alpha & 1+2\alpha^2+v_1^2 & 2\alpha \\ \alpha^2 & 2\alpha & 1+2\alpha^2-v_1^2 \end{bmatrix}$$

and

$$\tilde{\boldsymbol{G}}_3 = \begin{bmatrix} 1+2\alpha^2-v_1^2 & 2\alpha & 2\alpha^2 \\ 2\alpha & 1+2\alpha^2+v_1^2 & 4\alpha \\ \alpha^2 & 4\alpha & 2+4\alpha^2 \end{bmatrix}.$$

Consequently we have for any $\ell \geq 4$

$$\begin{aligned} 2d_\ell^2 \quad &\geq \quad \min_{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3} \boldsymbol{F}_1^3(\boldsymbol{e})\boldsymbol{G}_3\boldsymbol{F}_1^3(\boldsymbol{e})' \\ &+ \quad \min_{\boldsymbol{e}_N,\boldsymbol{e}_{N-1},\boldsymbol{e}_{N-2}} \boldsymbol{F}_{N-2}^N(\boldsymbol{e})\tilde{\boldsymbol{G}}_3\boldsymbol{F}_{N-2}^N(\boldsymbol{e})'. \quad (20) \end{aligned}$$

Since the solutions to the two optimization problem in (20) are identical, we conclude

$$d_\ell^2 \geq d_{\mathrm{LB},4}^2 \triangleq \min_{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3} \boldsymbol{F}_1^3(\boldsymbol{e})\boldsymbol{G}_3\boldsymbol{F}_1^3(\boldsymbol{e})'. \quad (21)$$

It should be pointed out that $\boldsymbol{G}_3$ in (21) can be generated from any factorization of $\boldsymbol{T}'\boldsymbol{T}$. In particular we could use $\boldsymbol{T}$. But such an approach leads to a weak bound. Instead we use the minimum phase factorization and obtain

$$\boldsymbol{G}_3 = \begin{bmatrix} 2+4\alpha^2 & 4\alpha & 2\alpha^2 \\ 4\alpha & 1+6\alpha^2 & 2\alpha \\ \alpha^2 & 2\alpha & 1-2\alpha^2 \end{bmatrix}.$$

Due to its construction, $\boldsymbol{G}_3$ is positive definite for any α.

At this point, we consider it to be a standard exercise, armed with Fact 3, to show that the solution of (21) is given by

$$d^2_{\mathrm{LB},4} \geq \begin{cases} \frac{3}{1+6\alpha^2}(1-4\alpha^2+12\alpha^4)d_0^2, & \frac{1}{2}-\frac{1}{2\sqrt{3}} \leq \alpha \leq \frac{1}{\sqrt{6}} \\ 2\frac{1-18\alpha^4+72\alpha^6-216\alpha^8}{(1+6\alpha^2)(1-12\alpha^4)}d_0^2, & \frac{1}{\sqrt{6}} \leq \alpha \leq \frac{1}{2}. \end{cases} \tag{22}$$

For $\alpha \leq 1/2 - 1/2\sqrt{3}$, $d^2_{\mathrm{LB},4} > (1+2\alpha^2)d_0^2$ and no error events involving four or more tracks can result in d^2_{min}.

We can now summarize, in the range $0 \leq \alpha \leq 1 - 1/\sqrt{2}$, d_1^2 is larger than any other d_ℓ^2, $\ell \geq 2$ and the first statement of Theorem 1 follows. Equating d_2^2 and d_3^2 we obtain, after some manipulations, the second statement follows (in the range $\alpha \leq \xi$, $d^2_{\mathrm{LB},4} > d_2^2$). Moreover, the error events that achieves d^2_{min} always exist and it follows that Theorem 1 states the exact d^2_{min}.

In the range $\xi < \alpha < 1/\sqrt{6}$, it can be shown that $d_3^2 < d_2^2 < d^2_{\mathrm{LB},4}$, and Theorem 3 follows. While the upper bound is achieved for example for $F(\boldsymbol{a}) = \boldsymbol{a}$ and a binary alphabet, the lower bound can only be achieved if error events of the type (19) exist.

In the range $1/\sqrt{6} < \alpha < 1/2$, $d^2_{\mathrm{LB},4} < d_\ell^2$, $\ell \leq 3$ and Theorem 3 follows. ■

We next prove Theorem 4.

Proof of Theorem 4 In this case we use the lower bound

$$\begin{aligned} d^2(\boldsymbol{E}) \geq & \sum_{k=0}^{1} \|v_0F(\boldsymbol{e}_k)+v_1F(\boldsymbol{e}_{k-1})+v_2F(\boldsymbol{e}_{k-2})\|^2 \\ & + \sum_{k=K}^{K+1} \|v_0F(\boldsymbol{e}_k)+v_1F(\boldsymbol{e}_{k-1})+v_2F(\boldsymbol{e}_{k-2})\|^2. \\ = & v_0^2\|F(\boldsymbol{e}_1)\|^2 + \|v_0F(\boldsymbol{e}_2)+v_1F(\boldsymbol{e}_1)\|^2 \\ & + \|v_1F(\boldsymbol{e}_N)+v_2F(\boldsymbol{e}_{N-1})\|^2 + v_2^2\|F(\boldsymbol{e}_N)\|^2. \end{aligned} \tag{23}$$

By proceeding in the same fashion as in the previous proof, where the key step is to use the minimum phase factorization, we obtain

$$d^2_{\mathrm{min}} \geq \min_{\boldsymbol{e}_1,\boldsymbol{e}_2} \boldsymbol{F}_1^2(\boldsymbol{e}) \begin{bmatrix} 1+6\alpha^2 & 2\alpha \\ 2\alpha & 1-2\alpha^2 \end{bmatrix} \boldsymbol{F}_1^2(\boldsymbol{e})'. \tag{24}$$

It is easy to see that to solve the optimization (24) we must have $\|F(\boldsymbol{e}_1)\|^2 = d_0^2$. Assume that $\|F(\boldsymbol{e}_2)\|^2 = k^2d_0^2$ and there is an angle θ between $F(\boldsymbol{e}_1)$ and $F(\boldsymbol{e}_2)$. Then we have

$$d^2_{\mathrm{min}} \geq \left(1+6\alpha^2+k^2(1-2\alpha^2)+4\alpha k\cos\theta\right)d_0^2. \tag{25}$$

In order for the right hand side of (25) to equal d_2^2, we must have

$$\cos\theta = \frac{2(k^2-1)\alpha - 4\alpha + (1-k^2)}{4k\alpha}. \tag{26}$$

Differentiating (26) with respect to α gives

$$\frac{\partial\cos\theta}{\partial\alpha} = (k^2-1)\left(\frac{1}{2k}+\frac{1}{4k\alpha^2}\right).$$

Since the derivative is always positive for $k \geq 1$, it follows that $\cos\theta$ in (26) is an increasing function of α. However, the existence of such k and θ is a property of F and not of α. It therefore suffices to consider $\alpha = 1/2$. It is worthwhile to point out that the fact that $\cos\theta$ is an increasing function of α is a consequence of the minimum phase factorization. For a non-minimum phase factorization it is not the case.

Now recall that the lower bound (23) measures the distance at tracks $0, 1, N$ and $N+1$. In the case of $\alpha = 1/2$ however, all the roots of $g(z)$ coincide. Consequently, all the phase factorizations coincide and we have $v(z) = h(z) = \alpha + z^{-1} + \alpha z^{-2}$. Because the ITI is symmetric, the minimum distance on the two first and the two last tracks coincide. Consequently there is no need to use Fact 2 at $\alpha = 1/2$. In order to have $d^2_{\mathrm{min}} = (2+4\alpha^2-4\alpha)d_0^2$, it is sufficient if we accumulate a distance amount $(2+4\alpha^2-4\alpha)d_0^2$ at tracks $0, 1, N$ and $N+1$. Since we will accumulate at least $2\alpha^2d_0^2$ at tracks 0 and $N+1$, we need to accumulate at least an amount $(1+\alpha^2-2\alpha)d_0^2$ at track 1 and an equal distance amount on track N. But since tracks 1 and N are identical from a minimum distance point of view at $\alpha = 1/2$, we have that if

$$\min_{\boldsymbol{e}_1,\boldsymbol{e}_2} \left\| F(\boldsymbol{e}_1) + \frac{1}{2}F(\boldsymbol{e}_2) \right\|^2 \geq \frac{d_o^2}{4},$$

then the F under investigation does not allow any k and θ such that (26) can be fulfilled and the theorem is proved. ■

IV. Conclusion

In this paper we have studied the minimum distance problem for two dimensional interference channels. By transforming the ITI response $[\alpha\, 1\, \alpha]$ into its minimum phase version, we have derived the exact minimum distance, for all possible ISI responses, up to $\alpha \leq 0.389$ (c.f. Theorem 1). In the range $0.389 \leq \alpha \leq 0.5$, we have lower-bounded the minimum distance (c.f. Theorems 2 and 3). In order to resolve the ambiguity in the latter range, we derived a sufficient condition on the ISI response, i.e., Theorem 4, in order to be able to state the exact minimum distance in the entire range $0 \leq \alpha \leq 0.5$. The results derived in this paper are more conclusive and they improve upon [2].

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