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Sjöberg, Daniel

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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

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Daniel Sjöberg

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
Sweden

Daniel Sjöberg

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
P.O. Box 118
SE-221 00 Lund
Sweden

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Abstract

Shock waves are discontinous solutions to quasi-linear partial differential equations, and can be studied through a singular perturbation known as the vanishing viscosity technique. The vanishing viscosity method is a means of smoothing the shock, and we study the case of electromagnetic waves in bianisotropic materials. We derive the conditions arising from this smoothing procedure for a traveling wave, and the waves are classified as fast, slow or intermediate shock waves.

1 Introduction

Electromagnetic waves propagating in an instantaneously reacting material can be modeled with a system of quasi-linear, partial differential equations. It is well known that such a model can exhibit shock solutions, *i.e.*, solutions which become discontinuous in finite time even if the initial/boundary data are smooth. This poses severe problems for numerical methods, such as finite difference schemes, which are often based on the assumption of continuous and differentiable solutions.

In order to overcome this problem, we can model the material on a finer scale, which requires a denser grid and thus increases the memory demands and the computation time. Another approach, is to develop more powerful numerical methods, which can handle discontinuous solutions. The development of these numerical schemes benefits from an understanding of the propagation of shock waves; for instance, Godunov's scheme is based on the solution of Riemann's problem [14, 15], where the shock wave is generated by discontinuous initial data. A variation of Godunov's scheme is Glimm's scheme, which is used to show global existence of solutions to certain systems of equations [13, 20].

The aim of this paper is to increase the understanding of electromagnetic shock waves, modeled with the Maxwell equations. Mainly using techniques from [31], we analyze the wave propagation in bianisotropic materials, *i.e.*, materials with different properties for different polarizations of the waves, and a possible coupling between the electric and the magnetic field [23, p. 7]. This adds insight not only to the numerical treatment of electromagnetic waves in complicated materials, but also provides some physical intuition.

The Maxwell equations can be considered as a hyperbolic system of conservation laws. A good introduction to the numerical approximation of such systems is given in [14], which introduces the analytical theory as well as some common schemes in one and two spatial dimensions. There is presently not a good mathematical understanding of systems of conservation laws in several dimensions, but some general references are [7, 14, 20, 30, 34].

Perhaps the most familiar kind of "electromagnetic" shock wave is in the field of magnetohydrodynamics, from which we give only a few references [1, 6, 10, 12, 17], [25, pp. 245–253]. Electromagnetic shock waves in isotropic media have previously been treated theoretically, see [25, pp. 388–391], [2] and references therein. Recently, a few papers on experiments concerning electromagnetic shock waves have been published [3, 4, 8]. In continuum mechanics, G. A. Maugin has recognized the similarity between shock waves and phase transition fronts as singular sets in irreversible motion, with a dissipation related to the power expanded by a driving force on the singularity set, see [27, 28].

In this paper, we study when the shock waves can be defined as the limit of continuous traveling wave solutions to an approximate problem, where the discontinuity is smoothed over a small region. This is the *shock structure* problem, which was introduced by Gel'fand [11], and is given an extensive treatment in [33]. A thorough treatment of this problem in magnetohydrodynamics is found in [10], and a recent paper deals with the structure of electromagnetic shock waves in anisotropic ferromagnetic media [19].

This paper is organized as follows. In Section 2 we introduce the Maxwell equations and the constitutive relations used to model the electromagnetic waves, as well as the general form of the entropy condition. In Section 3 we present the vanishing viscosity method of smoothing the solutions of quasi-linear, hyperbolic equations. The consequences of the vanishing viscosity method for traveling waves are studied in Sections 4 and 5, where we show that there exists three kinds of electromagnetic shock waves: the fast, the slow, and the intermediate shock wave. In Section 6 we also mention another form of dicontinuous solutions, contact discontinuities, which cannot be analyzed with the vanishing viscosity method for traveling waves. However, they exist only under the condition of linear degeneracy, and we present this condition and its opposite, genuine nonlinearity, in Section 6. The different kinds of shock waves are illustrated with phase portraits of a certain system of ordinary differential equations in Section 7, and some concluding remarks are made in Section 8.

2 The Maxwell equations, constitutive relations and the entropy condition

In this paper we use a slight modification of the Heaviside-Lorentz units for our fields [21, p. 781], *i.e.*, all electromagnetic fields are scaled to units of $\sqrt{\text{energy/volume}}$,

$$\begin{cases} \boldsymbol{E} = \sqrt{\epsilon_0}\boldsymbol{E}_{\rm SI} \\ \boldsymbol{H} = \sqrt{\mu_0}\boldsymbol{H}_{\rm SI} \end{cases} \qquad \begin{cases} \boldsymbol{D} = 1/\sqrt{\epsilon_0}\boldsymbol{D}_{\rm SI} \\ \boldsymbol{B} = 1/\sqrt{\mu_0}\boldsymbol{B}_{\rm SI}, \end{cases} \tag{2.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ is the electric and magnetic field strength, respectively, and $\boldsymbol{D}$ and $\boldsymbol{B}$ is the electric and magnetic flux density, respectively. The index SI is used to indicate the field in SI units. We use the scaled time $t = c_0 t_{\rm SI}$, where $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum, and the constants ϵ_0 and μ_0 are the permittivity and permeability of free space, respectively. The six-vector notation from [18, 31], *i.e.*,

$$\boldsymbol{e} = \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}, \quad \boldsymbol{d} = \begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix}, \quad \nabla\times\mathbf{J} = \begin{pmatrix} \mathbf{0} & -\nabla\times\mathbf{I} \\ \nabla\times\mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{2.2}$$

enables us to write the source free Maxwell equations in the compact form

$$\nabla \times \mathbf{J}\boldsymbol{e} + \partial_t \boldsymbol{d} = \mathbf{0}. \tag{2.3}$$

In this paper we treat the six-vectors as column vectors, *i.e.*, we write the scalar product as $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{d} = \sum_{i=1}^{6} e_i d_i$. This is merely for notational convenience and does not capture the full mathematical structure, which is not needed here. On occasions, we also consider the scalar product between two three-vectors, in which case we use the standard notation $\boldsymbol{E} \cdot \boldsymbol{D} = \sum_{i=1}^{3} E_i D_i$. For more ambitious attempts to construct a six-vector notation, we refer to [18, 26].

The Maxwell equations must be supplemented by a constitutive relation, whose purpose is to model the interaction of the electromagnetic field with the material. When the material reacts very fast to stimulance, we can model it with an instantaneous constitutive model, where the values of the electric flux density $\boldsymbol{D}$ and the magnetic flux density $\boldsymbol{B}$ are completely determined by the values of the electric field strength $\boldsymbol{E}$ and magnetic field strength $\boldsymbol{H}$ at the same point in spacetime. We write this as

$$\boldsymbol{d}(\boldsymbol{r}, t) = \boldsymbol{d}(\boldsymbol{e}(\boldsymbol{r}, t)), \tag{2.4}$$

where $\boldsymbol{d}(\boldsymbol{e})$ is the gradient of a scalar function $\phi(\boldsymbol{e})$ with respect to $\boldsymbol{e}$, *i.e.*, in terms of thermodynamics, the field gradient of the thermodynamic potential (or the free energy density or the free enthalpy density) [5, 25]. We use the notation $\boldsymbol{d}(\boldsymbol{e}) = \phi'(\boldsymbol{e})$ to denote this derivative, *i.e.*, $d_i(\boldsymbol{e}) = \partial\phi/\partial e_i$, $i = 1, \ldots, 6$. The model is passive if we require that the symmetric 6×6 matrix $\boldsymbol{d}'(\boldsymbol{e}) = \phi''(\boldsymbol{e})$, where $[\boldsymbol{d}'(\boldsymbol{e})]_{ij} = \partial^2\phi/\partial e_i \partial e_j$, is a positive definite matrix, which is the case if the scalar function $\phi(\boldsymbol{e})$ is a convex function.

The initial value problem for the Maxwell equations with an instantaneously reacting constitutive model is

$$\nabla \times \mathbf{J}\boldsymbol{e} + \boldsymbol{d}'(\boldsymbol{e})\partial_t \boldsymbol{e} = \mathbf{0}, \qquad \boldsymbol{e}(\boldsymbol{x}, 0) = \boldsymbol{e}_0(\boldsymbol{x}), \tag{2.5}$$

and since $\boldsymbol{d}'(\boldsymbol{e})$ is positive definite and symmetric, this is by definition a quasi-linear, symmetric, hyperbolic system of partial differential equations [34, p. 360]. This system has been extensively studied in [31], where it is shown that the equations in general support two waves, the ordinary and the extraordinary wave, each with its own refractive index.

Due to the quasi-linearity, the system (2.5) may exhibit shock solutions, *i.e.*, even if we give smooth initial data, the solution becomes discontinuous in finite time. This means that the derivatives cannot be classically defined everywhere, but we can make a weak formulation of the problem by requiring the equality

$$\int_0^\infty \int_{\mathbb{R}^3} [\boldsymbol{e}^{\mathrm{T}} \nabla \times \mathbf{J}\boldsymbol{\varphi} + \boldsymbol{d}(\boldsymbol{e})^{\mathrm{T}} \partial_t \boldsymbol{\varphi}] \, \mathrm{d}V \, \mathrm{d}t + \int_{\mathbb{R}^3} \boldsymbol{d}(\boldsymbol{e}_0)^{\mathrm{T}} \boldsymbol{\varphi}(\boldsymbol{x}, 0) \, \mathrm{d}V = 0 \tag{2.6}$$

to hold for all six-vector test functions $\boldsymbol{\varphi}$ defined on $\mathbb{R}^3 \times [0, \infty)$, *i.e.*, vector-valued functions which are infinitely differentiable with compact support. One problem with this weak formulation is that we lose uniqueness, *i.e.*, there are several weak solutions $\boldsymbol{e}$ which satisfy the above criteria.

If the solution $\boldsymbol{e}$ to (2.5) is smooth, we can multiply the equations from the left by $\boldsymbol{e}^{\mathrm{T}}$ to obtain the Poynting theorem, or energy conservation law,

$$\nabla \cdot \boldsymbol{S}(\boldsymbol{e}) + \partial_t \eta(\boldsymbol{e}) = 0, \tag{2.7}$$

where $\boldsymbol{S}(\boldsymbol{e}) = \boldsymbol{E} \times \boldsymbol{H}$ is the Poynting vector, and $\eta(\boldsymbol{e}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{d}(\boldsymbol{e}) - \phi(\boldsymbol{e})$ is the electromagnetic energy. When the solution $\boldsymbol{e}$ is not smooth, this inequality is no longer valid since the derivatives are not defined in the classical sense. However, as is shown in [32], it is reasonable to replace it with the inequality

$$\nabla \cdot \boldsymbol{S}(\boldsymbol{e}) + \partial_t \eta(\boldsymbol{e}) \leq 0, \tag{2.8}$$

which is interpreted in a weak sense, *i.e.*, for all scalar test functions $\varphi \geq 0$ defined on $\mathbb{R}^3 \times [0, \infty)$, the inequality

$$\int_0^\infty \int_{\mathbb{R}^3} [\boldsymbol{S}(\boldsymbol{e}) \cdot \nabla\varphi + \eta(\boldsymbol{e})\partial_t\varphi] \,\mathrm{d}V \,\mathrm{d}t + \int_{\mathbb{R}^3} \eta(\boldsymbol{e}_0)\varphi(\boldsymbol{x}, 0) \,\mathrm{d}V \geq 0 \tag{2.9}$$

holds. The inequality (2.8) is called an *entropy inequality*, and if $\boldsymbol{e}$ satisfies both (2.8) and (2.5), it is called an *entropy solution*. It is frequently conjectured that entropy solutions are unique [14, p. 32], and we refer to [32] for a discussion of the physical interpretation of this inequality. In the following section, we show how the entropy inequality is a natural consequence of the vanishing viscosity method.

3 Vanishing viscosity regularization

The loss of uniqueness for the weak solution is important to resolve if we want to make numerical approximations of the differential equations. This problem has been extensively studied for scalar conservation laws and systems of conservation laws in one space variable [9, 14, 20, 33, 34], where the conservation law is typically written $\partial_t u + \sum_i \partial_{x_i} f_i(u) = 0$. The knowledge of systems of conservation laws in several space variables is limited, but a common assumption is that reasonable (physical) solutions should arise as limits to the regularized equation $\partial_t u_\delta + \sum_i \partial_{x_i} f_i(u_\delta) = \delta\nabla^2 u_\delta$, when $\delta \to 0$. Since the second order derivative is often used as a model for a small viscous effect, this method is called the *vanishing viscosity method*. The benefit of the vanishing viscosity method is that for each $\delta > 0$ we can usually prove that the initial value problem is well posed, with unique, differentiable solutions. We can define a unique limit u as $\delta \to 0$ if we can find a convergent sequence of solutions $\{u_\delta\}$. However, this limit u must also be shown to satisfy the original conservation law, which is often nontrivial. For systems of conservation laws in several dimensions, this is still an unsolved problem [30].

We propose to use a similar method to define solutions to our quasi-linear system of equations, where we study the equations

$$\nabla \times \mathbf{J}\boldsymbol{e}_\delta + \partial_t \boldsymbol{d}(\boldsymbol{e}_\delta) = \delta\nabla^2 \boldsymbol{e}_\delta, \quad \boldsymbol{e}_\delta(\boldsymbol{x}, 0) = \boldsymbol{e}_0(\boldsymbol{x}), \quad \delta > 0. \tag{3.1}$$

Standard PDE theory guarantees a C^∞ solution $\boldsymbol{e}_\delta$ to this equation for every $\delta > 0$ for suitable $\boldsymbol{e}_0$, see [34, pp. 327–332]. An important result is that if the viscosity solution $\boldsymbol{e}_\delta$ converges boundedly almost everywhere in the limit $\delta \to 0$, the limit satisfies the entropy condition from the previous section. To see this, multiply (3.1) with $\boldsymbol{e}_\delta$ and observe

$$\begin{cases} \boldsymbol{e}_\delta^{\mathrm{T}} \partial_t \boldsymbol{d}(\boldsymbol{e}_\delta) = \partial_t(\boldsymbol{e}_\delta^{\mathrm{T}} \boldsymbol{d}(\boldsymbol{e}_\delta) - \phi(\boldsymbol{e}_\delta)) = \partial_t \eta(\boldsymbol{e}_\delta) \\ \boldsymbol{e}_\delta^{\mathrm{T}} \nabla \times \mathbf{J} \boldsymbol{e}_\delta = \nabla \cdot (\boldsymbol{E}_\delta \times \boldsymbol{H}_\delta) = \nabla \cdot \boldsymbol{S}(\boldsymbol{e}_\delta) \\ \boldsymbol{e}_\delta^{\mathrm{T}} \nabla^2 \boldsymbol{e}_\delta = -|\nabla \boldsymbol{e}_\delta|^2 + \nabla^2 |\boldsymbol{e}_\delta|^2/2, \end{cases} \tag{3.2}$$

where $\eta(\boldsymbol{e}_\delta)$ is the electromagnetic energy in the medium and $\boldsymbol{S}(\boldsymbol{e}_\delta)$ is the Poynting vector. Note that all the derivatives are classically defined, and we have the following scalar inequality,

$$\nabla \cdot \boldsymbol{S}(\boldsymbol{e}_\delta) + \partial_t \eta(\boldsymbol{e}_\delta) \leq \delta \nabla^2 |\boldsymbol{e}_\delta|^2/2. \tag{3.3}$$

It can be shown that if $\boldsymbol{e}_\delta$ is uniformly bounded in the supremum norm and converges almost everywhere to $\boldsymbol{e}$ as $\delta \to 0$, then this limit solution $\boldsymbol{e}$ is a weak solution to (2.5) and satisfies the inequality

$$\nabla \cdot \boldsymbol{S}(\boldsymbol{e}) + \partial_t \eta(\boldsymbol{e}) \leq 0 \tag{3.4}$$

almost everywhere, see [14, p. 27] and [34, p. 438]. In the following sections, we study the consequences of the vanishing viscosity method in the case of traveling waves, which provides us with a more precise means of writing the entropy condition.

4 Inner solutions and shock structure

In this section we largely follow the ideas presented in many textbooks, *e.g.*, [9, pp. 600–603], [14, pp. 79–83], [33, pp. 508–510] and [34, p. 431]. Dropping the index δ for brevity, we investigate the singularly perturbed Maxwell equations (3.1) for the existence of solutions in the form of traveling waves,

$$\boldsymbol{e} = \boldsymbol{e}\left(\frac{z - vt}{\delta}\right) = \boldsymbol{e}(\zeta), \tag{4.1}$$

where we have chosen z to be the coordinate in the propagation direction, and v is the speed of the shock wave. We also require the derivative $\boldsymbol{e}'(\zeta)$ to disappear as $\zeta \to \pm\infty$, and a typical situation is depicted in Figure 1. In the language of singular perturbation theory [22], the traveling wave corresponds to an *inner solution* of the problem (3.1), and is a means of analyzing the microscopic behavior of the solution at a scale of order δ. The microscopic properties of a number of discontinuities which are distant at a macroscopic scale can be treated by considering them as isolated traveling waves of the type (4.1). Observe that $\zeta \to \infty$ does not necessarily mean $z \to \infty$, it is sufficient that $z > vt$ and $\delta \to 0$.

The traveling wave solution (4.1) must satisfy the ordinary differential equation

$$\hat{\boldsymbol{z}} \times \mathbf{J}\boldsymbol{e}' - v\{\boldsymbol{d}(\boldsymbol{e})\}' = \boldsymbol{e}'', \tag{4.2}$$

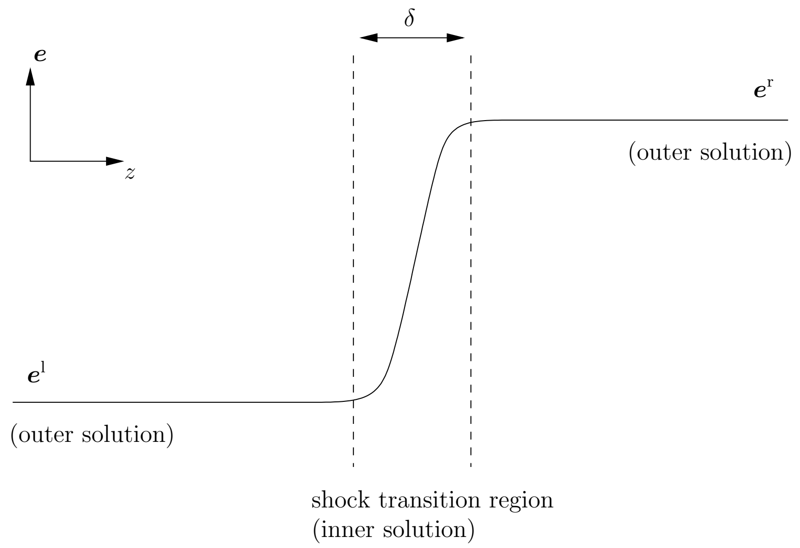

Figure 1: A typical traveling wave profile. The idea is that the inner solution shall provide a smooth transition between the outer solutions, the left and right constant states $\boldsymbol{e}_{\mathrm{l}}$ and $\boldsymbol{e}_{\mathrm{r}}$. The solution typically arises in Riemann's problem, where the initial values are $\boldsymbol{e}(\boldsymbol{x},0) = \boldsymbol{e}_0(\boldsymbol{x}) = \boldsymbol{e}^{\mathrm{l}}$ for $z < 0$ and $\boldsymbol{e}^{\mathrm{r}}$ for $z > 0$.

where $\hat{\boldsymbol{z}}$ denotes the unit vector in the z-direction. Observe that this equation does not involve the parameter δ, reflecting the fact that we are studying properties at a certain scale. Integrating the above equation once implies

$$\hat{\boldsymbol{z}} \times \mathbf{J}\boldsymbol{e} - v\boldsymbol{d}(\boldsymbol{e}) - \boldsymbol{e}' = \hat{\boldsymbol{z}} \times \mathbf{J}\boldsymbol{e}^{\mathrm{l,r}} - v\boldsymbol{d}(\boldsymbol{e}^{\mathrm{l,r}}) - (\boldsymbol{e}')^{\mathrm{l,r}}, \tag{4.3}$$

where $\boldsymbol{e}^{\mathrm{l,r}} = \lim_{\zeta\to\mp\infty} \boldsymbol{e}(\zeta)$. Taking the opposite limit $\boldsymbol{e} \to \boldsymbol{e}^{\mathrm{r,l}}$ in (4.3) implies the Rankine-Hugoniot jump condition

$$\hat{\boldsymbol{z}} \times \mathbf{J}[\![\boldsymbol{e}]\!] - v[\![\boldsymbol{d}(\boldsymbol{e})]\!] = \mathbf{0}, \tag{4.4}$$

where we use the notation $[\![\boldsymbol{e}]\!] = \boldsymbol{e}^{\mathrm{r}} - \boldsymbol{e}^{\mathrm{l}}$ and $[\![\boldsymbol{d}(\boldsymbol{e})]\!] = \boldsymbol{d}(\boldsymbol{e}^{\mathrm{r}}) - \boldsymbol{d}(\boldsymbol{e}^{\mathrm{l}})$ to indicate the jumps in the quantities $\boldsymbol{e}$ and $\boldsymbol{d}(\boldsymbol{e})$ over the shock. Note that the Rankine-Hugoniot condition is a vector identity, and that the jump in $\boldsymbol{d}(\boldsymbol{e})$ cannot have a component parallel to $\hat{\boldsymbol{z}}$, unless $v = 0$.

We use the assumption $(\boldsymbol{e}')^{\mathrm{l,r}} = \mathbf{0}$ to write (4.3) as a system of autonomous, ordinary differential equations,

$$\boldsymbol{e}' = \hat{\boldsymbol{z}} \times \mathbf{J}(\boldsymbol{e} - \boldsymbol{e}^{\mathrm{l,r}}) - v(\boldsymbol{d}(\boldsymbol{e}) - \boldsymbol{d}(\boldsymbol{e}^{\mathrm{l,r}})), \tag{4.5}$$

with the asymptotic boundary conditions $\lim_{\zeta\to\mp\infty} \boldsymbol{e}(\zeta) = \boldsymbol{e}^{\mathrm{l,r}}$. It is clear that these states are critical points for the system (4.5), *i.e.*, the right hand side is zero for these states. In the following section we investigate when the system (4.5) has a solution, and what conditions this infers on the speed v. The corresponding ODE for ferromagnets described by the Landau-Lifshitz constitutive equation is studied in detail in [19] .

5 The entropy condition for a traveling wave

A solution to (4.5) that connects its critical points $\boldsymbol{e}^{\mathrm{l}}$ and $\boldsymbol{e}^{\mathrm{r}}$, where $\boldsymbol{e}^{\mathrm{l}} \neq \boldsymbol{e}^{\mathrm{r}}$, is called a heteroclinic orbit [29]. Before investigating these orbits, we show that homoclinic orbits, *i.e.*, solutions where $\boldsymbol{e}^{\mathrm{l}} = \boldsymbol{e}^{\mathrm{r}}$ and $\boldsymbol{e} \neq \boldsymbol{e}^{\mathrm{l,r}}$ somewhere on the orbit, cannot exist. Multiplying (4.5) with $(\boldsymbol{e}')^{\mathrm{T}}$ we obtain

$$0 \leq |\boldsymbol{e}'|^2 = \Big(\boldsymbol{e}^{\mathrm{T}}\hat{\boldsymbol{z}} \times \mathbf{J}(\frac{\boldsymbol{e}}{2} - \boldsymbol{e}^{\mathrm{l,r}}) - v\phi(\boldsymbol{e}) + v\boldsymbol{e}^{\mathrm{T}}\boldsymbol{d}(\boldsymbol{e}^{\mathrm{l,r}})\Big)' = \psi(\boldsymbol{e})', \tag{5.1}$$

which shows there exists a scalar function $\psi(\boldsymbol{e})$ which is nondecreasing along the orbit. Such a function must be constant on a homoclinic orbit, implying $|\boldsymbol{e}'|^2 = 0$, and thus $\boldsymbol{e}$ must be constant throughout the orbit, which degenerates to a point.

The existence of a heteroclinic orbit for the system (4.5) requires that the unstable manifold of one critical point intersects the stable manifold of the other, where the unstable and the stable manifold is associated with the positive and the negative eigenvalues of the linearized problem, respectively. If the heteroclinic orbit is to be stable under small perturbations, then the sum of the dimensions of the stable and unstable manifold must exceed the dimension of the phase space [33, p. 509]. In our case, the relevant manifolds are the unstable manifold for $\boldsymbol{e}^{\mathrm{l}}$ and the stable manifold for $\boldsymbol{e}^{\mathrm{r}}$. The dimensions of these manifolds can be calculated from counting how many eigenvalues of the linearized equations that are greater/lesser than zero at each critical point. The linearized equations are

$$(\boldsymbol{e} - \boldsymbol{e}^{\mathrm{l,r}})' = \left[\hat{\boldsymbol{z}} \times \mathbf{J} - v\boldsymbol{d}'(\boldsymbol{e}^{\mathrm{l,r}})\right](\boldsymbol{e} - \boldsymbol{e}^{\mathrm{l,r}}). \tag{5.2}$$

Temporarily denoting the 6×6 matrix $\boldsymbol{d}'(\boldsymbol{e}^{\mathrm{l,r}})$ by $\mathbf{A}$, the problem of deducing the dimension of the stable and unstable manifolds consists in counting positive and negative eigenvalues for the matrix $\hat{\boldsymbol{z}} \times \mathbf{J} - v\mathbf{A}$. Since $\mathbf{A}$ is positive definite, the signs of the eigenvalues are the same as for the problem

$$[\hat{\boldsymbol{z}} \times \mathbf{J} - v\mathbf{A}]\boldsymbol{u}_i = \lambda_i \mathbf{A}\boldsymbol{u}_i \quad \Rightarrow \quad [\hat{\boldsymbol{z}} \times \mathbf{J} - (v + \lambda_i)\mathbf{A}]\boldsymbol{u}_i = \mathbf{0}. \tag{5.3}$$

Using the same technique as in [31], we formulate this eigenvalue problem as

$$c_i \boldsymbol{w}_i = \sqrt{\mathbf{A}}^{-1}[\hat{\boldsymbol{z}} \times \mathbf{J}]\sqrt{\mathbf{A}}^{-1}\boldsymbol{w}_i, \tag{5.4}$$

where $\sqrt{\mathbf{A}}$ is the symmetric, positive definite square root of $\mathbf{A}$, $c_i = v + \lambda_i$, and $\boldsymbol{w}_i = \sqrt{\mathbf{A}}\boldsymbol{u}_i$. The matrix in the right hand side is a congruence transformation of $\hat{\boldsymbol{z}} \times \mathbf{J}$, and it is well known that such a transformation does not change the signs of the eigenvalues [16, p. 251]. Since $\hat{\boldsymbol{z}} \times \mathbf{J}$ has the (double) eigenvalues ± 1 and 0, there are always two negative eigenvalues $c_{3,4} < 0$ and two zero eigenvalues $c_{5,6} = 0$. This implies $\lambda_{3,4,5,6} \leq -v < 0$. The argument concerning the dimensions of the stable and unstable manifolds can then involve only the positive eigenvalues c_1 and c_2, and the corresponding λ_1 and λ_2. In order for the sum of the dimension of the unstable manifold ($\lambda > 0$) at $\boldsymbol{e}^{\mathrm{l}}$ and the dimension of the stable manifold ($\lambda < 0$)

at $\boldsymbol{e}^{\mathrm{r}}$ to be larger than six (the dimension of the phase space), one of the following conditions must hold:

$$\begin{cases} 0 < \lambda_1(\boldsymbol{e}^{\mathrm{l}}) < \lambda_2(\boldsymbol{e}^{\mathrm{l}}) & \text{and} \quad \lambda_1(\boldsymbol{e}^{\mathrm{r}}) < 0 < \lambda_2(\boldsymbol{e}^{\mathrm{r}}), \quad \text{or} \\ \lambda_1(\boldsymbol{e}^{\mathrm{l}}) < 0 < \lambda_2(\boldsymbol{e}^{\mathrm{l}}) & \text{and} \quad \lambda_1(\boldsymbol{e}^{\mathrm{r}}) < \lambda_2(\boldsymbol{e}^{\mathrm{r}}) < 0, \quad \text{or} \\ 0 < \lambda_1(\boldsymbol{e}^{\mathrm{l}}) < \lambda_2(\boldsymbol{e}^{\mathrm{l}}) & \text{and} \quad \lambda_1(\boldsymbol{e}^{\mathrm{r}}) < \lambda_2(\boldsymbol{e}^{\mathrm{r}}) < 0. \end{cases} \tag{5.5}$$

Observe that the dimension of the unstable manifold at $\boldsymbol{e}^{\mathrm{l}}$ is calculated from the number of positive eigenvalues, *i.e.*, the number of positive eigenvalues in the left column of (5.5). The dimension of the stable manifold at $\boldsymbol{e}^{\mathrm{r}}$ is calculated from the number of negative eigenvalues, *i.e.*, the number of negative eigenvalues in the right column of (5.5) plus four, since we deduced earlier that $\lambda_{3,4,5,6}$ are always negative.

The two positive eigenvalues $c_{1,2} = v + \lambda_{1,2}$ are identified as the characteristic wave speeds in the material, which are determined from the eigenvalue problem (5.4) for each state $\boldsymbol{e}^{\mathrm{l,r}}$, as in [31]. The speeds are in general functions of both the state, $\boldsymbol{e}^{\mathrm{l}}$ or $\boldsymbol{e}^{\mathrm{r}}$, and the propagation direction, $\hat{\boldsymbol{z}}$, but we choose to suppress the dependence on the propagation direction since this is constant in this paper.

The conditions on $\lambda_{1,2}$ above can be written in terms of the shock speed v and the characteristic wave speeds $c_{1,2}$ as

$$\begin{cases} v < c_1(\boldsymbol{e}^{\mathrm{l}}) < c_2(\boldsymbol{e}^{\mathrm{l}}) & \text{and} \quad c_1(\boldsymbol{e}^{\mathrm{r}}) < v < c_2(\boldsymbol{e}^{\mathrm{r}}) \quad \text{(slow shock)} \\ c_1(\boldsymbol{e}^{\mathrm{l}}) < v < c_2(\boldsymbol{e}^{\mathrm{l}}) & \text{and} \quad c_1(\boldsymbol{e}^{\mathrm{r}}) < c_2(\boldsymbol{e}^{\mathrm{r}}) < v \quad \text{(fast shock)} \\ v < c_1(\boldsymbol{e}^{\mathrm{l}}) < c_2(\boldsymbol{e}^{\mathrm{l}}) & \text{and} \quad c_1(\boldsymbol{e}^{\mathrm{r}}) < c_2(\boldsymbol{e}^{\mathrm{r}}) < v \quad \text{(intermediate shock).} \end{cases} \tag{5.6}$$

These expressions constitute the entropy conditions for electromagnetic, plane shock waves. The nomenclature "fast shock" and "slow shock" is in accordance with [19] and [10], and "intermediate shock" is from [10]. Note that the fast and the slow shock are closely connected to the ordinary and extraordinary rays for anisotropic materials, see for instance [23, pp. 68–71] and [25, pp. 331–357].

To conclude this section, we note that our entropy condition is analogous to the Lax entropy condition for an n-dimensional, strictly hyperbolic system of conservation laws $u_t + f(u)_x = 0$. This condition is that there should exist an index k such that

$$\begin{cases} \lambda_1(u^{\mathrm{l}}) < \cdots < \lambda_{k-1}(u^{\mathrm{l}}) < v < \lambda_k(u^{\mathrm{l}}) < \cdots < \lambda_n(u^{\mathrm{l}}) \\ \lambda_1(u^{\mathrm{r}}) < \cdots < \lambda_k(u^{\mathrm{r}}) < v < \lambda_{k+1}(u^{\mathrm{r}}) < \cdots < \lambda_n(u^{\mathrm{r}}), \end{cases} \tag{5.7}$$

where $\lambda_1(u), \ldots, \lambda_n(u)$ are the eigenvalues of the $n \times n$ matrix $f'(u)$ and v is the shock speed (see for instance [9, p. 589], [14, p. 76], [20, p. 61], [33, p. 261]).

6 Genuine nonlinearity and contact discontinuities

When $c_i(\boldsymbol{e}^{\mathrm{l}}) = c_i(\boldsymbol{e}^{\mathrm{r}})$ for $i = 1$ and/or $i = 2$, one or several of the conditions (5.6) may not be applicable. This phenomenon occurs for a type of waves called *contact* *discontinuities*, which are characterized by

$$c_i(\boldsymbol{e}^{\mathrm{l}}) = c_i(\boldsymbol{e}^{\mathrm{r}}) = c_i(\boldsymbol{e}) \quad \text{and} \quad \hat{\boldsymbol{z}} \times \mathbf{J}(\boldsymbol{e} - \boldsymbol{e}^{\mathrm{l,r}}) - c_i(\boldsymbol{d}(\boldsymbol{e}) - \boldsymbol{d}(\boldsymbol{e}^{\mathrm{l,r}})) = \mathbf{0}, \tag{6.1}$$

for all $\boldsymbol{e} \in \gamma$, where γ is a smooth curve connecting $\boldsymbol{e}^{\mathrm{l}}$ to $\boldsymbol{e}^{\mathrm{r}}$ in $\mathbb{R}^6$. Differentiating the latter condition along the curve γ, implies $[\hat{\boldsymbol{z}} \times \mathbf{J} - c_i \boldsymbol{d}'(\boldsymbol{e})]\dot{\boldsymbol{e}} = \mathbf{0}$, where $\dot{\boldsymbol{e}}$ denotes the tangential derivative of $\boldsymbol{e}$ along this curve. This means $\dot{\boldsymbol{e}}$ is proportional to the eigenvector $\boldsymbol{e}_i$ by definition. That the speed is constant on the curve γ can also be written

$$0 = \dot{c}_i = (\mathrm{D}_e\, c_i)^{\mathrm{T}} \dot{\boldsymbol{e}} = (\mathrm{D}_e\, c_i)^{\mathrm{T}} \boldsymbol{e}_i, \tag{6.2}$$

where $\mathrm{D}_e\, c_i$ denotes the gradient of the speed c_i with respect to the six-vector $\boldsymbol{e}$, *i.e.*, $(\mathrm{D}_e\, c_i)_k = \partial c_i / \partial e_k$. This means that the eigenvector $\boldsymbol{e}_i$ must be orthogonal to $\mathrm{D}_e\, c_i$. We say the field $\boldsymbol{e}_i$ is *linearly degenerate* if $\boldsymbol{e}_i^{\mathrm{T}}\, \mathrm{D}_e\, c_i = 0$, and *genuinely nonlinear* if $\boldsymbol{e}_i^{\mathrm{T}}\, \mathrm{D}_e\, c_i \neq 0$, see *e.g.*, [14, p. 41]. One reason for the term linearly degenerate is that contact discontinuities travel along non-crossing characteristics, just as in the linear case. An interesting feature of contact discontinuities is that their structure is not captured by the traveling wave ansatz, since the right hand side of (4.5) is identically zero. In this paper, we restrict ourselves to investigating a few explicit examples.

Our first example is a constitutive relation which always has one linearly degenerate field. For an instantaneously reacting, isotropic, nonmagnetic material, we have the constitutive relations

$$\boldsymbol{D}(\boldsymbol{E}) = F(|\boldsymbol{E}|^2)\boldsymbol{E}, \qquad \boldsymbol{B} = \boldsymbol{H}. \tag{6.3}$$

It is not difficult to prove that the characteristic speeds are

$$c_1(\boldsymbol{E}) = \frac{1}{\sqrt{F(|\boldsymbol{E}|^2) + 2F'(|\boldsymbol{E}|^2)|\boldsymbol{E}|^2}}, \qquad c_2(\boldsymbol{E}) = \frac{1}{\sqrt{F(|\boldsymbol{E}|^2)}}, \tag{6.4}$$

with the corresponding eigenvectors defined by $\boldsymbol{e}_i = (\boldsymbol{E}_i, \boldsymbol{H}_i)^{\mathrm{T}}$, where $\boldsymbol{H}_i = \hat{\boldsymbol{z}} \times \boldsymbol{E}_i$ for $i = 1, 2$, and

$$\boldsymbol{E}_1(\boldsymbol{E}) = \boldsymbol{E}/|\boldsymbol{E}|, \qquad \boldsymbol{E}_2(\boldsymbol{E}) = \hat{\boldsymbol{z}} \times \boldsymbol{E}/|\boldsymbol{E}|. \tag{6.5}$$

Since the speed is independent of $\boldsymbol{H}$, we have $\boldsymbol{e}_i^{\mathrm{T}}\, \mathrm{D}_e\, c_i = \boldsymbol{E}_i \cdot \mathrm{D}_E\, c_i$ for $i = 1, 2$. From the explicit expressions (6.4) it is seen that $\mathrm{D}_E\, c_1 \sim \mathrm{D}_E\, c_2 \sim \boldsymbol{E}$, where the $\sim$ sign indicates proportionality. It is clear that $\boldsymbol{E}_1 \cdot \mathrm{D}_E\, c_1 \neq 0$ and $\boldsymbol{E}_2 \cdot \mathrm{D}_E\, c_2 = 0$, *i.e.*, the field $\boldsymbol{E}_1$ is genuinely nonlinear and $\boldsymbol{E}_2$ is linearly degenerate. We interpret a wave where the change in $\boldsymbol{E}$ is orthogonal to $\boldsymbol{E}$, *i.e.*, $\partial_t \boldsymbol{E} \sim \boldsymbol{E}_2$, as a circularly polarized wave. This is motivated by the fact that the amplitude $|\boldsymbol{E}|$ does not change, but the vector $\boldsymbol{E}$ appears to rotate when observed as a function of time at a given point in space. Thus, we have found that circularly polarized waves in an isotropic medium are linearly degenerate.

Our second example is a constitutive model where there are no linearly degenerate fields. The model is

$$\boldsymbol{D}(\boldsymbol{E}) = (1 + \boldsymbol{C}\cdot\boldsymbol{E})\boldsymbol{E} + \frac{|\boldsymbol{E}|^2}{2}\boldsymbol{C}, \qquad \boldsymbol{B} = \boldsymbol{H}, \tag{6.6}$$

which is not valid for all $\boldsymbol{E}$, since $\boldsymbol{D}'(\boldsymbol{E}) = (1 + \boldsymbol{C}\cdot\boldsymbol{E})\mathbf{I} + \boldsymbol{C}\boldsymbol{E} + \boldsymbol{E}\boldsymbol{C}$ is not positive definite everywhere. However, it is positive definite if $|\boldsymbol{C}||\boldsymbol{E}| < 1/3$, and thus the model suffices as an approximation for $\boldsymbol{E}$ small enough. For this model, the three-vector $\boldsymbol{C}$ represents a "nonlinear axis" of the material, which is obviously anisotropic. It is straightforward to show that when both $\boldsymbol{C}$ and $\boldsymbol{E}$ are orthogonal to $\hat{\boldsymbol{z}}$, we have

$$c_{1,2} = \frac{1}{\sqrt{1 + 2\boldsymbol{C}\cdot\boldsymbol{E} \pm |\boldsymbol{E}||\boldsymbol{C}|}}, \quad \text{and} \quad \boldsymbol{E}_{1,2} = \frac{\boldsymbol{E}}{|\boldsymbol{E}|} \pm \frac{\boldsymbol{C}}{|\boldsymbol{C}|}, \tag{6.7}$$

where the upper sign corresponds to c_1 and $\boldsymbol{E}_1$, and $\boldsymbol{H}_{1,2} = \hat{\boldsymbol{z}} \times \boldsymbol{E}_{1,2}$. The scalar product $\boldsymbol{E}_i \cdot \mathrm{D}_E\, c_i$ from which we analyze genuine nonlinearity can be shown to be

$$\boldsymbol{E}_{1,2}\cdot \mathrm{D}_E\, c_{1,2} = -\frac{3}{2}\frac{\boldsymbol{C}\cdot\boldsymbol{E}/|\boldsymbol{E}| \pm |\boldsymbol{C}|}{(1 + 2\boldsymbol{C}\cdot\boldsymbol{E} \pm |\boldsymbol{C}||\boldsymbol{E}|)^{3/2}}. \tag{6.8}$$

We see that one of these quantities is zero if $\boldsymbol{E}$ is parallel or antiparallel to $\boldsymbol{C}$, but any situation inbetween means $\boldsymbol{E}_1 \cdot \mathrm{D}_E\, c_1 \neq 0$ and $\boldsymbol{E}_2 \cdot \mathrm{D}_E\, c_2 \neq 0$. This shows that this model usually has no linearly degenerate fields, and contact discontinuities occurs only when the electric field is parallel or antiparallel to the axis $\boldsymbol{C}$. We conclude this example by noting the peculiarity that when the scalar product $\boldsymbol{C}\cdot\boldsymbol{E}$ is negative, the characteristic speeds $c_{1,2}$ may be larger than one, which is the speed of light in vacuum in our units. This may further restrict the validity of this model.

7 Numerical demonstration of shock structure for an anisotropic material

In this section we show numerically that there exists a structure (an inner solution dissipatively connecting two states) for a nonlinear anisotropic material. In order to present a concise example, we regularize the Maxwell equations in the electric field only, *i.e.*,

$$\begin{cases} -\nabla\times\boldsymbol{H} + \partial_t\boldsymbol{D} = -\delta\nabla^2\boldsymbol{E} \\ \nabla\times\boldsymbol{E} + \partial_t\boldsymbol{B} = \mathbf{0}. \end{cases} \tag{7.1}$$

The benefit of this approach is to reduce the phase space of the resulting system of ordinary differential equations to two dimensions, which enables us to plot the phase space easily. The approach is reasonable if we consider the Faraday law to be exact, and a similar technique is sometimes used for equations describing gas dynamics [9, p. 602]. The anisotropic material is described by the constitutive equation

$$\boldsymbol{D}(\boldsymbol{E}) = \begin{pmatrix} (2+|\boldsymbol{E}|^2)E_x \\ (3+|\boldsymbol{E}|^2)E_y \\ (4+|\boldsymbol{E}|^2)E_z \end{pmatrix}, \quad \boldsymbol{B} = \boldsymbol{H}, \tag{7.2}$$

where in this section the fields are dimensionless, see [31] for details on the scaling. This model has an anisotropic linear part and an isotropic nonlinear part, *i.e.*, practically the same example material as in [31].

The system of ordinary differential equations corresponding to (4.5) becomes

$$\begin{pmatrix} E_x' \\ E_y' \\ E_z' \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x - E_x^{\mathrm{l,r}} \\ E_y - E_y^{\mathrm{l,r}} \\ E_z - E_z^{\mathrm{l,r}} \\ H_x - H_x^{\mathrm{l,r}} \\ H_y - H_y^{\mathrm{l,r}} \\ H_z - H_z^{\mathrm{l,r}} \end{pmatrix} \\ - v \begin{pmatrix} (2+|\boldsymbol{E}|^2)E_x - (2+|\boldsymbol{E}^{\mathrm{l,r}}|^2)E_x^{\mathrm{l,r}} \\ (3+|\boldsymbol{E}|^2)E_y - (3+|\boldsymbol{E}^{\mathrm{l,r}}|^2)E_y^{\mathrm{l,r}} \\ (4+|\boldsymbol{E}|^2)E_z - (4+|\boldsymbol{E}^{\mathrm{l,r}}|^2)E_z^{\mathrm{l,r}} \\ H_x - H_x^{\mathrm{l,r}} \\ H_y - H_y^{\mathrm{l,r}} \\ H_z - H_z^{\mathrm{l,r}} \end{pmatrix}. \tag{7.3}$$

With $E_z^{\mathrm{l,r}} = H_z^{\mathrm{l,r}} = 0$ we have $E_z = H_z = 0$ throughout the shock, and $|\boldsymbol{E}|^2 = E_x^2 + E_y^2$. By eliminating the magnetic field and the z-components, we obtain the following 2×2 system of ordinary differential equations,

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix}' = \frac{1}{v} \begin{pmatrix} E_x - E_x^{\mathrm{l,r}} \\ E_y - E_y^{\mathrm{l,r}} \end{pmatrix} - v \begin{pmatrix} (2+E_x^2+E_y^2)E_x - (2+(E_x^{\mathrm{l,r}})^2+(E_y^{\mathrm{l,r}})^2)E_x^{\mathrm{l,r}} \\ (3+E_x^2+E_y^2)E_y - (3+(E_x^{\mathrm{l,r}})^2+(E_y^{\mathrm{l,r}})^2)E_y^{\mathrm{l,r}} \end{pmatrix}, \tag{7.4}$$

which contains all the qualitative information we need. We remark that this system can be integrated exactly for certain values of $E_x^{\mathrm{l,r}}$ and $E_y^{\mathrm{l,r}}$, but we refrain from exploiting this possibility in this paper. Phase portraits, *i.e.*, plots of the vector fields on the right hand side of the equations above, are found in Figures 2, 3 and 4 for a fast shock, an intermediate shock and a slow shock, respectively. Figure 5 depicts the phase portrait for a shock with mixed polarization, and Table 1 lists the relevant numbers used in each phase portrait. It is clearly seen from the figures that there exists a path connecting the critical points.

8 Discussion and conclusions

By studying a parabolic regularization of the quasi-linear Maxwell equations, we have proposed a classification of electromagnetic shock waves into three categories:

Fig.	E^{l}_x	E^{l}_y	E^{r}_x	E^{r}_y	v	c^{l}_1	c^{l}_2	c^{r}_1	c^{r}_2	Δ
2	0.000	0.000	0.500	0.000	0.667	0.577	0.707	0.555	0.603	−0.011
3	0.200	0.000	1.000	0.000	0.556	0.573	0.687	0.447	0.500	−0.087
4	0.788	0.000	1.000	0.000	0.476	0.509	0.525	0.447	0.500	−0.002
5	0.617	0.472	1.000	1.000	0.400	0.478	0.574	0.342	0.475	−0.059

Table 1: Relevant values for the phase portraits. The last column is the entropy difference $\Delta = \hat{\boldsymbol{z}}\cdot(\boldsymbol{S}(\boldsymbol{e}^{\mathrm{r}}) - \boldsymbol{S}(\boldsymbol{e}^{\mathrm{l}})) - v(\eta(\boldsymbol{e}^{\mathrm{r}}) - \eta(\boldsymbol{e}^{\mathrm{l}}))$, and since all the numbers in the column are negative, we see that all the waves satisfy the original entropy condition (2.8).

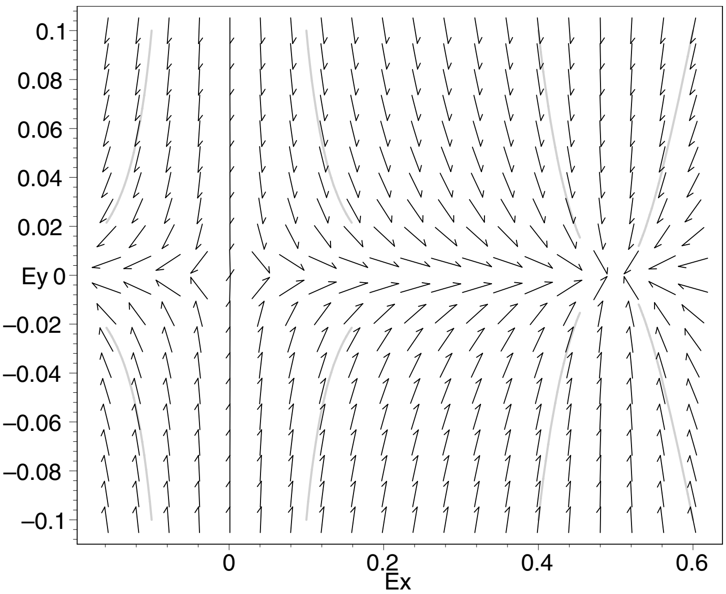

Figure 2: Phase portrait of a fast shock wave structure problem. The critical points are $(E^{\mathrm{l}}_x, E^{\mathrm{l}}_y) = (0.000, 0.000)$ and $(E^{\mathrm{r}}_x, E^{\mathrm{r}}_y) = (0.500, 0.000)$.

slow, fast and intermediate. This classification depends on how the shock speed relates to the characteristic speeds in the material, which in turn depend on the field strengths on both sides of the shock. These shock conditions can probably be improved with the help of Conley's index theory, as in [10, 33].

There also exists an additional kind of discontinuity, the contact discontinuity, which only occurs for linearly degenerate fields. In particular, we have showed that circularly polarized waves in isotropic, nonlinear media, exhibits contact discontinuities. The further study of contact discontinuities is beyond the scope of this paper, but it is seen from the analysis in Section 6 that it is important to understand which constitutive relations that permit a linearly degenerate field.

We consider the parabolic regularization term $\nabla^2 \boldsymbol{e}$ merely as a mathematical technique used in order to obtain a well posed problem, and do not require it to have a physical interpretation. Though, it is noteworthy that it may arise as a consequence of a multiple scale analysis of a more detailed constitutive relation,

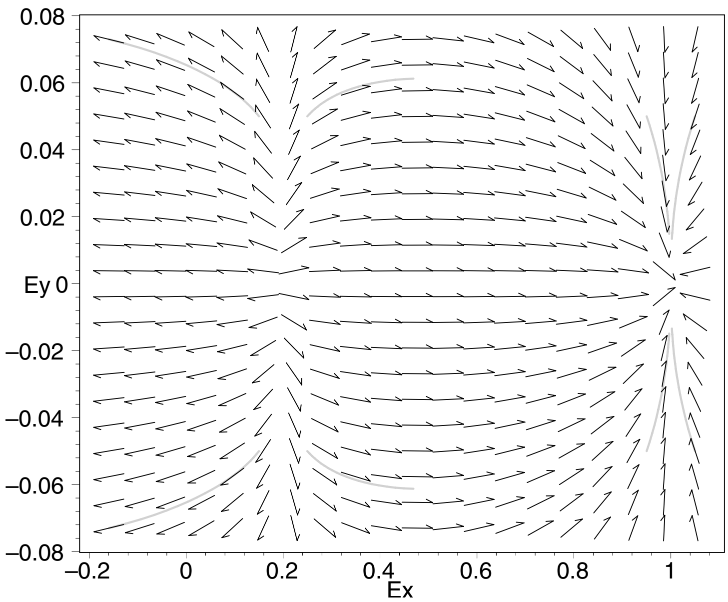

Figure 3: Phase portrait of an intermediate shock wave structure problem. The critical points are $(E_x^{\mathrm{l}}, E_y^{\mathrm{l}}) = (0.200, 0.000)$ and $(E_x^{\mathrm{r}}, E_y^{\mathrm{r}}) = (1.000, 0.000)$.

for instance when temporal and/or spatial dispersion is taken into account. The dispersion can be modeled with a convolution, for instance $\boldsymbol{d} = \chi_1 * \boldsymbol{e} + \chi_2 * \boldsymbol{e} * \boldsymbol{e}$, where $*$ denotes temporal and/or spatial convolution. Introducing a microscopic and a macroscopic time or space variable and performing a formal multiple scale expansion, it is found that the leading order term of the solution should satisfy $\nabla \times \mathbf{J}\boldsymbol{e} + \partial_t \boldsymbol{d}(\boldsymbol{e}) = \delta\, \mathrm{D}^2\, \boldsymbol{e}$, where D^2 is a second order differential operator in time and/or space. In the case of $\mathrm{D}^2 = \partial_t^2$, *i.e.*, "temporal viscosity", we note that even though we obtain exactly the same analysis for a traveling wave profile as for the term $\nabla^2 \boldsymbol{e}$ used in this paper, this version of the Maxwell equations is noncausal, and very difficult to treat in more than one spatial dimension. A similar system of equations in one dimension is studied as a boundary value problem in [24], and the influence of the noncausality is found to be small when δ is small.

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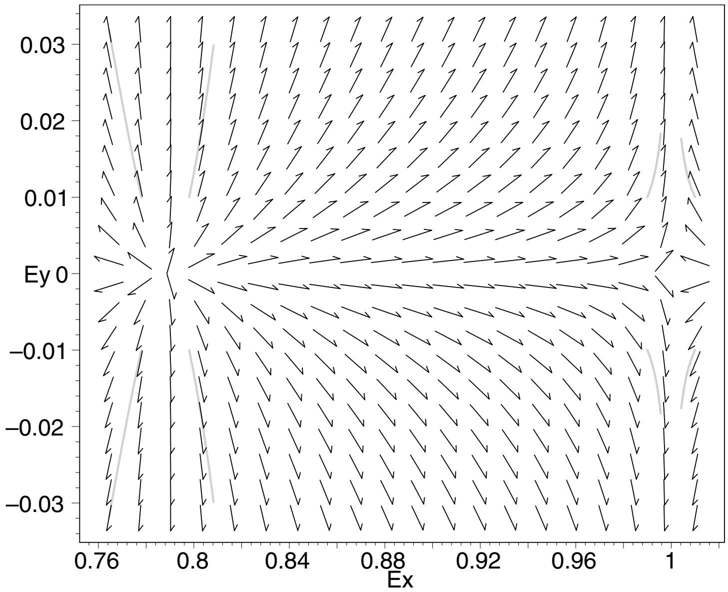

Figure 4: Phase portrait of a slow shock wave structure problem. The critical points are $(E_x^{\mathrm{l}}, E_y^{\mathrm{l}}) = (0.788, 0.000)$ and $(E_x^{\mathrm{r}}, E_y^{\mathrm{r}}) = (1.000, 0.000)$.

valuable discussions on this paper.

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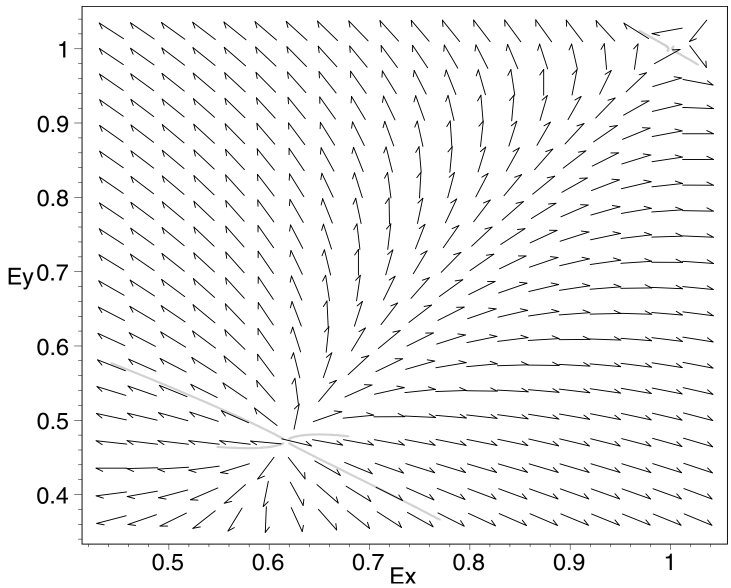

Figure 5: Phase portrait of a slow shock wave structure problem with mixed polarizations. The critical points are $(E_x^{\mathrm{l}}, E_y^{\mathrm{l}}) = (0.617, 0.472)$ and $(E_x^{\mathrm{r}}, E_y^{\mathrm{r}}) = (1.000, 1.000)$.

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