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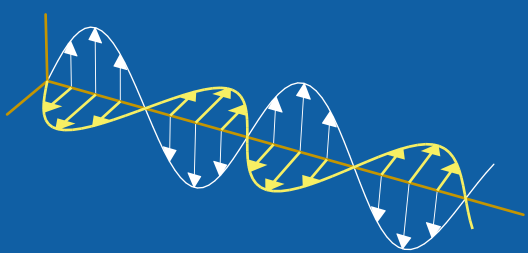

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Abstract

In this paper transient electromagnetic wave propagation in an inhomogeneous, cold plasma is considered. It is assumed that a constant magnetic induction is present and that the plasma is spatially inhomogeneous in the direction of the magnetic induction. Losses in the plasma are modeled with a collision frequency ν. The direct problem, which is to calculate the reflected and transmitted responses of the plasma, is considered in this paper. Special attention is paid to the precursor effects in the plasma and several examples of precursor effects in an inhomogeneous plasma are showed.

1 Introduction

One-dimensional propagation of transient waves in dispersive media has traditionally been treated mostly in the frequency domain. In some cases, e.g. when precursor effects are pronounced, this is probably the only way to treat these problems, see [4, pp. 316–326] and [7], since the stationary phase method and related methods then can be used to obtain approximate solutions. However, there are a number of wave propagation problems in media where these effects are less pronounced, e.g. wave propagation in the ionosphere. In these cases, time domain methods are an alternative to frequency domain methods. Some advantages with time domain methods versus frequency domain methods are that they facilitates the physical interpretation of the results, the numerical implementation is simpler and the solutions obtained in the time domain automatically satisfy causality.

Recently, transient wave propagation in isotropic dispersive media has been treated in the time domain using wave splitting and invariant imbedding techniques, see [1] and [5]. The main purpose for the development of a time domain method for dispersive media was to solve the inverse problem, i.e. finding the memory function which characterizes the medium, from reflection or transmission data. This is equivalent of finding the dispersion relation in the frequency domain. The same method can be applied to direct scattering from isotropic inhomogeneous dispersive media and in some cases to the corresponding inverse problem, see [3].

In this paper scattering of transient electromagnetic fields from gyrotropic inhomogeneous dispersive media is considered. The problem is one-dimensional with the pulse propagating normal to the stratification of the medium. The dispersion is modeled by a generalized Ohm's law where the current density is expressed as a convolution of a conductivity kernel and the electric field. In the direct problem the explicit expression for the conductivity kernel is not crucial. However, in order to have a realistic model, e.g. of the ionosphere, a neutral plasma is considered. The conductivity kernel is then expressed in terms of the electron density and the collision frequency which are functions of depth in the medium. This choice of model is well suited for the inverse problem. Since the medium is anisotropic there are two independent reflected components of the field which are sufficient for the reconstruction of the electron density and the collision frequency. This inverse problem will be treated in a subsequent paper.

This model of the ionosphere is, of course, too simple to be realistic. There are, however, a number of generalizations that can be made in order to make the model more realistic. Thus, the direction and the strength of the magnetic field may vary in the vertical direction. It is reasonable to include the effects of the motion of the ions in the conductivity kernel since they are of importance for the long time behavior of the scattered fields, see [2, p. 4]. An important extension of the method itself is to consider transient waves at oblique incidence. The phenomena that occur at oblique incidence are of course strongly frequency dependent and it is of interest to see how these phenomena show up in a time domain treatment of the problem. These extensions are addressed in a subsequent paper.

In appendix A, an alternative method of solving the direct problem for a homogeneous medium is discussed. Two Volterra equations of the second kind are derived which solve the direct scattering problem and also the inverse scattering problem for a spatially homogeneous conductivity kernel. These Volterra equations are considerably much simpler and faster to solve numerically than the integro-differential equations which solve the inhomogeneous problem.

In a second appendix, appendix B, a reciprocity relationship for gyrotropic media is presented. This relationship is used to show how the transmitted field from an incident field from one side is related to the transmitted field from an incident field from the other side.

2 Gyrotropic medium without restoring force

In this section a generalized Ohm's law is defined in the time domain for a gyrotropic medium.

Consider a neutral plasma characterized by an electron density $N(\boldsymbol{r})$, an effective collision frequency $\nu(\boldsymbol{r})$, and a constant magnetic induction $\boldsymbol{B}_0$. If the positive charges are considered to be heavy and if the restoring force between the electrons and the positive charges is neglected, the polarization of the plasma is governed by the equation of motion for the electrons

$$m\left(\partial_t^2\boldsymbol{r}(t)+\nu(\boldsymbol{r})\partial_t\boldsymbol{r}(t)\right)=q\left(\boldsymbol{E}(\boldsymbol{r},t)+\partial_t\boldsymbol{r}\times\boldsymbol{B}_0\right),$$

where m and q are the mass and the charge of the electron, respectively. The magnetic field is oriented along the z direction, $\boldsymbol{B}_0=\hat{z}B_0$, and is assumed static (i.e. the static field $\boldsymbol{B}_0$ is assumed to be much greater than the induced magnetic field due to the electromagnetic wave). By introducing the definition of current density $\boldsymbol{J}$

$$\boldsymbol{J}(\boldsymbol{r},t)=N(\boldsymbol{r})q\partial_t\boldsymbol{r}$$

the equation of motion becomes

$$\partial_t\boldsymbol{J}(\boldsymbol{r},t)+\nu(\boldsymbol{r})\boldsymbol{J}(\boldsymbol{r},t)+\omega_g\hat{z}\times\boldsymbol{J}(\boldsymbol{r},t)=\omega_p^2(\boldsymbol{r})\epsilon_0\boldsymbol{E}(\boldsymbol{r},t),$$

where the gyrotropic frequency ω_g and the plasma frequency $\omega_p(\boldsymbol{r})$ are defined as

$$\omega_g=\frac{qB_0}{m},\qquad \omega_p^2(\boldsymbol{r})=\frac{N(\boldsymbol{r})q^2}{\epsilon_0 m}.$$

and ϵ_0 is the permittivity of vacuum.

A generalized Ohm's law may be introduced as ($i = 1, 2, 3$ correspond to the x, y and z-directions, respectively)[1]

$$\begin{aligned} J_i(\boldsymbol{r}, t) &= \int_{-\infty}^{t} \Sigma_{ij}(\boldsymbol{r}, t - t') E_j(\boldsymbol{r}, t')\, dt' = (\Sigma_{ij}(\boldsymbol{r}, \cdot) * E_j(\boldsymbol{r}, \cdot))(t) \\ &= (\Sigma_{ij} * E_j)(\boldsymbol{r}, t). \end{aligned} \tag{2.1}$$

Notice the shorthand notation for the convolution integral introduced in this equation. When the Ohm's law is inserted into the equation of motion the following equation is obtained

$$\left(\Sigma_{ij}(\boldsymbol{r}, 0) - \epsilon_0 \omega_p^2(\boldsymbol{r})\delta_{ij}\right) E_j(\boldsymbol{r}, t) + (f_{ij} * E_j)(\boldsymbol{r}, t) = 0,$$

where

$$f_{ij}(\boldsymbol{r}, t) = \partial_t \Sigma_{ij}(\boldsymbol{r}, t) + \nu(\boldsymbol{r})\Sigma_{ij}(\boldsymbol{r}, t) + \omega_g \epsilon_{i3k}\Sigma_{kj}(\boldsymbol{r}, t),$$

and ϵ_{ijk} is the Levi-Civita density. Since the electric field is arbitrary, the conductivity kernel $\Sigma_{ij}(\boldsymbol{r}, t)$ is determined by

$$\begin{cases} \Sigma_{ij}(\boldsymbol{r}, 0) = \epsilon_0 \omega_p^2(\boldsymbol{r})\delta_{ij} \\ \partial_t \Sigma_{ij}(\boldsymbol{r}, t) + \nu(\boldsymbol{r})\Sigma_{ij}(\boldsymbol{r}, t) + \omega_g \epsilon_{i3k}\Sigma_{kj}(\boldsymbol{r}, t) = 0. \end{cases}$$

The unique solution to these equations is

$$\begin{cases} \Sigma_{11}(\boldsymbol{r}, t) = \Sigma_{22}(\boldsymbol{r}, t) = \epsilon_0 \omega_p^2(\boldsymbol{r}) e^{-\nu(\boldsymbol{r})t} \cos \omega_g t \\ \Sigma_{12}(\boldsymbol{r}, t) = -\Sigma_{21}(\boldsymbol{r}, t) = \epsilon_0 \omega_p^2(\boldsymbol{r}) e^{-\nu(\boldsymbol{r})t} \sin \omega_g t \\ \Sigma_{33}(\boldsymbol{r}, t) = \epsilon_0 \omega_p^2(\boldsymbol{r}) e^{-\nu(\boldsymbol{r})t} \\ \Sigma_{13}(\boldsymbol{r}, t) = \Sigma_{23}(\boldsymbol{r}, t) = \Sigma_{31}(\boldsymbol{r}, t) = \Sigma_{32}(\boldsymbol{r}, t) = 0. \end{cases} \tag{2.2}$$

There is no polarization or magnetization of the medium and thus the Maxwell equations are

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -\partial_t \boldsymbol{B} \\ \nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J} + \tfrac{1}{c^2}\partial_t \boldsymbol{E} \end{aligned}$$

where μ_0 is the permeability of vacuum and where $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light in vacuum.

From these equations and the Ohm's law, eq. (2.1), the wave equation in a source free region is obtained

$$\left(\partial_k \partial_k \delta_{ij} - \partial_i \partial_j - \frac{1}{c^2}\partial_t^2 \delta_{ij}\right) E_j(\boldsymbol{r}, t) - \frac{1}{c^2}\partial_t (\Sigma_{ij} * E_j)(\boldsymbol{r}, t) = 0. \tag{2.3}$$

[1]The more general assumption $J_i = aE_i + \epsilon_0 \Sigma_{ij} * E_j$ implies that $a = 0$.

3 Scattering kernels and imbedding equations

In this section we treat one-dimensional wave propagation in a stratified medium. In the region $z < 0$ there is vacuum and in the region $z > 0$ the medium is assumed to be stratified in the vertical direction, i.e. in the z-direction. The medium is characterized by the conductivity kernel introduced in the generalized Ohm's law, eq. (2.1). The electromagnetic field is assumed to be vertically propagating and the following shorthand notation for the electric field will be used

$$E(z,t) = \hat{x}E_1(z,t) + \hat{y}E_2(z,t).$$

Since the z-component of the field is zero, all matrices are 2×2 and will be written boldface without indices. The following short-hand notation for a matrix product and a convolution of two matrices will be used

$$\begin{aligned} \boldsymbol{AB} &= A_{i1}B_{1j} + A_{i2}B_{2j} \\ \boldsymbol{A} * \boldsymbol{B} &= A_{i1} * B_{1j} + A_{i2} * B_{2j}. \end{aligned}$$

All matrices can be shown to commute with each other and the equations can be written formally using this shorthand matrix notation. The wave equation (2.3) simplifies to

$$\partial_z^2 E(z,t) - c^{-2}\partial_t^2 E(z,t) - c^{-2}\partial_t(\boldsymbol{\Sigma} * E)(z,t) = 0.$$

The two coupled second order PDE's can be rewritten as four coupled first order PDE's, which in a matrix notation read

$$\begin{aligned} \partial_z \begin{pmatrix} E(z,t) \\ \partial_z E(z,t) \end{pmatrix} &= c^{-2} \left\{ \begin{pmatrix} \mathbf{0} & c^2\mathbf{1} \\ \mathbf{1}\partial_t^2 & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\Sigma} * \partial_t & \mathbf{0} \end{pmatrix} \right\} \begin{pmatrix} E(z,t) \\ \partial_z E(z,t) \end{pmatrix} \\ &= A \begin{pmatrix} E(z,t) \\ \partial_z E(z,t) \end{pmatrix}, \end{aligned}$$

where $(\boldsymbol{\Sigma} * \partial_t E)(z,t) = \int_{-\infty}^{t} \boldsymbol{\Sigma}(z, t-t')\partial_{t'}E(z,t')\,dt'$, $\mathbf{1}$ is the 2×2 unit matrix and $\mathbf{0}$ is the 2×2 zero matrix.

In the homogeneous region, $z < 0$, the electromagnetic field can be split up in a left-moving and a right-moving part. Based on this splitting, the following more general splitting can be defined which splits the field into two parts even in the inhomogeneous region, $z > 0$

$$E^{\pm}(z,t) = \frac{1}{2}\left[E(z,t) \mp c\partial_t^{-1}\partial_z E(z,t)\right]$$

where $\partial_t^{-1}\partial_z f(z,t) = \int_{-\infty}^{t} \partial_z f(z,t')\,dt'$.

In a matrix notation this change of basis reads

$$\begin{aligned} \begin{pmatrix} E^+(z,t) \\ E^-(z,t) \end{pmatrix} &= \tfrac{1}{2} \begin{pmatrix} \mathbf{1} & -\mathbf{1}c\partial_t^{-1} \\ \mathbf{1} & \mathbf{1}c\partial_t^{-1} \end{pmatrix} \begin{pmatrix} E(z,t) \\ \partial_z E(z,t) \end{pmatrix} \\ &= P \begin{pmatrix} E(z,t) \\ \partial_z E(z,t) \end{pmatrix}. \end{aligned} \tag{3.1}$$

The formal inverse of the operator P is

$$P^{-1} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1}c^{-1}\partial_t & \mathbf{1}c^{-1}\partial_t \end{pmatrix}.$$

The wave equation can now be rewritten in terms of four coupled first order PDE's for the plus and minus fields

$$\begin{aligned} \partial_z \begin{pmatrix} E^+(z,t) \\ E^-(z,t) \end{pmatrix} &= PAP^{-1} \begin{pmatrix} E^+(z,t) \\ E^-(z,t) \end{pmatrix} \\ &= \left\{ c^{-1} \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \partial_t + \frac{1}{2} \begin{pmatrix} -\boldsymbol{\Sigma} & -\boldsymbol{\Sigma} \\ \boldsymbol{\Sigma} & \boldsymbol{\Sigma} \end{pmatrix} \right\} \begin{pmatrix} E^+(z,t) \\ E^-(z,t) \end{pmatrix}. \end{aligned} \tag{3.2}$$

From Duhamel's principle it can be shown that there exists a linear relation between the plus and the minus parts of the electric field

$$E^-(z,t) = (\boldsymbol{R} * E^+)(z,t). \tag{3.3}$$

The kernel $\boldsymbol{R}(z,t)$ is referred to as the reflection matrix kernel. For $z < 0$ the reflection matrix kernel then relates the reflected field to the incident field.

An integro-differential equation, referred to as the imbedding equation, can be obtained for the reflection matrix kernel by utilizing the equations (3.2) and (3.3). Differentiate eq. (3.3) with respect to z and use the dynamics of the fields $E^{\pm}$, eq. (3.2). After lengthy calculations the following equations are obtained:

$$2c\partial_z \boldsymbol{R} - 4\partial_t \boldsymbol{R} = \boldsymbol{\Sigma} + \boldsymbol{\Sigma} * \boldsymbol{R} + \boldsymbol{R} * \boldsymbol{\Sigma} + \boldsymbol{R} * \boldsymbol{\Sigma} * \boldsymbol{R} \tag{3.4}$$

and

$$\boldsymbol{R}(z,0) = \mathbf{0}, \qquad z \geq 0.$$

Assuming unique solvability of these equations gives $R_{11} = R_{22}$ and $R_{12} = -R_{21}$, which also can be concluded from symmetry arguments, and the explicit equations read

$$\begin{aligned} 2c\partial_z R_{11} - 4\partial_t R_{11} &= \Sigma_{11} + (\Sigma_{11} * \cdot)\left[2R_{11} + R_{11} * R_{11} - R_{12} * R_{12}\right] \\ &\quad -2(\Sigma_{12} * \cdot)\left[R_{12} + R_{11} * R_{12}\right] \\ 2c\partial_z R_{12} - 4\partial_t R_{12} &= \Sigma_{12} + 2(\Sigma_{11} * \cdot)\left[R_{12} + R_{11} * R_{12}\right] \\ &\quad +(\Sigma_{12} * \cdot)\left[2R_{11} + R_{11} * R_{11} - R_{12} * R_{12}\right], \end{aligned} \tag{3.5}$$

If the medium is of finite length, L, the R-kernels satisfy the boundary conditions

$$R_{11}(L,t) = R_{12}(L,t) = 0.$$

The transmitted field through a medium of finite length L is related to the incident field as follows

$$E^+(L, t + L/c) = \boldsymbol{D}(0)E^+(0, t) + (\boldsymbol{T} * E^+)(0, t).$$

This representation can be generalized to the region $0 < z < L$ as

$$E^+(L, t + (L - z)/c) = \boldsymbol{D}(z)E^+(z, t) + (\boldsymbol{T} * E^+)(z, t). \tag{3.6}$$

As was the case for the reflection operator the above representation can be justified by the Duhamel's principle.

Differentiate eq. (3.6) with respect to z and use the dynamics of the fields $E^\pm$, eq. (3.2). After lengthy calculations the following imbedding equations are obtained:

$$2c\partial_z \boldsymbol{T} = \boldsymbol{D}\boldsymbol{\Sigma} + \boldsymbol{D}\boldsymbol{\Sigma} * \boldsymbol{R} + \boldsymbol{T} * \boldsymbol{\Sigma} + \boldsymbol{T} * \boldsymbol{\Sigma} * \boldsymbol{R}. \tag{3.7}$$

Any discontinuity in $\boldsymbol{T}(z, t)$ along $t = d_i(z) > 0$ has to satisfy $d_i'(z) = 0$. The matrix $\boldsymbol{D}(z)$ satisfies

$$\partial_z \boldsymbol{D}(z) = 0.$$

The matrix $\boldsymbol{D}$ thus is the unit matrix

$$\boldsymbol{D} = \mathbf{1},$$

since $\boldsymbol{D}(L) = \mathbf{1}$. From the symmetries in the conductivity kernel $\boldsymbol{\Sigma}$ and the reflection matrix kernel $\boldsymbol{R}$ and from eq. (3.7) it follows that $T_{11} = T_{22}$ and $T_{12} = -T_{21}$. This also follows from the axial symmetry. The transmission kernels T_{11} and T_{12} then satisfy

$$\begin{aligned}
2c\partial_z T_{11} &= \Sigma_{11} + (\Sigma_{11} * \cdot)\left[R_{11} + T_{11} + T_{11} * R_{11} - T_{12} * R_{12}\right] \\
&\quad -(\Sigma_{12} * \cdot)\left[R_{12} + T_{12} + T_{11} * R_{12} + T_{12} * R_{11}\right] \\
2c\partial_z T_{12} &= \Sigma_{12} + (\Sigma_{11} * \cdot)\left[R_{12} + T_{12} + T_{11} * R_{12} + T_{12} * R_{11}\right] \\
&\quad +(\Sigma_{12} * \cdot)\left[R_{11} + T_{11} + T_{11} * R_{11} - T_{12} * R_{12}\right].
\end{aligned} \tag{3.8}$$

4 The Green function approach

In the section above, the reflected or transmitted fields were obtained using invariant imbedding technique. To obtain the internal fields a method which is not based on the invariant imbedding technique will be used. The basis for this method is the following representation, which can be verified from the Duhamel's principle,

$$\begin{pmatrix} E^+ \\ E^- \end{pmatrix}(z, t + z/c) = \begin{pmatrix} E^+ \\ 0 \end{pmatrix}(0, t) + \begin{pmatrix} \boldsymbol{G}^+(z, t) * E^+(0, t) \\ \boldsymbol{G}^-(z, t) * E^+(0, t) \end{pmatrix} \tag{4.1}$$

Here $E^+(z,t+z/c)$ and $E^-(z,t+z/c)$ are the splitted fields inside the medium, i.e. $E^+(z,t+z/c)$ and $E^-(z,t+z/c)$ are related to $E(z,t+z/c)$ and $\partial_z E(z,t+z/c)$, by eq. (3.1), and the kernels $\boldsymbol{G}^+$ and $\boldsymbol{G}^-$ are the Green functions for the time dependent problem. It should be noted that time t is measured from the wavefront. It is straightforward to derive equations for the Green functions using a similar technique as in the previous section.

The z-derivative of (4.1) reads

$$\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} (z,t+z/c) + \frac{1}{c}\partial_t \begin{pmatrix} E^+ \\ E^- \end{pmatrix} (z,t+z/c) = \begin{pmatrix} \boldsymbol{G}_z^+(z,t) * E^+(0,t) \\ \boldsymbol{G}_z^-(z,t) * E^+(0,t) \end{pmatrix}$$

By using the dynamics for $E^+(z,t+z/c)$ and $E^-(z,t+z/c)$, eq. (3.2), the following system of PDE's is obtained

$$-\frac{1}{c}\left(\partial_t + \frac{1}{2}\begin{pmatrix} \boldsymbol{\Sigma}* & -\boldsymbol{\Sigma}* \\ -\boldsymbol{\Sigma}* & \boldsymbol{\Sigma}* \end{pmatrix}\right)\left[\begin{pmatrix} E^+ \\ 0 \end{pmatrix}(0,t) + \begin{pmatrix} \boldsymbol{G}^+(z,t) * E^+(0,t) \\ -\boldsymbol{G}^-(z,t) * E^+(0,t) \end{pmatrix}\right] +$$
$$+\frac{1}{c}\begin{pmatrix} E_t^+ \\ 0 \end{pmatrix} + \frac{1}{c}\partial_t \begin{pmatrix} \boldsymbol{G}^+(z,t) * E^+(0,t) \\ \boldsymbol{G}^-(z,t) * E^+(0,t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{G}_z^+(z,t) * E^+(0,t) \\ \boldsymbol{G}_z^-(z,t) * E^+(0,t) \end{pmatrix}.$$

Since the incoming field, $E^+(0,t)$, is an arbitrary function of t, the Green functions have to satisfy the following integro-differential equations

$$\begin{aligned} \boldsymbol{G}_z^+ &= -\frac{1}{2c}\left(\boldsymbol{\Sigma} + \boldsymbol{\Sigma} * \left(\boldsymbol{G}^+ + \boldsymbol{G}^-\right)\right) \\ \boldsymbol{G}_z^- - \frac{2}{c}\boldsymbol{G}_t^- &= \frac{1}{2c}\left(\boldsymbol{\Sigma} + \boldsymbol{\Sigma} * \left(\boldsymbol{G}^+ + \boldsymbol{G}^-\right)\right), \end{aligned}$$

and the initial conditions

$$\begin{aligned} \boldsymbol{G}^-(z,0) &= 0 \\ \boldsymbol{G}^+(z,0) &= -\frac{1}{2c}\int_0^z \boldsymbol{\Sigma}(0,z')dz'. \end{aligned}$$

Thus the Green functions satisfy a system of four coupled first order PDE's. Unlike the imbedding equations for the reflection and transmission kernels, the equations for the Green functions do not contain any double convolutions. Numerically, these equations are then one order faster to solve than the imbedding equations. It should be noted that the boundary values of the Green functions are

$$\begin{aligned} \boldsymbol{G}^+(0,t) &= 0 \\ \boldsymbol{G}^-(0,t) &= \boldsymbol{R}(0,t) \\ \boldsymbol{G}^+(L,t+L/c) &= \boldsymbol{T}(0,t) \\ \boldsymbol{G}^-(L,t) &= 0. \end{aligned}$$

A limitation of the Green function technique is that numerically it requires the storage of gridpoints in both time and space. Thus $N \times N$ arrays have to be stored whereas in the numerical solution of the imbedding equations for the reflection and transmission kernels only $N \times 1$ arrays have to be stored.

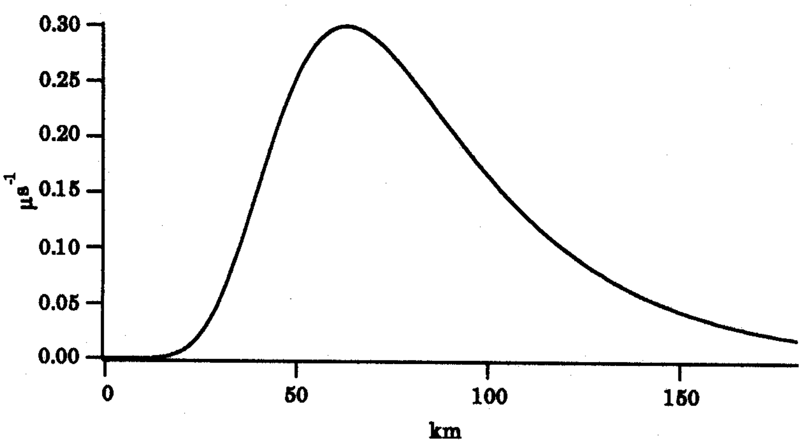

Figure 1: The plasma frequency given by the Chapman profile.

5 Numerical examples

In the numerical examples presented in this section it was necessary to use around 1000 points in both the $z-$ and $t-$discretizations. The Green function technique would require on the order of 30 MB of internal memory and thus only the invariant imbedding technique has been used.

A straightforward discretization of the equations for the reflection and transmission kernels, eqs. (3.5) and (3.8) is obtained by an application of the trapezoidal rule. The equations are integrated along their characteristics, i.e. along $2z/c-t =$constant and $z =$constant, respectively, by the trapezoidal rule. The stepsize in z and in t are then coupled to each other by $\Delta_z = \Delta_t c/2$. As shown in the first example below, the coupling of the discretization in space and time sometimes leads to time consuming numerical calculations.

In the first example the $z-$dependence of the conductivity kernel is chosen according to the Chapman law, cf. [2, p. 9]. The Chapman law is a simplified model of the ionosphere where the electron density is given by

$$N(z) = N_0 \exp\{\frac{1}{2}(1 - \frac{(z-z_0)}{H} - \exp(\frac{z_0 - z}{H})\}.$$

In this example, the reference height, H, is choosen to be $10\,km$, the height for the maximum electron density, z_0, is $115\,km$ and the maximum electron density, N_0, is $2.8 \times 10^{11}\,m^{-3}$.

The corresponding plasma frequency as a function of z is shown in figure 1. A constant collision frequency, $\nu = 100\,Hz$, is assumed and the gyrotropic frequency is $\omega_g = 10^6\,rad/s$ which is in accordance with the earth magnetic induction. There are two time-scales involved in this problem. The first time-scale is defined by the period of the gyrotropic frequency and the other time-scale is defined by the time it takes for the wave-front to pass through the main part of the medium. The time-scales then differ with three orders of magnitude and it is the smallest of these scales that will determine the step-size in both time and space.

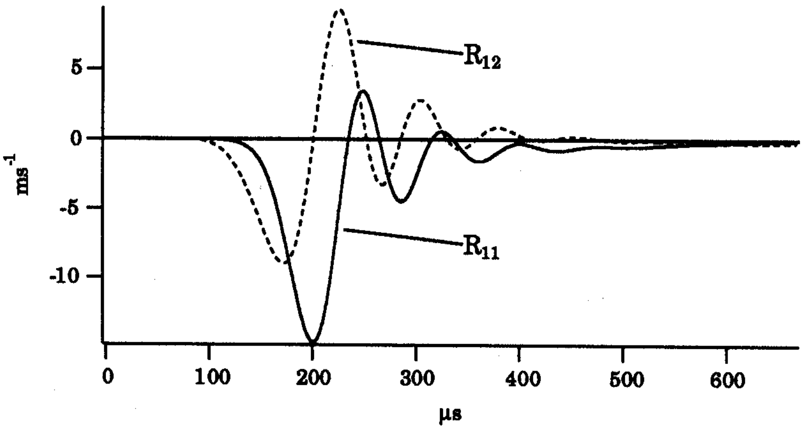

Figure 2: The reflection kernels.

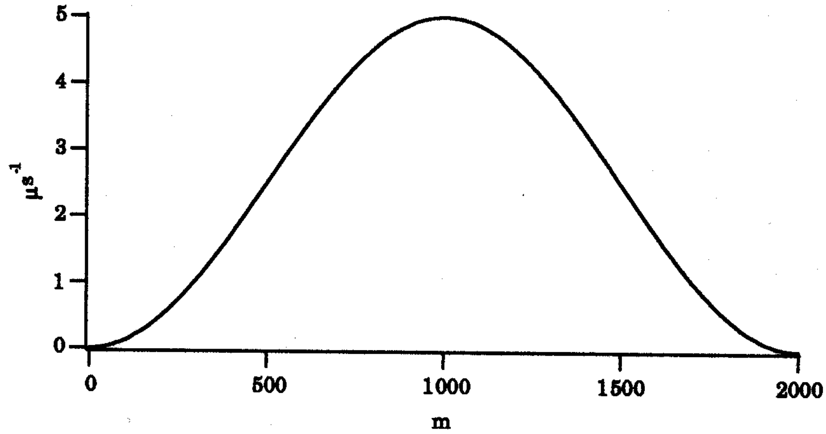

Figure 3: The plasma frequency $\omega_p(z) = 2.5 \times 10^6 - 2.5 \times 10^6 \cos(2\pi z/L)\, rad/s$.

As expected the kernels reflect the smooth behavior of the profile. However, when the curves are magnified it is possible to see a signal, superimposed on the smooth curve, that oscillates with the gyrotropic frequency. From figure 2 it is seen that the reflection kernels R_{11} and R_{12} are out of phase. This can be understood from eq. (3.5) where it is seen that the time derivative of the reflection R_{12} has the same source term as R_{11}.

It is likely that precursor effects can occur in a gyrotropic medium since it has the same structure as a Lorentz medium, cf. [4]. The first precursor is essentially determined by the initial values of the conductivity kernel and its first derivative, whereas the second precursor is caused by the internal resonance frequency of the medium, i.e in this case the gyrotropic frequency. In the second example the magnitude of the material parameters and the length of the medium were chosen so that both the first and the second precursor could be seen in the transmission kernels.

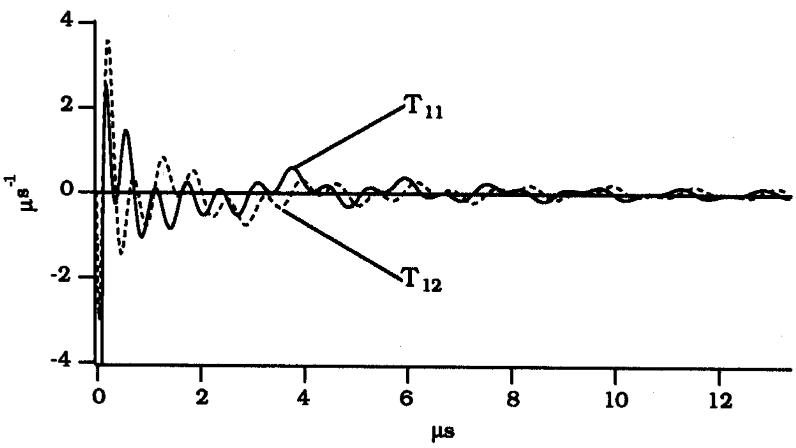

Figure 4: The transmission kernels.

The plasma frequency varies like

$$\omega_p(z) = 2.5 \times 10^6 - 2.5 \times 10^6 \cos(2\pi z/L)\ rad/s$$

where $L = 2\,km$, as seen in figure 3, and the gyrotropic frequency is $\omega_g = 5 \times 10^6\,rad/s$.

The values are realistic for the ionosphere, i.e. it should be possible to see both the first and the second precursor in a time domain experiment. As seen in figure 4, the transmission kernel first has an oscillatory behavior, where the frequency of the oscillations decreases with time. This part of the transmission kernel corresponds to the first precursor. In the case of a homogeneous plasma it is approximately equal to $\omega_p\sqrt{\frac{L}{2ct}}J_1(\omega_p\sqrt{\frac{2Lt}{c}})$, where L is the lenghth of the medium and J_1 is the Bessel function of the first order. After a certain time there is a second oscillation of the transmission kernel that appears. It corresponds to the second precursor and has a less rapid variation then the first precursor. According to [4] the second precursor is delayed a time $t_s \simeq \frac{L}{2c}(\frac{\omega_p}{\omega_g})^2 \simeq 3\mu s$ relative to the wave-front which is in good agreement with figure 4.

6 Conclusions

In the present paper, reflection and transmission from a gyrotropic medium are analyzed. A constitutive relation, based upon the equation of motion for the electrons, is introduced and it is seen that in the time domain the medium is characterized by a conductivity kernel which acts as a memory function for the medium. The reflection and transmission kernels for the plasma are calculated using the wave splitting and the invariant imbedding technique. The solution is implicitly given as coupled integro-differential equations for the scattering kernels. Numerical solutions to these equations are presented for a simplified model of the ionosphere and also to a denser plasma with a large magnetic induction where precursor effects appear.

An alternative method to the invariant imbedding method is also discussed in this paper. In this method coupled PDE's for the Green functions are obtained. This method is used in a subsequent paper to obtain the internal fields in a gyrotropic medium at oblique incidence, and where also the precursors in the ionosphere are addressed in more detail.

Appendix A The homogeneous plasma

In this appendix it is shown that the reflection and transmission kernels for a homogeneous plasma, i.e. where ω_p and ν are constant, can be obtained by solving two Volterra equations of the second kind. Numerically, these Volterra equations are one order faster to solve than the integro-differential equations. The equations are derived by a Laplace transformation of eqs. (3.4) and (3.7).

For a plasma of finite length, L, one round-trip, τ, is defined as the time it takes for the wave-front to go back and forth once in the medium, i.e. $\tau = 2L/c$. For a homogeneous plasma the reflection matrix kernel is independent of z for times less than one round-trip and thus the equation for the reflection matrix kernel, eq. (3.4) can be integrated in time to give

$$4\boldsymbol{R} + \boldsymbol{G} + \boldsymbol{G} * \boldsymbol{R} + \boldsymbol{R} * \boldsymbol{G} + \boldsymbol{R} * \boldsymbol{G} * \boldsymbol{R} = 0, \quad 0 < t < \tau, \tag{A.1}$$

where $\boldsymbol{G} = \partial_t^{-1}\boldsymbol{\Sigma}$. The matrices $\boldsymbol{R}$ and $\boldsymbol{G}$ are here assumed to commute, which can be verified for the case of a gyrotropic medium. To obtain the reflection matrix kernel for times larger than one round-trip eq. (3.4) is Laplace transformed,

$$2c\partial_z\tilde{\boldsymbol{R}} = \tilde{\boldsymbol{\Sigma}} + 2\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{R}} + \tilde{\boldsymbol{R}}\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{R}} + 4s\tilde{\boldsymbol{R}}.$$

A tilde over a character denotes the Laplace transform, i.e. $\tilde{\boldsymbol{R}}(s)$ is the Laplace transform of the matrix $\boldsymbol{R}(t)$ where s is the variable of the transform. The formal solution to this equation at $z = 0$ reads

$$\tilde{\boldsymbol{R}}(s) = \tilde{\boldsymbol{r}}(s)[1 - \exp(-\tilde{\boldsymbol{\Phi}}(s) - s\tau)][1 - \tilde{\boldsymbol{r}}(s)^2 \exp(-\tilde{\boldsymbol{\Phi}}(s) - s\tau)]^{-1},$$

where

$$\tilde{\boldsymbol{\Phi}}(s) = s\tau((1 + \tilde{\boldsymbol{G}}(s))^{\frac{1}{2}} - 1). \tag{A.2}$$

and

$$\tilde{\boldsymbol{r}}(s) = [1 - (1 + \tilde{\boldsymbol{G}}(s))^{\frac{1}{2}}][1 + (1 + \tilde{\boldsymbol{G}}(s))^{\frac{1}{2}}]^{-1}. \tag{A.3}$$

The interpretation of the formal expressions is then given by a Taylor series expansion. By a formal expansion, $\tilde{\boldsymbol{R}}(s)$ can be written as

$$\tilde{\boldsymbol{R}}(s) = \tilde{\boldsymbol{r}}(s) + \sum_{n=0}^{\infty}((\tilde{\boldsymbol{r}}(s))^2\tilde{\boldsymbol{v}}(s) - \tilde{\boldsymbol{v}}(s))(\tilde{\boldsymbol{r}}(s)\tilde{\boldsymbol{r}}(s)\tilde{\boldsymbol{v}}(s))^n \exp(-(n+1)s\tau),$$

where

$$\tilde{\boldsymbol{v}}(s) = \tilde{\boldsymbol{r}}(s)\exp(-\tilde{\boldsymbol{\Phi}}(s)).$$

By introducing

$$\tilde{\boldsymbol{e}}(s) = \exp(-\tilde{\boldsymbol{\Phi}}(s)/2) - 1,$$

$\tilde{\boldsymbol{v}}(s)$ can be written as

$$\tilde{\boldsymbol{v}}(s) = \tilde{\boldsymbol{v}}(s)(\tilde{\boldsymbol{e}}(s))^2 + 2\tilde{\boldsymbol{r}}(s)\tilde{\boldsymbol{e}}(s) + \tilde{\boldsymbol{r}}(s).$$

In the time domain the reflection matrix kernel thus has the following series expansion

$$\boldsymbol{R}(0,t) = \boldsymbol{r}(t) + \sum_{n=0}^{\infty}[(\boldsymbol{r}*\boldsymbol{r}*\boldsymbol{v} - \boldsymbol{v})(*\boldsymbol{r}*\boldsymbol{v})^n](t-(n+1)\tau), \tag{A.4}$$

where $\boldsymbol{r}(t)$ and $\boldsymbol{v}(t)$ are the inverse Laplace transforms of $\tilde{\boldsymbol{r}}(s)$ and $\tilde{\boldsymbol{v}}(s)$, respectively, and where the terms on the right hand side are understood to be zero for negative arguments. The series expansion then corresponds to a multiple scattering expansion. From eq. (A.4) it is obvious that $\boldsymbol{r}(t)$ is the reflection matrix kernel for times less than one round-trip. This can also be seen by deriving eq. (A.1) from the expression in eq. (A.3). Thus $\boldsymbol{r}(t)$ satisfies the equation

$$4\boldsymbol{r} + \boldsymbol{G} + 2\boldsymbol{G}*\boldsymbol{r} + \boldsymbol{r}*\boldsymbol{G}*\boldsymbol{r} = 0. \tag{A.5}$$

The matrix $\boldsymbol{v}(t)$ is obtained from

$$\boldsymbol{v} = \boldsymbol{r}*\boldsymbol{e}*\boldsymbol{e} + 2\boldsymbol{r}*\boldsymbol{e} + \boldsymbol{r}. \tag{A.6}$$

The matrix $\boldsymbol{e}(t)$ satisfies the following Volterra equation of the second kind

$$2t\boldsymbol{e}(t) + [\boldsymbol{f}*\boldsymbol{e}](t) + \boldsymbol{f}(t) = 0, \tag{A.7}$$

where

$$\boldsymbol{f}(t) = t\boldsymbol{\Phi}(t).$$

and $\boldsymbol{\Phi}(t)$ is the inverse Laplace transform of $\tilde{\boldsymbol{\Phi}}(s)$, which can be written as

$$\boldsymbol{\Phi}(t) = \frac{\tau}{2}\{(\boldsymbol{\Sigma}*\boldsymbol{r})(t) + \boldsymbol{\Sigma}(t)\}$$

as seen from eqs. (A.2) and (A.3). The Volterra equation (A.7) is easy to derive by noticing that

$$2\partial_s\tilde{\boldsymbol{e}}(s) = -(\partial_s\tilde{\boldsymbol{\Phi}}(s))\exp(-\tilde{\boldsymbol{\Phi}}(s)/2) = -(\partial_s\tilde{\boldsymbol{\Phi}}(s))\tilde{\boldsymbol{e}}(s) - \partial_s\tilde{\boldsymbol{\Phi}}(s)$$

and using that the inverse Laplace transform of $\partial_s\tilde{\boldsymbol{\Phi}}(s)$ is $-t\boldsymbol{\Phi}(t)$.

It is seen that the reflection matrix kernel can be obtained by solving two Volterra equations of the second kind, eqs. (A.5) and (A.7), and by performing the convolutions and summations in eqs. (A.6) and (A.4).

The Laplace transform of the equation for the transmission kernels, eq. (3.7), reads

$$2c\partial_3\tilde{\boldsymbol{T}} = \tilde{\boldsymbol{\Sigma}}(1+\tilde{\boldsymbol{R}})(1+\tilde{\boldsymbol{T}}).$$

If $\tilde{\boldsymbol{R}}$ and $\tilde{\boldsymbol{T}}$ commute, the formal solution to this equation at $z=0$ reads

$$\tilde{\boldsymbol{T}}(0,s) = -\tilde{\boldsymbol{r}}(s)\tilde{\boldsymbol{R}}(s) + \tilde{\boldsymbol{e}}(s)(1-\tilde{\boldsymbol{r}}(s)\tilde{\boldsymbol{R}}(s))$$

and thus in the time domain

$$\boldsymbol{T}(0,t) = \boldsymbol{e}(t) - [\boldsymbol{r} * \boldsymbol{R}](t) - [\boldsymbol{e} * \boldsymbol{r} * \boldsymbol{R}](t).$$

Appendix B A reciprocity relationship

A gyrotropic medium is in general not reciprocal. It is, however, possible to prove a reciprocity relationship by introducing a complementary medium. This is the same idea that is used in [6] for bianisotropic media. The reciprocity theorem will then be used to prove that the transmission kernel for an incident wave propagating in the positive $z-$direction is identical to the transmission kernel for an incident wave traveling in the negative $z-$direction.

Let $\{\boldsymbol{E},\boldsymbol{H}\}$ be the electromagnetic field satisfying the Ohm's law, eq. (2.1), and let $\{\boldsymbol{E}^c,\boldsymbol{H}^c\}$ be an independent electromagnetic field satisfying the Ohms law for a complementary medium

$$J_i^c = \Sigma_{ji} * E_j^c \tag{B.1}$$

i.e. with the transpose of the original conductivity kernel. This corresponds to reversing the direction of the magnetic induction $\boldsymbol{B}_0$. Both media are otherwise identical and the conductivity kernel is given by eq. (2.2). From the Maxwell equations, the Ohm's laws given by eqs. (2.1) and (B.1) for the medium and the complementary medium, respectively, and by use of Gauss' law the following reciprocity relationship is seen to hold

$$\iint_S \int_0^t \{\epsilon_{ijk}(E_j^c(t')H_k(t-t') + H_j^c(t')E_k(t-t'))\}dt'\hat{n}_i dS = 0. \tag{B.2}$$

The surface S is closed and $\hat{n}_i$ is its outward pointing normal.

Now, let the medium be stratified in the z-direction and extend from $z=0$ to $z=L$. Outside the medium there is vacuum. Let $\boldsymbol{E}(z,t)$ be the total field from an incident field propagating in the negative $z-$direction for $z>L$, i.e. from $\boldsymbol{E}^-(z,t) = \hat{x}E_x^-(L,t+(z-L)/c)$, and let $\boldsymbol{E}^c(z,t)$ be the total field for the complementary medium from an incident field propagating in the positive $z-$direction for $z<0$, i.e. from $\boldsymbol{E}^{c+}(z,t) = \hat{x}E_x^{c+}(0,t-z/c)$. Both incident fields are assumed to impinge on the slab at time $t = 0$, i.e. $\boldsymbol{E}(L,t) = \boldsymbol{E}^c(0,t) = 0$ for $t < 0$. If the surface S is choosen as the surface of a straight cylinder extending from $z = 0$ to $z = L$ the reciprocity theorem implies

$$\epsilon_{3jk}\{[E_j^c * H_k](0,t) + [H_j^c * E_k](0,t)\} = \epsilon_{3jk}\{[E_j^c * H_k](L,t) + [H_j^c * E_k](L,t)\}$$

By differentiating this equation with respect to t and using

$$\partial_t H_i(z,t) = -\frac{1}{\mu_0}\epsilon_{i3k}E_{k,3}(z,t),$$

where $E_{k,3}(z,t) = (\partial_z E_3)(z,t)$ one gets

$$\begin{aligned} &[E_1^c * E_{1,3}](0,t) + [E_2^c * E_{2,3}](0,t) - [E_{1,3}^c * E_1](0,t) - [E_{2,3}^c * E_2](0,t) = \\ &[E_1^c * E_{1,3}](L,t) + [E_2^c * E_{2,3}](L,t) - [E_{1,3}^c * E_1](L,t) + [E_{2,3}^c * E_2](L,t) \end{aligned} \tag{B.3}$$

Now

$$\begin{cases} E_i^c(0,t) = E_i^{c+}(0,t) + E_i^{c-}(0,t) \\ E_i^c(L,t) = E_i^{c+}(L,t) = E_i^{c+}(0,t-L/c) + [T_{ij}^{c+} * E_j^{c+}](0,t-L/c) \\ E_i(L,t) = E_i^{+}(L,t) + E_i^{-}(L,t) \\ E_i(0,t) = E_i^{-}(0,t) = E_i^{-}(L,t-L/c) + [T_{ij}^{-} * E_j^{-}](L,t-L/c) \end{cases}$$

where $T_{ij}^{c+}(t)$ is the physical transmission kernel for the complementary medium for an incident field propagating in the positive $z-$direction and $T_{ij}^{-}(t)$ is the physical transmission kernel for the medium for an incident wave propagating in the negative $z-$direction. By inserting these representations in eq. (B.3) and using that for $z = 0$ and $z = L$

$$\begin{aligned} E_{i,3}^{+} &= -\tfrac{1}{c}\partial_t E_i^{+} \\ E_{i,3}^{-} &= \tfrac{1}{c}\partial_t E_i^{-}, \end{aligned}$$

it follows that

$$T_{ij}^{-}(t) = T_{ji}^{c+}(t).$$

From the imbedding equation for the transmission kernel, eq. (3.6), and for the reflection matrix kernel (3.3) it follows that that reflection matrix kernel and transmission matrix kernel for the complementary medium is equal to the transpose of the corresponding kernels for the medium, i.e.

$$\begin{aligned} R_{ij}^{c+}(0,t) &= R_{ji}^{+}(0,t) \\ T_{ij}^{c+}(0,t) &= T_{ji}^{+}(0,t) \end{aligned}$$

and thus

$$T_{ij}^{-}(t) = T_{ij}^{+}(t).$$

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