A Note on Generator Reuse in the Component Codes of Low Rate Turbo Codes

Leanderson, Carl Fredrik; Hokfelt, Johan; Edfors, Ove; Maseng, Torleiv

Published in:
Proc. International Symposium on Turbo codes

2000

A Note on Generator Reuse in the Component Codes of Low Rate Turbo Codes

Carl Fredrik Leanderson, Johan Hokfelt, Ove Edfors and Torleiv Maseng

Dept. of Applied Electronics, Lund University, PO Box 118, SE-221 00 LUND, Sweden
Phone: +46 (0)46 222 30 21, Fax: +46 (0)46 222 47 18
E-mail: Carl_Fredrik.Leanderson@tde.lth.se

Abstract: *In this paper we study the design of low-rate Turbo codes efficient at very low signal-to-noise ratios (SNRs). The alternative to use low-rate component codes where repeated use of some generator polynomials is allowed is compared to the case when component code generator polynomials are restricted to be unique. It is observed that correlation between the extrinsic information of consecutively transmitted bits, which has a negative impact on the performance of the iterative Turbo decoding algorithm, is higher for component codes with repeated generator polynomials than for component codes with unique generator polynomials. It is concluded that correlation of the extrinsic information should be considered as a complement to distance criteria in order to identify component codes that yield efficient low rate turbo codes in the region of very low SNRs.*

Keywords - low-rate Turbo codes, generator polynomials, iterative decoding, extrinsic information.

1. INTRODUCTION

Turbo codes [1], good for relatively high signal-to-noise ratios (SNRs) can be derived for a wide range of code rates using component code search criteria based on distance arguments [2, 3]. However, for the additive white gaussian noise (AWGN) channel, in the region of SNRs where Turbo codes are most efficient, distance criteria do not always produce the most powerful Turbo codes. For instance, when low-rate component encoders are obtained by repeated use of generator polynomials good distance properties can be achieved but, in general, component codes where the generator polynomials are restricted to be unique yield better performance in the waterfall region of SNR [3, 6], cf Figure 1.

In order to find a design methodology for low-rate Turbo codes with high performance at very low SNRs there is a need to extend the previously proposed distance based criteria.

In this paper we adopt three different perspectives on the design of low-rate Turbo codes: 1) distance properties, 2) extrinsic information transfer (EIT) characteristics and 3) correlation of the extrinsic information. The tangential sphere bound [4] is used to derive upper bounds on the maximum likelihood (ML) decoding performance of different Turbo codes with random interleavers. The upper limits on the ML-decoding performances provided by the tangential sphere bounds indicate that the best ML-

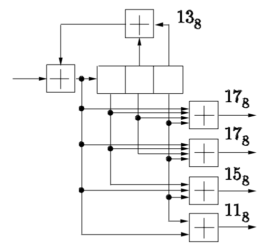

Figure 1: Low-rate component encoder with repeated use of generator polynomial 17_8.

decoding performance in the whole range of SNRs is achieved with component codes defined by repeated generator polynomials. However, the iterative decoding algorithm of Turbo codes is not ML-decoding and therefore these results should not be taken too far.

The application of EIT analysis [5] indicates that it occurs a loss of mutual information between the transmitted bit and the extrinsic values, output from an a posteriori probability (APP) [8] decoding step, when repeated- instead of unique generator polynomials are used in the component encoders. Further, the correlation between the extrinsic information of consecutively transmitted bits is observed to be higher for codes with repetition of generator polynomials.

Correlation between the extrinsic information of consecutively transmitted bits violates the assumption of independent a priori information inherent in the Turbo decoding algorithm. This is taken as an explanation to the performance degradation arising in the waterfall region of SNRs when using repeated generator polynomials in the component codes.

2. DISTANCE CRITERIA

We consider a Turbo encoder structure with two component encoders and an intermediate interleaver. While most previous work has been focused on low bit-error rates (BERs) in the error-floor region [2, 6], we focus on the frame-error rate (FER) in the waterfall region. In [3] it was suggested that codes suitable for these requirements should be non-systematic with no repetition of generator polynomials. It was seen that the minimum distances corresponding to low-weight input sequences, cf Table 1, did not determine the performance in the waterfall region. An impor-

	$g_1,g_2,\ldots,g_n$	d_2,N_2	d_3,N_3	d_4,N_4	d_5,N_5	d_6,N_6
A	17,15,11	18,1	9,1	10,1	13,2	14,3
B	17,17,15	18,1	11,2	8,1	13,6	10,1
C	17,16,15,11	22,1	12,1	12,1	18,2	16,1
D	17,17,15,11	24,1	13,1	12,1	17,2	16,1

Table 1: Minimum weight of code sequences and their multiplicities, (d_i, N_i), $2 \le i \le 6$, for convolutional codes with memory 3, rate 1/3 and 1/4 and feedback polynomial 13_8.

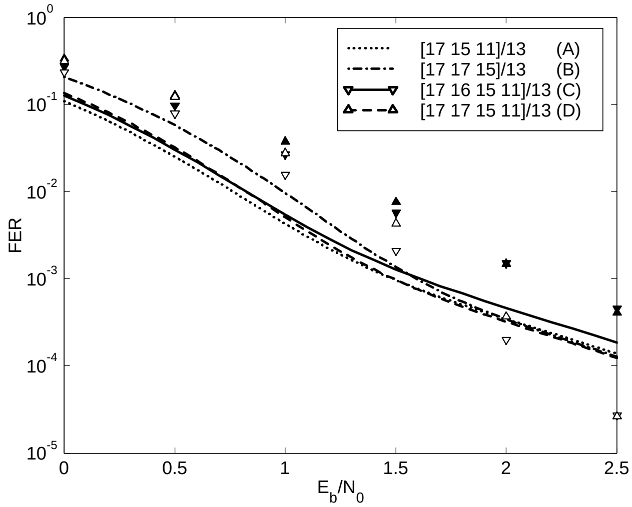

Figure 2: Bounds and simulations for Turbo codes with interleaver length 100 bits using the component codes in Table 1. Filled- and white faced markers correspond to Turbo codes with random- and correlation-designed interleavers respectively.

tant property for this region of SNRs was rather the multiplicity of the larger distance codewords.

Under ML-decoding the best asymptotic FER performance is obtained using the component codes with the best weight distribution properties, i.e. the component codes with repeated generator polynomials. However, with low-rate Turbo codes, the most important performance improvement is achieved below the cut-off rate. In order to upper bound the ML-decoding performance of Turbo codes with component codes according to Table 1, we use the tangential sphere bound [4] since it, in contrast to the union bound, provides a useful upper bound on the error performance for ML-decoding also below the cut-off rate of the channel. Due to space limitations, the equations of the tangential sphere bound will not be restated here.

Using the low-rate component codes in Table 1 with uniform interleaver of blocksize 100 bits we obtain rate 1/6 and rate 1/8 Turbo codes with performance bounds according to Figure 2.

For rate 1/3 component codes, code A results in the lowest bound on the FER performance below 1.5 dB. However, asymptotically, component code B, having the best minimum distance properties according to Table 1, yields the best ML-performance. Similar observations are made for the rate 1/4 component codes C and D. Accordingly, simulation results for Turbo codes with random- as well as correlation-designed [7] interleavers indicate that the use of component codes with repeated generator polynomials, having the best minimum distance properties, gives asymptotically good performance. However, there is a large region of SNRs of practical interest where it is possible to improve the FER performance choosing component codes defined by unique- instead of repeated generator polynomials. In Figure 2, simulations of a Turbo code using component code D, with good distance properties but repeated generator polynomials, yield worse performance up to 2.5 dB in spite of the fact that the upper bound indicates the opposite relation. Due to this inconsistency there is a need for a modified design criterion.

3. ITERATIVE DECODING ISSUES

3.1. Turbo Decoding

The symbolwise log-likelihood ratio $\Lambda_t^{(k)}$ of the t^{th} output at the k^{th} decoding step from an APP decoder [8] for a non-systematic Turbo code is calculated according to

$$\begin{aligned}\Lambda_t^{(k)} &= \ln \frac{\Pr\{u_t = 1|R_1^N\}}{\Pr\{u_t = 0|R_1^N\}} \quad (1)\\ &= \ln \frac{\sum_m \sum_{m'} \alpha_{t-1}^{(k)}(m')\gamma_t^{(k),1}(m',m)\beta_t^{(k)}(m)}{\sum_m \sum_{m'} \alpha_{t-1}^{(k)}(m')\gamma_t^{(k),0}(m',m)\beta_t^{(k)}(m)}\\ &= L_{a,t}^{(k)} + L_{e,t}^{(k)}.\end{aligned}$$

At the end of the iterative decoding process, the value of $\Lambda_t^{(k)}$ is compared with a threshold value in order to give a decision about the transmitted bit at position t.

The $\alpha_t^{(k)}, \beta_t^{(k)}$ and $\gamma_t^{(k)}$ values are produced during the forward/backward recursions according to

$$\gamma_t^{(k),u}(m',m) = e^{\frac{\sum_{j=1}^{n} y_t(2c_{t,j}-1)}{\sigma^2}} \cdot e^{uL_{a,t}^{(k)}}, \quad (2)$$

$$\alpha_t^{(k)}(m) = \sum_{m'} \sum_{u=0}^{1} \alpha_{t-1}^{(k)}(m') \cdot \gamma_t^{(k),u}(m',m) \text{ and} \quad (3)$$

$$\beta_{t-1}^{(k)}(m') = \sum_{m} \sum_{u=0}^{1} \beta_t^{(k)}(m) \cdot \gamma_t^{(k),u}(m',m). \quad (4)$$

Note that the values of $L_a^{(k)}$ correspond to the interleaved values of $L_e^{(k-1)}$.

3.2. Extrinsic Information Transfer

In [5], a tool for prediction of the convergence of the iterative decoding process of Turbo codes was presented. The methodology relies on the assumption of a large interleaver which keeps the a priori information values relatively uncorrelated from their respective channel values during the decoding iterations. This assumption allows limitation of the analysis to a study of the relation between the a priori values and the extrinsic values of single decoding steps.

The decoding process is described in terms of the development of the information content of the extrinsic values, $L_{e,t}^{(k)}$, along with an increasing number of decoding steps k. In Figure 3, the values of the mutual information, $I(L_{a,t}^{(k)}; X)$, between the a priori values, $L_{a,t}^{(k)}$, at the input of an a APP decoder and the transmitted bits X are given along the x-axis. For the case of an AWGN channel, the corresponding mutual informations, $I(L_{e,t}^{(k)}; X)$, between the extrinsic values, $L_{e,t}^{(k)}$, at the output of the APP decoder and the transmitted bits X are given along the y-axis. All possible values of the mutual information pairs $[I(L_{a,t}^{(k)}; X),\ I(L_{e,t}^{(k)}; X)]$ constitute an extrinsic information transfer (EIT) chart.

If the EIT chart follows the diagonal, no information about the transmitted bits, X, additional to that contained in the a priori values, $L_{a,t}^{(k)}$, is gained during an APP decoding step. As long as the EIT trajectory does not intersect with the diagonal the mutual information of the extrinsic values is refined in each decoding step and convergent behavior is possible. In Figure 3, the EIT trajectory is mirrored in the diagonal to reflect the fact that the extrinsic values of one decoding step will be fed as a priori values to the other decoder. For low SNRs, when convergence is most uncertain, a component code with good EIT characteristics has large separation between the upper and lower EIT trajectories. If the EIT trajectory coincides with the diagonal, the Turbo decoding algorithm does not converge under the above assumptions.

In Figure 3 it is shown that the component code D, using repeated generator polynomials has worse EIT characteristics than the component code C with unique generator polynomials and it is argued that this translates into worse performance of the iterative decoding process. Interestingly, we have noted that it is possible to obtain further improved EIT characteristics using low-rate component codes with non-primitive feedback polynomials, such as for instance 17_8 in a rate 1/4 component encoder with memory 3.

3.3. Correlation of Extrinsic Values

Since the state metric $\alpha_t^{(k)}(m)$ in (3) is defined for a memoryless channel, the a priori values, $L_{a,t}^{(k)}$, are required to be independent [9]. Further the a priori value, $L_{a,t}^{(k)}$, should be independent of the channel value y_t at time t. Since the a priori values are obtained by interleaving the extrinsic values, correlation between the extrinsic values causes correlation between the a priori values which makes the iterative algorithm described in (1) suboptimum. An important task of the Turbo code interleaver is therefore to permute the correlated extrinsic values in such a way that they appear uncorrelated when used as a priori values at the succeeding decoder [7]. Adopting the assumption of a large interleaver, made in Section 3.2., we have investigated this correlation for component codes with repeated- and non-repeated generator polynomials by estimating the covariance between the extrinsic values of different bits after an APP decoding step, cf Equation 5. In (5), the extrinsic value of the transmitted bit at the j^{th} position in the block, output from the k^{th} decoding step, is denoted by $L_{e,j}^{(k)}$. The estimated mean of the extrinsic values at the k^{th} decoding step is represented by $\mu_{L_e^{(k)}}$ and the estimated variance is denoted $\sigma^2_{L_e^{(k)}}$.

$$A(m) = E\left[\frac{(L_{e,j+m}^{(k)} - \mu_{L_e^{(k)}})(L_{e,j}^{(k)} - \mu_{L_e^{(k)}})}{\sigma^2_{L_e^{(k)}}}\right] \quad (5)$$

The correlation between adjacent extrinsic values is dependent on the memory, the feedback- and the feedforward polynomials of the component encoder. In Figure 4, the correlation coefficients after one APP decoding, of a 1000 bit block, are plotted for different component codes given that the mutual information between the a priori values and the transmitted bits, $I(L_a^{(k)}; X)$, defined in [5], is equal to 0.55. This corresponds to the region of mutual information content in the a priori values where the EIT trajectories are most narrow and convergence should be most likely to cease. When distinct generator polynomials are

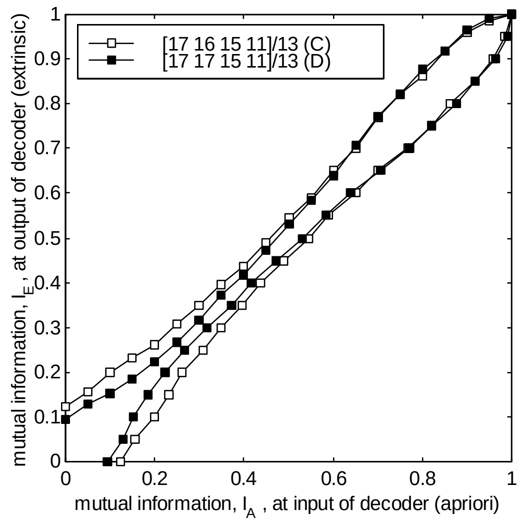

Figure 3: Extrinsic information transfer chart of turbo code component codes with rate 1/4 at −0.5 dB SNR and blocklength 1000 bits.

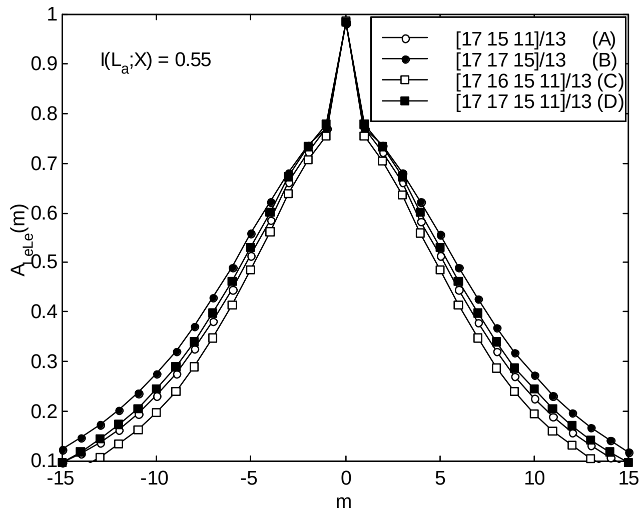

Figure 4: Correlation of extrinsic values after one APP decoding step for different component encoders. The extrinsic values are separated by m trellis steps, E_b/N_0 is equal to 1.0 dB and the mutual information $I(L_a^{(k)}; X)$ equals 0.55.

	Generators	Distance Multiplicity (N_d)				
	$g_1, g_2, \ldots, g_n$	N_0	N_1	N_2	N_3	N_4
A	$17_8, 15_8, 11_8$	1	6	6	2	-
B	$17_8, 17_8, 15_8$	3	4	4	4	
C	$17_8, 16_8, 15_8, 11_8$	-	4	6	4	1
D	$17_8, 17_8, 15_8, 11_8$	1	4	4	4	2

Table 2: Distance spectra of transition labels

used, the correlation decreases with the decreasing code rate. However when the code rate is decreased by repetition of generator polynomials the correlation instead increases, as illustrated in Figure 4.

For low SNRs the correlation between the extrinsic values appears to have a significant impact on the Turbo decoding performance. Higher correlation between the extrinsic values in combination with short-length interleaving would degrade the performance of the Turbo decoding algorithm defined by (1) since it implicitly assumes independent a priori values.

In Table 2, the distributions of Hamming distances between the transition labels are listed for different component encoders. The component codes with repeated generator polynomials are seen to have smaller minimum distance between the transitions labels or higher multiplicity of the transition label minimum distance. In general, we have noted that the correlation decreases when the distances between the trellis transitions increases. If this distance is large, it is reasonable that the correlation between extrinsic values corresponding to consecutively transmitted bits should be smaller since high trellis transition distance reduces the impact of the memory in the code.

4. CONCLUSIONS

We have studied different types of component codes in low-rate Turbo codes using repeated and non-repeated generator polynomials. It is seen that superior distance properties can be obtained using repeated generator polynomials. However, simulations show that repetition gives rise to performance losses in the waterfall region of SNRs. Using extrinsic information transfer analysis it is argued that the convergence of the iterative decoding is degraded at low SNRs when component codes using repeated generator polynomials are used. Further, it is observed that the correlation of the extrinsic values is higher for component codes with repeated generator polynomials. Ideally, the extrinsic values should be independent when used as a priori values in the succeeding decoding step. It is therefore inferred that an increase of correlation between the extrinsic values has a negative effect on the iterative decoding of the Turbo code. This is an interesting issue that, according to our knowledge, has not previously been addressed and further studies will be done.

REFERENCES

[1] C. Berrou and A. Glavieux, "Near Optimum Error Correcting Coding and Decoding: Turbo Codes", IEEE Transactions on Communications, vol. 44 (10), pp. 1261-1271, Oct., 1996.

[2] S. Benedetto, R. Garello and G. Montorsi, "A Search for Good Convolutional Codes to be used in the Construction of Turbo Codes", IEEE Transactions on Communications, 46 (9), pp. 1101-1105, Sep., 1998.

[3] C. F. Leanderson, J. Hokfelt, O. Edfors and T. Maseng, "On the Design of Low Rate Turbo Codes", In Proc. Vehicular Technology Conference, Tokyo, May, 2000.

[4] G. Poltyrev, "Bounds on the Decoding Error Probability of Binary Linear Codes via their Spectra", IEEE Transactions on Information Theory, vol. 40 (4), pp. 1284-1292, Jul. 1994.

[5] S. Ten Brink, "Iterative Decoding Trajectories of Parallel Concatenated Codes", In Proc. 3rd IEEE/ITG Conference on Source and Channel Coding, Munich, Germany, Jan., 2000.

[6] D. Divsalar and R. J. McEliece, "Effective Free Distance of Turbo Codes", Electronics Letters, vol. 32 (5), pp. 445-446, Feb., 1996.

[7] J. Hokfelt, O. Edfors and T. Maseng, "A Turbo Code Interleaver Design Criterion Based on the Performance of Iterative Decoding", IEEE Communications Letters. vol. 4 (5), May, 2000.

[8] L. R. Bahl, J. Cocke, F. Jelinek and J. Raviv, "Optimal Decoding of Linear Codes for minimizing Symbol Error Rate", *IEEE Trans. Inform. Theory*, vol. IT-20, pp. 284-287, Mar. 1974.

[9] P. Thitimajshima, "Recursive Systematic Convolutional Codes and their Application to Parallel Concatenation", l'Université de Bretagne Occidentale, Dec., 1993.