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Abstract

The recently developed technique for computation of piecewise quadratic Lyapunov functions is further developed for performance analysis and controller synthesis for nonlinear systems. In this way, degree of observability is estimated, L2 induced gain is computed and optimal control problems are solved. The computations are based on convex optimization in terms of linear matrix inequalities.

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Piecewise Linear Quadratic Optimal Control

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1. Introduction

A powerful model class for nonlinear systems is the class of piecewise affine systems [Sontag, 1981; Pettit and Wellstead, 1995]. Such systems arise naturally in many applications, for example in presence of saturations. Piecewise affine systems can also be used for approximation of other nonlinear systems.

A new framework for stability analysis of piecewise affine systems was developed in [Johansson and Rantzer, 1996] and similar ideas were reported in [Pettersson, 1996]. It was suggested to search for piecewise quadratic Lyapunov functions using convex optimization. The approach is considerably more powerful than ordinary quadratic stability [Corless, 1994] and another special case is polytopic Lyapunov functions, as defined in [Blanchini and Miani, 1996] and the references therein.

In this paper, the method is developed further, to treat performance analysis and optimal control. We show that several concepts from linear systems theory, such as observability Gramians, linear quadratic regulators and L_2 induced gain can be generalized using the framework of piecewise quadratic Lyapunov functions.

Quadratic control of piecewise linear systems has earlier been addressed in [Banks and Khathur, 1989]. The treatment was there based on backward solutions of Riccati differential equations, and the optimum had to be recomputed for each new final state. It is also known that the L_2 gain of a nonlinear system is determined by a Hamilton-Jacobi-Bellman equation or inequality [van der Schaft, 1992]. The linear matrix inequalities presented here give piecewise quadratic upper and lower bounds on the solutions to these inequalities and can be viewed as alternative numerical methods based on finite difference schemes [James and Yuliar, 1995].

An important feature of the approach is that a local linear-quadratic analysis near an equilibrium point of a nonlinear system can be improved step by step, by splitting the state space into more regions, thereby increasing the flexibility in the nonlinearity description and enlarging the validity domain for the analysis. In principle, any smooth nonlinear system can be approximated to an arbitrary accuracy in this way, so the tradeoff between precision and computational complexity can be addressed directly.

The paper is organized as follows. The basic setup for system representation and stability analysis is described in Section 2. This analysis is refined in the next section, to estimate the transient properties of the system. Optimal control problems are studied in Section 4 and applied to gain computations and other integral quadratic constraints in Section 5. Simplex partitions are discussed in Section 7 and used to prove a converse theorem on existence of piecewise quadratic Lyapunov functions.

2. Stability Analysis

Consider piecewise affine systems of the form

$$\begin{cases} \dot{x} = a_i + A_i x + B_i u \\ y = c_i + C_i x + D_i u \end{cases} \qquad \text{for } x \in X_i \tag{1}$$

Here, $\{X_i\}_{i\in I} \subseteq \mathbb{R}^n$ is a partition of the state space into a number of closed (possibly unbounded) polyhedral cells. The index set of the cells is denoted I. Let $x(t) \in \cup_{i\in I} X_i$ be a continuous piecewise C^1 function on the time interval $[t_0, t_1]$. We say that $x(t)$ is a *trajectory* of the system (1), if for every $t \in [t_0, t_1]$ such that the derivative $\dot{x}(t)$ is defined, the equation $\dot{x}(t) = A_i x(t) + a_i + B_i u(t)$ holds for all i with $x(t) \in X_i$. Given $u \equiv 0$, the system is said to have an *attractive sliding mode* at x_0, if there exists a system trajectory with $x(t_1) = x_0$, but no trajectory with $x(t_0) = x_0$.

We let $I_0 \subseteq I$ be the set of indices for the cells that contain origin, and $I_1 \subseteq I$ be the set of indices for cells that do not contain the origin. It is assumed that $a_i = c_i = 0$, $i \in I_0$. For convenient notation, we introduce

$$\left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] = \left[\begin{array}{cc|c} A_i & a_i & B_i \\ 0 & 0 & 0 \\ \hline C_i & c_i & D_i \end{array}\right] \qquad \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{2}$$

Then

$$\begin{cases} \dot{\bar{x}} = \bar{A}_i \bar{x} + \bar{B}_i u \\ y = \bar{C}_i \bar{x} + \bar{D}_i u \end{cases} \qquad \text{for } x \in X_i.$$

The cells are polyhedrons, so we can construct matrices

$$\bar{E}_i = [\,E_i \quad e_i\,] \qquad\qquad \bar{F}_i = [\,F_i \quad f_i\,]$$

with $e_i = 0$ and $f_i = 0$ for $i \in I_0$ and such that

$$\bar{E}_i\bar{x} \geq 0 \qquad\qquad x \in X_i, \quad i \in I \tag{3}$$
$$\bar{F}_i\bar{x} = \bar{F}_j\bar{x} \qquad\qquad x \in X_i \cap X_j \quad i, j \in I \tag{4}$$

The vector inequality $z \geq 0$ means that each entry of z is non-negative. Construction of the constraint matrices E_i and F_i will be further discussed in Section 7. The following result on stability analysis was proved in [Johansson and Rantzer, 1997].

PROPOSITION 1—PIECEWISE QUADRATIC STABILITY
Consider symmetric matrices T, U_i and W_i, such that U_i and W_i have non-negative entries, while $P_i = F_i^T T F_i$ for $i \in I_0$ and $P_i = F_i^T T F_i$ for $i \in I$ satisfy

$$\begin{cases} 0 > A_i' P_i + P_i A_i + E_i' U_i E_i \\ 0 < P_i - E_i' W_i E_i \end{cases} \qquad i \in I_0 \tag{5}$$

$$\begin{cases} 0 > \bar{A}_i' \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i' U_i \bar{E}_i \\ 0 < \bar{P}_i - \bar{E}_i' W_i \bar{E}_i \end{cases} \qquad i \in I_1 \tag{6}$$

Then $x(t)$ tends to zero exponentially for every continuous piecewise C^1 trajectory in $\cup_{i\in I} X_i$ satisfying (1) with $u \equiv 0$ for $t \geq 0$. □

Remark 1. Note that the assumption that the system equation holds for all $t \geq 0$ prevents application to trajectories ending in attractive sliding modes. It is however possible to modify the result to cover also such cases. □

In the absence of attractive sliding modes, the above conditions assure that

$$V(x) = \bar{x}' \bar{P}_i \bar{x} \qquad\qquad x \in X_i, \quad i \in I \tag{7}$$

is a Lyapunov function for the system. Any level set of $V(x)$ that is fully contained in the cell partition $\cup_{i\in I} X_i$, is a region of attraction for the equilibrium $x = 0$. In particular, if $\cup_{i\in I} X_i$ covers the whole state space, then the system is globally exponentially stable.

Proposition 1 can be used for systematic analysis of nonlinear systems based on piecewise approximations. A linear model valid locally around an equilibrium point can be refined by splitting the state space into more regions, each with different affine dynamics. Splitting a given partition also increases the flexibility of the piecewise quadratic Lyapunov function. The approach is illustrated in the following example.

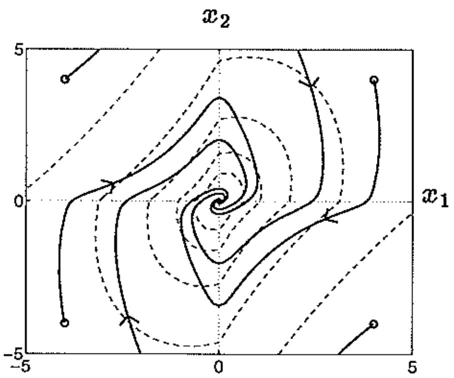

Figure 1 Simulations (full) and Lyapunov function level surfaces (dashed).

EXAMPLE 1—PIECEWISE LINEAR ANALYSIS
Simulations indicate that the following nonlinear system is stable

$$\begin{aligned}\dot{x}_1 &= -2x_1 + 2x_2 + \text{sat}(x_1x_2)x_1 \\ \dot{x}_2 &= -2x_1 - \text{sat}(x_1x_2)(x_1 + 4x_2).\end{aligned}$$

We would like to verify exponential stability of the origin by computing a piecewise quadratic Lyapunov function for the system. A simple technique for rigorous analysis of the system is to explore bounds on the nonlinearity

$$p_{min} \leq \text{sat}(x_1x_2) \leq p_{max}$$

and re-write the model as the differential inclusion

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ -2 & 0 \end{bmatrix} x + p(t) \begin{bmatrix} 1 & 0 \\ -1 & -4 \end{bmatrix} x \tag{8}$$

with $p_{min} \leq p(t) \leq p_{max}$. We notice that analysis using global model based on the bound $-1 \leq p(t) \leq 1$ is futile, since $p(t) = -1$ gives the unstable system

$$\dot{x} = \begin{bmatrix} -3 & 2 \\ -1 & 4 \end{bmatrix} x$$

Taking the step from linear analysis to piecewise linear analysis, we can obtain a refined model by exploring the fact that

$$0 \leq \text{sat}(x_1x_2) \leq 1$$

in the first and third quadrant, and

$$-1 \leq \text{sat}(x_1x_2) \leq 0$$

in the second and fourth quadrant. This observation motivates a model with four regions, each covering one quadrant. The dynamics in each region is given by a linear differential inclusion on the form (8). To assure stability of the original system, we search for a piecewise quadratic Lyapunov function that is simultaneously valid for all extreme systems in each region [Johansson and Rantzer, 1997]. Note that one of the extreme systems in the second and fourth quadrant is unstable. The numerical routines return the Lyapunov function with the level curves indicated in Figure 1. This proves global exponential stability. □

3. Transient Analysis

The objective of this section is to refine the stability analysis by estimating the "ouput energy" $\int_{t=0}^{\infty} |y(t)|^2 dt$ as a function of the initial state $x(0)$. Such an estimate can be obtained using a minor modification of the Lyapunov inequalities, taking the output function into account.

THEOREM 1—UPPER BOUND ON TRANSIENT
Let $x(t)$ with $x(0) \in X_{i_0}$, $x(\infty) = 0$ be a continuous piecewise C^1 trajectory of the system (1) with $u \equiv 0$ for $t \geq 0$. Consider symmetric matrices T and U_i, such that U_i have non-negative entries, while $P_i = F_i' T F_i$ and $\bar{P}_i = \bar{F}_i' T \bar{F}_i$ satisfy

$$
\begin{aligned}
0 &> P_i A_i + A_i' P_i + C_i' C_i + E_i' U_i E_i \qquad & i \in I_0 \\
0 &> \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{C}_i' \bar{C}_i + \bar{E}_i' U_i \bar{E}_i \qquad & i \in I_1
\end{aligned}
$$

Then

$$
\int_0^{\infty} |y|^2 dt \leq \inf_{T,U_i} \bar{x}(0)' P_{i_0} \bar{x}(0)
$$

□

Proof. It follows directly from the two inequalities that

$$
0 \geq \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{C}_i' \bar{C}_i + \bar{E}_i' U_i \bar{E}_i \qquad i \in I
$$

Multiplying this inequality from left and right by $\bar{x}$ and removing the nonnegative terms including U_i gives

$$
0 \geq 2\bar{x}(t)' \bar{P}_{i(t)} \bar{A}_{i(t)} \bar{x}(t) + |y(t)|^2
$$

where $i(t)$ is chosen so that $X_{i(t)} \ni x(t)$. Integration from $t = 0$ to $t = \infty$ gives the desired result. □

A lower bound can be obtained similarly.

THEOREM 2—LOWER BOUND ON TRANSIENT
Let $x(t)$ with $x(0) \in X_{i_0}$, $x(\infty) = 0$ be a continuous piecewise C^1 trajectory of the system (1) with $u \equiv 0$ for $t \geq 0$. Consider symmetric matrices S and W_i,

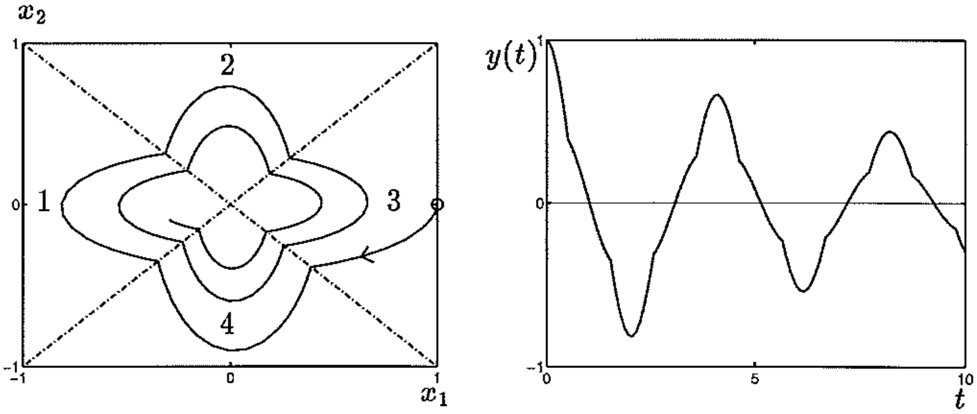

Figure 2 Trajectory of a simulation (left) and corresponding output (right).

such that W_i have non-negative entries, while $O_i = F_i' S F_i$ and $\bar{O}_i = \bar{F}_i' S \bar{F}_i$ satisfy

$$\begin{aligned} 0 &< O_i A_i + A_i' O_i + C_i' C_i - E_i' W_i E_i & i \in I_0 \\ 0 &< \bar{O}_i \bar{A}_i + \bar{A}_i' \bar{O}_i + \bar{C}_i' \bar{C}_i - \bar{E}_i' W_i \bar{E}_i & i \in I_1 \end{aligned}$$

Then

$$\sup_{S, W_i} \bar{x}(0)' O_{i_0} \bar{x}(0) \leq \int_0^\infty |y|^2 dt$$

□

Proof. The proof is analogous to the previous one. □

EXAMPLE 2—TRANSIENT IN FLOWER EXAMPLE
Consider the piecewise linear system with the cell partition shown in Figure 2 (left) and dynamics given by the matrices

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix}, \qquad A_2 = A_4 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix},$$

and $C_i = [1 \quad 0]$, $i \in I$. The trajectory of a simulation with initial value $x_0 = (\, 1, \quad 0\,)^T$ moves towards the origin in a flower-like trajectory, as shown in Figure 2 (left). The corresponding output is shown in Figure 2 (right). This output has the total energy $\int_0^\infty |y|^2 dt = 1.88$, while solving the linear matrix inequalities in Theorem 1 and Theorem 2 with the initial cell partition gives the estimate $0.60 \leq \int_0^\infty |y|^2 \, dt \leq 2.50$.

To improve the bounds, we introduce new cells by repeatedly splitting every cell in two. This simple-minded refinement procedure, illustrated in Figure 3, is repeated three times yielding the bounds shown in Table 1. Note that the bounds on the output energy optimized for the initial state $(1, 0)$ match closely over the the whole state space, giving good estimates of the output energy also for other initial states. The computation time for the final partition is comparable to the computation time for a simulation giving the same accuracy.
□

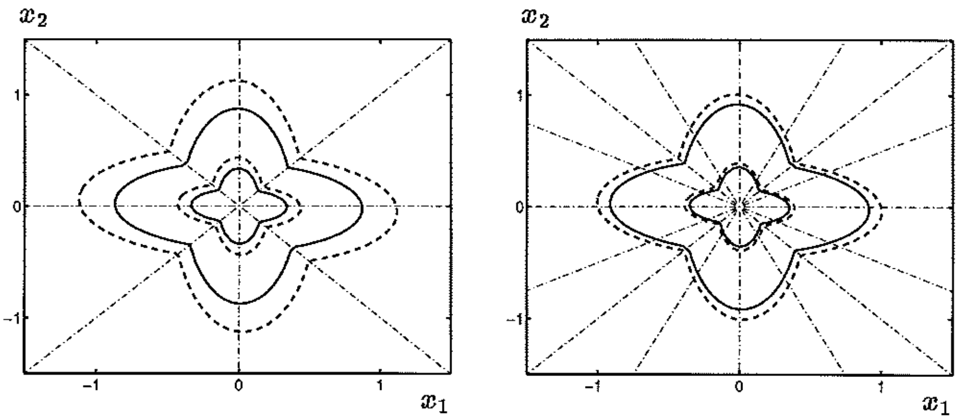

Figure 3 Upper (full) and lower (dashed) bounds on the storage function computed in Example 2. The bounds get increasingly tight when we move from 8 cells (left) to 16 cells (right).

Number of Cells	Lower bound	Upper bound
4	0.60	2.50
8	1.33	2.18
16	1.65	1.98
32	1.78	1.88

Table 1 Lower and upper bounds for output energy.

It should be noted that systems with discontinuous dynamics require special attention in analysis and simulation. All simulation examples in this paper were performed in Omsim [Andersson, 1994] with proper treatment of discrete events .

In duality with transient estimation, which can be viewed as an observability problem, one may also consider reachability. The problem is then to estimate the input energy $\int_0^\tau |u(t)|^2 dt$ that is needed to reach a certain state $x(\tau)$ starting from $x(0) = 0$. However, rather than the reachability problem, we will next consider a more general class of optimal control problems.

4. Piecewise Linear Quadratic Optimal Control

Consider the following general form of optimal control problem.

$$\begin{aligned} &\text{Minimize} \quad \int_0^\infty L(x,u)dt \\ &\text{subject to} \quad \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(0) = x_0 \end{cases} \end{aligned}$$

It is well known that solutions of this problem can be characterized in terms of the Hamilton—Jacobi—Bellman (H-J-B) equation

$$0 = \inf_u \left(\frac{\partial V}{\partial x} f(x,u) + L(x,u) \right) \tag{9}$$

In fact, by integrating the inequality

$$0 < \frac{\partial V}{\partial x} f(x,u) + L(x,u) \qquad \forall x, u \tag{10}$$

assuming that $x(\infty) = 0$, we get

$$V(x_0) - V(0) = -\int_0^\infty \frac{\partial V}{\partial x} f(x,u) dt \leq \int_0^\infty L(x,u) dt$$

Hence, every V that satisfies (10) gives a lower bound on the optimal value of the loss function. In fact, the maximization of $V(x_0) - V(0)$ subject to (10) is a convex optimization problem in V with an infinite number of constraints parameterized by x and u. The objective of this section is to solve this problem in some special cases.

Let us consider the case where f is piecewise linear and L is piecewise quadratic. Then, the objective is to bring the system to $x(\infty) = 0$ from an arbitrary initial state $x(0)$, while limiting the cost

$$J(x_0, u) = \int_0^\infty \left(\bar{x}' \bar{Q}_i \bar{x} + u' R_i u \right) dt$$

Here $i(t)$ is defined so that $x(t) \in X_{i(t)}$. Under the assumption that

$$\bar{Q}_i = \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{for } i \in I_0 \tag{11}$$

this can be done in analogy with the previous results as follows.

Theorem 3—Lower Bound on Optimal Cost

Consider symmetric matrices T and U_i, such that U_i have non-negative entries, while $P_i = F_i' T F_i$ and $\bar{P}_i = \bar{F}_i' T \bar{F}_i$ satisfy

$$0 < \begin{bmatrix} P_i A_i + A_i' P_i + Q_i - E_i' U_i E_i & P_i B_i \\ B_i' P_i & R_i \end{bmatrix} \qquad i \in I_0$$

$$0 < \begin{bmatrix} \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{Q}_i - \bar{E}_i' U_i \bar{E}_i & \bar{P}_i \bar{B}_i \\ \bar{B}_i' \bar{P}_i & R_i \end{bmatrix} \qquad i \in I_1$$

Then, every continuous piecewise C^1 trajectory $x(t)$ of (1) with $x(\infty) = 0$, $x(0) = x_0 \in X_{i_0}$ satisfies

$$J(x_0, u) \geq \sup_{T, U_i} \bar{x}_0' \bar{P}_{i_0} \bar{x}_0$$

□

Proof. It follows directly from the two matrix inequalities in Theorem 3 that

$$0 \leq \begin{bmatrix} \bar{P}_i\bar{A}_i + \bar{A}_i'\bar{P}_i + \bar{Q}_i - \bar{E}_i'U_i\bar{E}_i & \bar{P}_i\bar{B}_i \\ \bar{B}_i'\bar{P}_i & R_i \end{bmatrix} \qquad i \in I$$

Multiplying from left and right by $(\bar{x}, u)$ and removing the nonnegative terms including U_i gives

$$0 \leq 2\bar{x}'\bar{P}_i\bar{A}_i\bar{x} + \bar{x}'\bar{Q}_i\bar{x} + u'R_iu = \frac{d}{dt}\left(\bar{x}'\bar{P}_i\bar{x}\right) + \bar{x}'\bar{Q}_i\bar{x} + u'R_iu$$

Integration from 0 to ∞ gives the desired result. □

Theorem 3 gives a lower bound on the minimal value of the cost function J. It is natural to also search for a control law that achieves a low cost. Consider the control law obtained by the minimization

$$\inf_u \left(\frac{\partial V}{\partial x} f(x,u) + L(x,u)\right) \tag{12}$$

If the H-J-B equation (9) holds, then V has a decay rate given by $-L(x,u)$, which is typically negative, so V may serve as a Lyapunov function to prove that the control law is stabilizing.

However, if only the inequality (10) holds, for example as a result of solving the matrix inequalities in Theorem 3, then there is now guarantee that the control law (12) is even stabilizing. Still, this control law will be the basis for our further analysis.

In analogy with ordinary linear quadratic control, we therefore introduce the following notation.

$$\begin{aligned} L_i &= -R_i^{-1}B_i'P_i & \bar{L}_i &= -R_i^{-1}\bar{B}_i'\bar{P}_i \\ \mathcal{A}_i &= A_i + B_iL_i & \bar{\mathcal{A}}_i &= \bar{A}_i + \bar{B}_i\bar{L}_i \\ \mathcal{Q}_i &= Q_i + P_iB_iR_i^{-1}B_i'P_i & \bar{\mathcal{Q}}_i &= \bar{Q}_i + \bar{P}_i\bar{B}_iR_i^{-1}\bar{B}_i'\bar{P}_i \end{aligned}$$

The control law can then be written as

$$u(t) = \hat{u}(t) := \bar{L}_i\bar{x} \qquad x \in X_i$$

It should be noted that even if the piecewise linear dynamics $\bar{\mathcal{A}}_i\bar{x}$ is continuous in x, the control law may be discontinuous and give rise to attractive sliding modes.

THEOREM 4—UPPER BOUND ON OPTIMAL COST

Assume that the system $\dot{x} = \mathcal{A}_ix$, $x \in X_i$ is asymptotically stable and has no attractive sliding modes. Consider symmetric matrices S and W_i, such that W_i have non-negative entries, while $O_i = F_i'SF_i$ and $\bar{O}_i = \bar{F}_i'S\bar{F}_i$ satisfy

$$\begin{aligned} 0 &> O_i\mathcal{A}_i + \mathcal{A}_i'O_i + \mathcal{Q}_i + E_i'W_iE_i & i &\in I_0 \\ 0 &> \bar{O}_i\bar{\mathcal{A}}_i + \bar{\mathcal{A}}_i'\bar{O}_i + \bar{\mathcal{Q}}_i + \bar{E}_i'W_i\bar{E}_i & i &\in I_1 \end{aligned}$$

Then for every initial state x_0

$$J(x_0, \hat{u}) \leq \inf_{S, W_i} \bar{x}_0' \bar{O}_{i_0} \bar{x}_0$$

□

Proof. Analogous to the proof of Theorem 3. □

EXAMPLE 3—LQ CONTROL OF AN INVERTED PENDULUM
Consider the following simple model of an inverted pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.1x_2 + \sin(x_1) + u \end{aligned} \tag{13}$$

We are interested in applying the proposed technique to find a feedback control that brings the pendulum from rest at the stable equilibrium $(\pi, 0)'$ to the upright position $(0, 0)'$ while minimizing the criteria

$$J(x, u) = \int_{t=0}^{\infty} 4x_1^2(t) + 4x_2(t)^2 + u^2 dt.$$

A piecewise linear model of the system (13) can be constructed by finding piecewise affine bounds on the system nonlinearity $\sin(x_1)$. For the purpose of this example, we divide the interval $[-4, 4]$ into five segments and compute the bounds illustrated in Figure 4 (left). This description of the system nonlinearity induces the partition shown by dotted lines in Figure 4 (right). The partition can be viewed as a simplex partition in the x_1 variable, while x_2 is independent of the partition. We apply Theorem 3 to compute a lower bound on the achievable performance as $J(x_0, u) \geq 15.2$.

It is easy to verify that the closed loop system obtained by applying the control suggested in Theorem 4 has no attractive sliding modes. To illustrate the use of local analysis, we compute a piecewise quadratic Lyapunov function for the closed loop system using Proposition 1. The level surfaces of the Lyapunov function are shown in Figure 4(right). The guaranteed region of attraction is given by the outermost level set, which contains the initial value $x_0 = (\pi, 0)'$. Theorem 4 can now be applied to compute the upper bound on the performance to be $J(x_0, u) \leq 16.6$. We conclude that both the optimal and the computed control law satisfy

$$15.2 \leq J(x_0, u) \leq 16.6.$$

The level surfaces of the upper and lower bounds on the value function is shown in Figure 5. Although the bounds are valid for all initial values within the estimated region of attraction, they match most closely for the optimized initial value. In addition, the computed control law is evaluated on the pendulum model (13) by simulation. The value of the loss function computed in this way is $J(x_0, u) = 15.4$. □

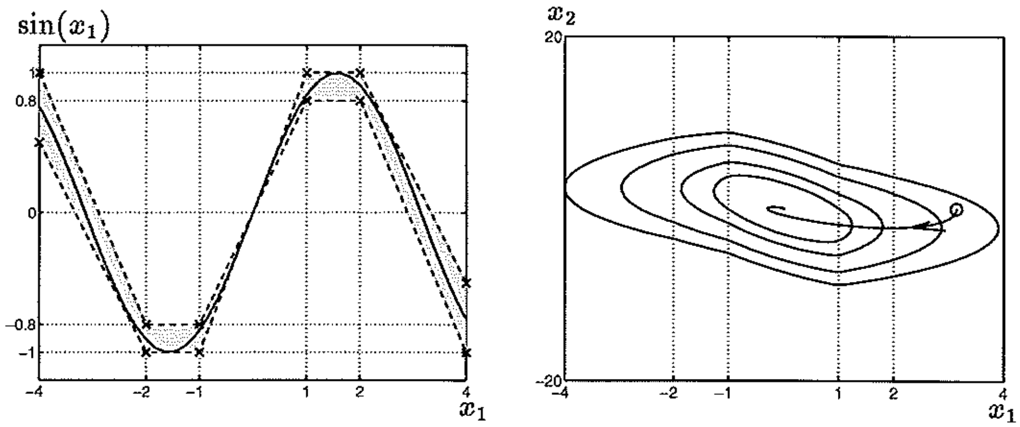

Figure 4 The left figure shows bounds on the system nonlinearity. The right figure shows the region of attraction for the closed loop system, as estimated by a piecewise quadratic Lyapunov function.

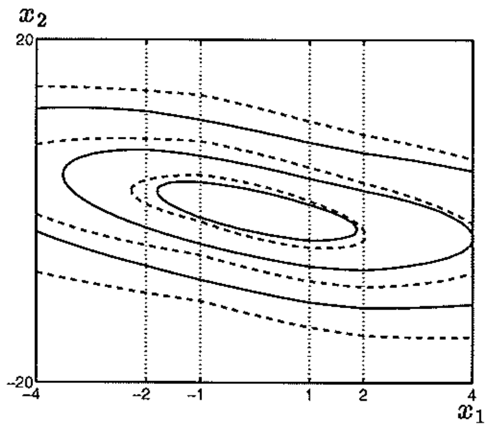

Figure 5 Lower (dashed) and upper (full) bounds on the optimal cost.

5. Input-Output Gain

As a another application of the central idea, we shall compute bounds on the L_2 induced gain of a piecewise linear system as well as other integral quadratic constraints.

THEOREM 5—UPPER BOUND ON L_2 GAIN
Suppose that the system Consider symmetric matrices T, U_i and W_i such that U_i and W_i have non-negative entries, while $P_i = F_i'TF_i$ and $\bar{P}_i = \bar{F}_i'T\bar{F}_i$ satisfy

$$
\begin{aligned}
0 &> \begin{bmatrix} P_iA_i + A_i'P_i + C_i'C_i + E_i'U_iE_i & P_iB_i \\ B_i'P_i & -\gamma^2 I \end{bmatrix} \\
0 &< P_i - E_i'W_iE_i
\end{aligned}
$$

for $i \in I_0$ and

$$
\begin{aligned}
0 &> \begin{bmatrix} \bar{P}_i\bar{A}_i + \bar{A}_i'\bar{P}_i + \bar{C}_i'\bar{C}_i + \bar{E}_i'U_i\bar{E}_i & \bar{P}_i\bar{B}_i \\ \bar{B}_i'\bar{P}_i & -\gamma^2 I \end{bmatrix} \\
0 &< \bar{P}_i - \bar{E}_i'W_i\bar{E}_i
\end{aligned}
$$

for $i \in I_1$. Then every continuous piecewise C^1 trajectory $x(t)$, $t \in [0, \tau]$ with $x(0) = 0$ satisfies

$$\int_0^\tau |y|^2 dt \leq \gamma^2 \int_0^\tau |u|^2 dt$$

□

The best upper bound on the L_2 induced gain is achieved by minimizing γ subject to the constraints defined by the inequalities.

Proof. It follows as in the proof of Theorem 3 that

$$0 \geq \frac{d}{dt}\left(\bar{x}'\bar{P}_i\bar{x}\right) + |y|^2 - \gamma^2|u|^2$$

Integration from 0 to τ gives

$$0 \geq \bar{x}(\tau)'\bar{P}_i\bar{x}(\tau) + \int_0^\tau (|y|^2 - \gamma^2|u|^2)dt \geq \int_0^\tau (|y|^2 - \gamma^2|u|^2)dt$$

and the proof is complete. □

In analogy with the previous section, it is possible to compute a lower bound from an explicit formula for u. To verify that a given number γ is smaller than the L_2 gain, one may introduce

$$\mathcal{A}_i = A_i + \gamma^{-2}B_iB_i'P_i \qquad \bar{\mathcal{A}}_i = \bar{A}_i + \gamma^{-2}\bar{B}_i\bar{B}_i'\bar{P}_i$$
$$\mathcal{Q}_i = C_i'C_i - \gamma^{-2}P_iB_iB_i'P_i \qquad \bar{\mathcal{Q}}_i = \bar{C}_i'\bar{C}_i - \gamma^{-2}\bar{P}_i\bar{B}_i\bar{B}_i'\bar{P}_i$$

and apply the following result.

THEOREM 6—LOWER BOUND ON L_2 GAIN

Consider positive scalars τ_i, symmetric matrices S, U_i, W_i, such that U_i and W_i have non-negative entries, while $O_i = F_i'SF_i$, $\bar{O}_i = \bar{F}_i'S\bar{F}_i$ satisfy

$$\begin{cases} 0 > O_i\mathcal{A}_i + \mathcal{A}_i'O_i + E_i'U_iE_i \\ 0 < \tau_i\mathcal{Q}_i + O_i - E_i'W_iE_i \end{cases} \qquad \text{for } i \in I_0$$

$$\begin{cases} 0 > \bar{O}_i\bar{\mathcal{A}}_i + \bar{\mathcal{A}}_i'\bar{O}_i + \bar{E}_i'U_i\bar{E}_i \\ 0 < \tau_i\bar{\mathcal{Q}}_i + \bar{O}_i - \bar{E}_i'W_i\bar{E}_i \end{cases} \qquad \text{for } i \in I_1$$

$$0 > \bar{x}_0'\bar{O}_{i_0}\bar{x}_0 \text{ for some } i_0 \in I \text{ and some } x_0 \in X_{i_0}$$

Assume that the state x_0 is reachable from zero in finite time and that the system $\dot{\bar{x}} = \mathcal{A}_i\bar{x}$ has no attractive sliding modes. Then

$$\int_0^T |y|^2 dt > \gamma^2 \int_0^T |u|^2 dt \qquad \text{for some } u, y, T$$

□

Proof. Let $\tau = \max_{i\in I}\{\tau_i\}$. Select $u(t)$ for $t \in [0, t_0]$ such that $u(t_0) = x_0$ and for $t > t_0$ use the control law $u(t) = \gamma^{-2}\bar{B}_i'\bar{P}_i\bar{x}(t)$. Define

$$V(t) = \bar{x}(t)'\bar{O}_i\bar{x}(t) \qquad\qquad x(t) \in X_i$$

Then $V(t_0) < 0$ and $V'(t) \le 0$ for $t > t_0$. Hence

$$\begin{aligned}\int_0^T \left(|y|^2 - \gamma^2|u|^2\right) dt &\ge \int_0^{t_0} \left(|y|^2 - \gamma^2|u|^2\right) dt - \tau^{-1}\int_{t_0}^T V(t)dt \\ &\ge \int_0^{t_0} \left(|y|^2 - \gamma^2|u|^2\right) dt - V(t_0)(T - t_0)/\tau\end{aligned}$$

The right hand side is positive for sufficiently large T so the statement is proved. □

EXAMPLE 4—ANALYSIS OF A SATURATED CONTROL SYSTEM
Consider the control system shown in Figure 6. The output of the system $G_1(s)$ is subject to a unit saturation. The closed loop dynamics is piecewise affine, with three cells induced by the saturation limits $u = \pm 1$. We set $r = 0$ and

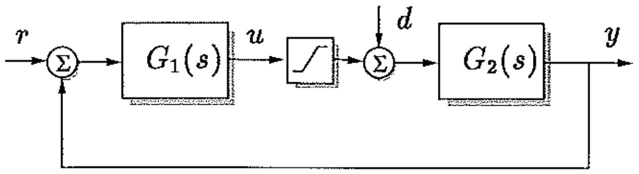

Figure 6 Saturated control system.

estimate the L_2 induced gain from the disturbance d to the output y. Consider the transfer functions

$$G_1(s) = \frac{5s - 3}{5s^2 + 4s + 4}, \qquad G_2(s) = \frac{6s - 1}{15s^2 + 5s + 6}$$

Estimating the gain using a quadratic storage function [Boyd *et al.*, 1994] does not yield a feasible solution. Using the computations of Theorem 5 we obtain the upper bound 9.63. A lower bound on the induced gain is computed using Theorem 6 to 4.81. □

6. Validation of Integral Quadratic Constraints

The results of the previous section can be generalized in a natural way to validate or invalidate arbitrary integral quadratic constraints (IQC's) for the nonlinear system.

THEOREM 7—VALIDATION OF INTEGRAL QUADRATIC CONSTRAINT
Consider symmetric matrices T, U_i and W_i such that U_i and W_i have non-negative entries, while $\bar{P}_i = \bar{F}_i' T \bar{F}_i$ satisfy

$$0 \geq \begin{bmatrix} \bar{C}_i' & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} \bar{C}_i & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{E}_i' U_i \bar{E}_i & \bar{P}_i \bar{B}_i \\ \bar{B}_i' \bar{P}_i & 0 \end{bmatrix}$$

for $i \in I$. Then every continuous piecewise C^1 trajectory $x(t)$, $t \in [0, \tau]$ with $x(0) = 0$, $\int_0^\infty (|x|^2 + |u|^2) dt < \infty$ satisfies

$$0 \geq \int_0^\infty \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}' M \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} dt$$

□

Proof. Multiplying from left and right by $(\bar{x}, u)$ gives

$$0 \leq \begin{bmatrix} u \\ y \end{bmatrix}' M \begin{bmatrix} u \\ y \end{bmatrix} + \frac{d}{dt} \left(\bar{x}' \bar{P}_i \bar{x} \right)$$

and the result follows by integration over $[0, \infty]$. □

Notice that IQC's with a frequency dependent weight instead of the constant matrix M can be verified with the same theorem by first introducing a state space realization of the weight and include these dynamics in the system description.

Instead of asking a yes/no-question about the validity of a particular IQC one may try to validate the inequality

$$0 \geq \int_0^\infty \begin{bmatrix} y \\ \gamma u \end{bmatrix}' M \begin{bmatrix} y \\ \gamma u \end{bmatrix} dt$$

for as small values of γ as possible. Assume that $M_{22} > 0$ and that an upper bound on the optimal γ has been found using Theorem 7. A lower bound can then again be obtained based on a "control law" defined in term of the matrices $\bar{P}_i$. Let $u = \bar{L}_i \bar{x}$ be defined by minimization of the expression

$$\begin{bmatrix} y \\ \gamma u \end{bmatrix}' M \begin{bmatrix} y \\ \gamma u \end{bmatrix} + 2\bar{x}' \bar{P}_i (\bar{A}_i \bar{x} + \bar{B}_i u)$$

with respect to u. Define $\bar{\mathcal{A}}_i$ and $\bar{\mathcal{Q}}_i$ as

$$\bar{\mathcal{A}}_i = \bar{A}_i + \bar{B}_i \bar{L}_i \qquad \bar{\mathcal{Q}}_i = \begin{bmatrix} \bar{C}_i \\ \gamma \bar{L}_i \end{bmatrix}' M \begin{bmatrix} \bar{C}_i \\ \gamma \bar{L}_i \end{bmatrix} + 2\bar{P}_i (\bar{A}_i + \bar{B}_i \bar{L}_i)$$

Then we get the following analogy of Theorem 6.

Theorem 8—Invalidation of Integral Quadratic Constraints
Consider positive scalars τ_i, symmetric matrices S, U_i, W_i, such that U_i and W_i have non-negative entries, while $\bar{O}_i = \bar{F}_i' S \bar{F}_i$ satisfy

$$\begin{cases} 0 > O_i \mathcal{A}_i + \mathcal{A}_i' O_i + E_i' U_i E_i \\ 0 < \tau_i \mathcal{Q}_i + O_i - E_i' W_i E_i \end{cases} \qquad \text{for } i \in I_0$$

$$\begin{cases} 0 > \bar{O}_i \bar{\mathcal{A}}_i + \bar{\mathcal{A}}_i' \bar{O}_i + \bar{E}_i' U_i \bar{E}_i \\ 0 < \tau_i \bar{\mathcal{Q}}_i + \bar{O}_i - \bar{E}_i' W_i \bar{E}_i \end{cases} \qquad \text{for } i \in I_1$$

$$0 > \bar{x}_0' \bar{O}_{i_0} \bar{x}_0 \text{ for some } i_0 \in I \text{ and some } x_0 \in X_{i_0}$$

Assume that the state x_0 is reachable from zero in finite time and that the system $\dot{\bar{x}} = \mathcal{A}_i \bar{x}$ has no attractive sliding modes. Then

$$0 < \int_0^T \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}' M \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} dt \qquad \text{for some } u, y, T$$

□

7. Simplex Partitions

So far, we did not pay much attention to the partitioning of the state space and the specification of the matrices $\bar{E}_i$ and $\bar{F}_i$. For a piecewise linear system, with a given state space partition for the dynamics, it is natural to so the initial analysis using the same partition for the Lyapunov function or loss function. However, there are many examples where a more refined partition is needed for the analysis. The purpose of this section is to introduce some convenient concepts for this purpose and discuss their properties.

An *n-dimensional polytope* is defined as the convex hull of a finite number of corner points in $\mathbb{R}^n$. It is called an *n-dimensional simplex* if the number of corner points is $n+1$. Note that any polytope which is not a simplex can be partitioned into two polytopes, each with fewer corners than the original one. Repeating this procedure eventually generates a partition of the original polytope into simplices (See Figure 7).

A simple and flexible way to partition the state space is to divide it in to simplices. In fact, every region with continuous boundary can be approximated by a polytope built from a finite number of simplices.

Let $X \subset \mathbb{R}^n$ be a polytope with the simplex partition $X = \cup_{i \in I} X_i$, where all the simplices have nonempty interior and $x = 0$ is a simplex vertex. Let $\nu_0, \nu_1, \ldots, \nu_p$ with $\nu_0 = 0$ be the collection of vertices and define

$$\mathcal{V} = [\, \nu_0 \quad \ldots \quad \nu_p \,]$$
$$\bar{\mathcal{V}} = [\, \bar{\nu}_0 \quad \ldots \quad \bar{\nu}_p \,]$$

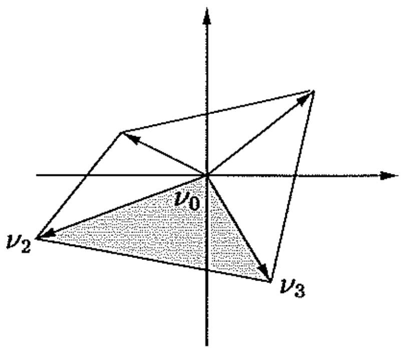

Figure 7 Simplex partition of a compact domain of the state space.

Then, each $\bar{x}$ has a unique representation as a convex combination $\bar{x} = \sum_{k=0} z_k \bar{\nu}_k$ with $z_k \geq 0$ for all k, $\sum_k z_k = 1$ and $z_k \neq 0$ if and only if $\nu_k \in X_i$. Define $z = [0\, z_1 \ldots z_p]'$. Then

$$\begin{aligned} x &= \mathcal{V} z & x &\in X_i \quad i \in I \\ \bar{x} &= \bar{\mathcal{V}} z & x &\in X_i \quad i \in I_1 \end{aligned}$$

For each simplex X_i, define an *extraction matrix* $\mathcal{E}_i \in \mathbb{R}^{(p+1)\times(n+1)}$ of X_i as follows. The k:th row of $\mathcal{E}_i$ is zero for all k such that $\nu_k \notin X_i$ and the remaining rows of $\mathcal{E}_i$ are equal to the rows of an identity matrix.

The extraction matrix then has the property that $z = \mathcal{E}_i \mathcal{E}_i' z$ for all z corresponding to $x \in X_i$. In addition, the matrix $\bar{\mathcal{V}} \mathcal{E}_i$ is invertible, due to the nonempty interior of X_i. Let $\bar{E}_i$ and $\bar{F}_i$ be defined by

$$\begin{aligned} \bar{F}_i &= [\, 0 \quad I_p \,]\, \mathcal{E}_i (\bar{\mathcal{V}} \mathcal{E}_i)^{-1} \\ \bar{E}_i &= \mathcal{E}_i' \begin{bmatrix} 0 \\ \bar{F}_i \end{bmatrix} \end{aligned}$$

for all $i \in I$. Then (3) and (4) are implied by the following proposition.

PROPOSITION 2

$$\bar{E}_i \bar{x} = \mathcal{E}_i' z \qquad \bar{F}_i \bar{x} = [\, z_1 \quad \ldots \quad z_p \,]' \qquad \text{for } x \in X_i,\, i \in I \tag{14}$$

In particular, $e_i = 0$ and $f_i = 0$ for $i \in I_0$. □

Proof. Let $\bar{z} = [z_0\, z_1 \ldots z_p]'$. Then

$$\begin{aligned} \bar{x} &= \bar{\mathcal{V}} \bar{z} = \bar{\mathcal{V}} \mathcal{E}_i \mathcal{E}_i' \bar{z} \\ \bar{z} &= \mathcal{E}_i \mathcal{E}_i' \bar{z} = \mathcal{E}_i \left(\bar{\mathcal{V}} \mathcal{E}_i \right)^{-1} \bar{x} \\ \bar{F}_i \bar{x} &= [\, 0 \quad I_p \,]\, \bar{z} = [\, z_1 \quad \ldots \quad z_p \,]' \\ \bar{E}_i \bar{x} &= \mathcal{E}_i' \begin{bmatrix} 0 \\ \bar{F}_i \end{bmatrix} \bar{x} = \mathcal{E}_i' z \end{aligned}$$

The last column of $\bar{F}_i$, denoted f_i, is identical to the z that corresponds to $\bar{x} = [0 \ldots 0\, 1]' = \bar{\nu}_0$. Hence $f_i = [0 \ldots 0]'$. □

Remark 2. In applications, it is often advantageous to extend the $\bar{F}_i$ further, so that $\bar{F}_i\bar{x} = [z_1 \dots z_p\, x_1 \dots x_n]'$. □

Polyhedral Partitions

As a generalization of "polytope", that also allows corners at infinity, $X_i \subset \mathbb{R}^n$ is called a *polyhedron*, if every $x \in X_i$ can be written as

$$x = \sum_{k=0}^{q} z_k \nu_k + \sum_{k=q+1}^{p} z_k \nu_k \tag{15}$$

with $z_k \geq 0$, $\sum_{k=0}^{q} z_k = 1$. The vectors $\nu_1, \dots, \nu_q$ are finite vertices, while $\nu_{q+1}, \dots, \nu_p$ define vertex directions at infinity. A *generalized simplex* is a polyhedron with $p = n$.

Let $X = \cup_{i\in I} X_i$ be a polyhedron partitioned into generalized simplices, each with nonempty interior. Let the partition be given by the finite vertices $\nu_0, \nu_1, \dots, \nu_q$ with $\nu_0 = 0$ and the infinite vertex directions $\nu_{q+1}, \dots, \nu_p$. Then, with $\mathcal{V}$, $\bar{\mathcal{V}}$, $\bar{E}_i$ and $\bar{F}_i$ defined as in the previous subsection, all the earlier statements remain valid, except that the identity $\sum_k z_k = 1$ does not include terms with $k > q$.

Partitioning a Subspace of the State Space

In some cases, it is natural to partition only a subspace of the state space. This can be done conveniently by replacing x with Cx for some matrix $C \in \mathbb{R}^{m\times n}$ everywhere in the discussion of simplex partitions. Then $\nu_0, \dots, \nu_p \in \mathbb{R}^m$

$$Cx = \mathcal{V}z \qquad\qquad x \in X_i \quad i \in I$$

and Proposition 2 holds with

$$\begin{aligned} \bar{F}_i &= \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix} \mathcal{E}_i(\bar{\mathcal{V}}\mathcal{E}_i)^{-1} \begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix} \\ \bar{E}_i &= \mathcal{E}_i' \bar{F}_i \end{aligned}$$

8. Approximation of Smooth Systems

One motivation for the study of piecewise linear systems is that they can be used to approximate smooth nonlinear systems. The purpose of this section is to show how the approximation error can be explicitly taken into account, in order to generate formal results also for smooth systems. Moreover, we prove a converse result for smooth nonlinear systems on the existence and computability of piecewise quadratic Lyapunov functions.

In [Johansson and Rantzer, 1997], it was suggested that upper and lower bounds of the smooth nonlinearity are used in each polyhedral region. Stability of the original system follows if it is possible to find a Lyapunov function that is valid for the bounding systems in all regions. Another good alternative, particularly for multivariable nonlinearities, is to use a norm bound of the approximation error in the following manner.

THEOREM 9
Let $x(t)$ be a piecewise C^1 trajectory of the system $\dot{x} = f(x)$ and assume that

$$|f(x) - A_i x - a_i| \leq \epsilon_i |x| \qquad\qquad i \in I$$

If there exist numbers $\gamma_i > 0$, symmetric matrices U_i and V_i with non-negative entries, and a symmetric matrix T such that $\bar{P}_i = \bar{F}_i^T T \bar{F}_i$ and $P_i = F_i^T T F_i$ satisfy

$$E_i^T U_i E_i < P_i < \gamma_i I \tag{16}$$

$$-E_i^T V_i E_i > A_i^T P_i + P_i A_i + 2\epsilon_i \gamma_i I \tag{17}$$

for $i \in I_0$ and

$$\bar{E}_i^T U_i \bar{E}_i < \bar{P}_i < \gamma_i I \tag{18}$$

$$-\bar{E}_i^T V_i \bar{E}_i > \bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + 2\epsilon \gamma_i I \tag{19}$$

for $i \in I_1$, then $x(t)$ tends to zero exponentially. □

Proof. Define

$$\bar{V}(x) = \bar{x}' \bar{P}_i \bar{x} \qquad\qquad x \in X_i, \quad i \in I \tag{20}$$

The inequalities (16) and (18) imply that

$$c_1 |x|^2 \leq \bar{V}(x) \leq c_2 |x|^2$$

for some $c_1, c_2 > 0$. Let the approximation error be

$$\widetilde{a}_i(x) = \begin{bmatrix} f(x) - A_i x - a_i \\ 0 \end{bmatrix} \qquad\qquad x \in X_i,\ i \in I$$

Then, (17) and (16) together with the assumption $|\widetilde{a}_i(x)| \leq \epsilon_i |x|$ imply that

$$\begin{aligned} \frac{d}{dt}\bar{V}(x) &= \bar{x}(\bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i)\bar{x} + 2x^T \bar{P}_i \widetilde{a}_i(x) \\ &< -2(\epsilon_i \gamma_i + \delta)|x|^2 + 2\gamma_i |x| \cdot |\widetilde{a}_i(x)| \qquad (21) \\ &\leq -\delta |x|^2 \leq -\delta \bar{V}(x)/c_2 \qquad (22) \end{aligned}$$

for some $\delta > 0$. This proves the exponential decay. □

Theorem 9 quantifies the trade-off between computational effort and precision in the analysis. If no solution to the above inequalities is found, one may refine the state space partition for the piecewise linear system approximation and the piecewise quadratic Lyapunov function, and try again.

It is natural to ask how restrictive this approach is, compared to a theorem based on arbitrary continuous Lyapunov functions. The answer is given by the following result, showing that in principle, whenever a Lyapunov function exists, there also exists a solution to the relevant matrix inequalities.

THEOREM 10
Let $f \in C^1(X, \mathbb{R}^n)$ and assume that $\partial f/\partial x$ is bounded on X. Suppose that the system $\dot{x} = f(x)$ is globally exponentially stable. Then for every sufficiently refined partition $\{X_i\}_{i=1}^N$ with corresponding matrices $\bar{E}_i$, $\bar{F}_i$ and $A_i = f(w_i)$, there exists a solution γ_i, U_i, $\bar{U}_i$, V_i, $\bar{V}_i$ and T to the inequalities (16)-(19). □

Proof. First note that by a standard converse Lyapunov theorem, see Theorem 3.12 in [Khalil, 1996] for example, there exists a C^1 Lyapunov function $V(x)$ that satisfies

$$c_1|x|^2 \le V(x) \le c_2|x|^2 \tag{23}$$

$$\frac{\partial V}{\partial x} f(x) \le -c_3|x|^2 \tag{24}$$

$$\left|\frac{\partial V}{\partial x}\right| \le c_4|x|$$

for some positive constants c_1, c_2, c_3, c_4. The function V can be approximated by a function $\bar{V}$ of the form (20) by letting $\bar{P}_i = \bar{F}_i^T T \bar{F}_i$ with $\bar{F}_i$ defined by (14) and

$$T = [1 \dots 1]'[V(\nu_1) \dots V(\nu_p)]/2 + [V(\nu_1) \dots V(\nu_p)]'[1 \dots 1]/2$$

Then

$$\bar{V}(\nu_i) = T_{ii} := V(\nu_i), \qquad i \in I$$

and $\bar{V}$ and $\partial \bar{V}/\partial x$ become arbitrarily accurate approximations of V and $\partial V/\partial x$ as the partition is refined. Let γ_i be defined by the size of V. For sufficiently small approximation errors ϵ_i the inequalities (23) and (24) imply that

$$\frac{c_1|x|^2}{2} \le \begin{bmatrix} x \\ 1 \end{bmatrix}' \bar{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \le 2c_2|x|^2 \qquad x \in X_i$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}' (\bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i) \begin{bmatrix} x \\ 1 \end{bmatrix} \le -2\epsilon_i \gamma_i |x|^2 \qquad x \in X_i$$

What remains is to find U_i, $\bar{U}_i$, W_i and $\bar{W}_i$ with non-negative entries such that (16)-(19) hold. By the C^1 condition on f, it can be assumed without restriction that V and $\bar{V}$ are quadratic and positive definite in a neighborhood of $x = 0$. Hence U_i and W_i are not needed and can be put to zero. In the regions that do not contain the origin, $\bar{V}$ is linear, so $\bar{U}_i$ and $\bar{W}_i$ exist by Farkas lemma [Schrijver, 1986]. This completes the proof. □

9. Conclusions

A flexible and powerful approach to analysis and optimization of control systems has been developed using a combination of piecewise linear system descriptions and picewise quadratic Lyapunov functions and loss functions.

Local analysis of nonlinear systems near an equilibrium is usually done based on linearization. The linear approximation is good close to the equilibrium and there is a powerful theory for control and performance analysis of linear systems. However, as the region of investigation is extended, it becomes desirable to take the nonlinear effects more explicitly into account. Using the framework of this paper this can be done incrementally. Starting from the purely linear analysis, one can add more and more partitions of the state space in order to extend the investigated region of state space, piece by piece.

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