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BOUNDS ON THE DIRECT SCATTERING PROBLEMS OF ACOUSTIC AND ELECTROMAGNETIC WAVES

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ABSTRACT

This paper presents new bounds on the scattering and absorption cross sections for acoustic and electromagnetic waves. The main result states that the weighted extinction cross section integrated over all frequencies is equal to certain components of the static polarizability dyadics of the target. Various isoperimetric bounds are introduced and illustrated by numerical simulations in a finite frequency interval using the method of moments.

1. INTRODUCTION

Consider the direct scattering problem of a plane wave, $e^{ik\hat{\boldsymbol{k}}\cdot\boldsymbol{x}}$ or $e^{ik\hat{\boldsymbol{k}}\cdot\boldsymbol{x}}\hat{\boldsymbol{e}}$, depending on whether the problem is of acoustic or electromagnetic nature, of unit amplitude impinging in the $\hat{\boldsymbol{k}}$-direction on a bounded target V of arbitrary shape, see Fig. 1. The target is assumed to be linear and time-translational invariant with passive material parameters modeled by isotropic and heterogeneous constitutive relations.[1] To this end, introduce the following notation relative to $\mathbb{R}^3 \setminus \bar{V}$: the mass density $\rho_{\rm rel}(\boldsymbol{x}, k)$ and the homogeneous compressibility $\kappa_{\rm rel}(k)$, and the permittivity $\epsilon_{\rm rel}(\boldsymbol{x}, k)$ and the permeability $\mu_{\rm rel}(\boldsymbol{x}, k)$. Here, k denotes the angular wave number of the background $\mathbb{R}^3 \setminus \bar{V}$ which is assumed to be a homogeneous, isotropic, and lossless medium that supports either acoustic or electromagnetic waves.

In a series of papers, see Refs. 1–4, published in 2007–2008, a forward dispersion relation for the combined effect of scattering and absorption (*i.e.*, the extinction cross section $\sigma_{\rm ext} = \sigma_{\rm s} + \sigma_{\rm a}$) is derived from the holomorphic properties of the scattering amplitude in the forward direction. The result for the acoustic transmission problem reads

$$\int_0^\infty \frac{\sigma_{\rm ext}(k;\hat{\boldsymbol{k}})}{k^2}\,\mathrm{d}k = \frac{\pi}{2}\left((\kappa_{\rm s}-1)|V| - \hat{\boldsymbol{k}}\cdot\boldsymbol{\gamma}(1/\rho_{\rm s})\cdot\hat{\boldsymbol{k}}\right), \quad \text{(A)}$$

where $|V|$ denotes the volume of supp V. This identity can easily be modified to also include the Neumann

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[1] The results in this paper can also be modified to include anisotropic or bi-anisotropic materials.
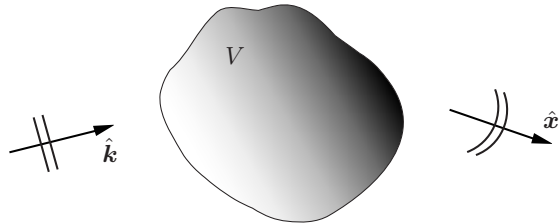
Fig. 1. Geometry for the direct scattering problem.

boundary condition. The corresponding result for the electromagnetic transmission problem is

$$\int_0^\infty \frac{\sigma_{\rm ext}(k;\hat{\boldsymbol{k}})}{k^2}\,\mathrm{d}k =$$
$$\frac{\pi}{2}\left(\hat{\boldsymbol{e}}^*\cdot\boldsymbol{\gamma}(\epsilon_{\rm s})\cdot\hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}}^*)\cdot\boldsymbol{\gamma}(\mu_{\rm s})\cdot\hat{\boldsymbol{k}}\right). \quad \text{(EM)}$$

It is indeed surprising that the right hand sides of (A) and (EM) only depend on the static material parameters of V, *viz.*, $\epsilon_{\rm s} = \epsilon_{\rm rel}(\boldsymbol{x},0)$, $\mu_{\rm s} = \mu_{\rm rel}(\boldsymbol{x},0)$, $\rho_{\rm s} = \rho_{\rm rel}(\boldsymbol{x},0)$, and $\kappa_{\rm s} = \kappa_{\rm rel}(0)$. Here, the static polarizability dyadic $\boldsymbol{\gamma}$ is defined by ($\alpha_{\rm s}$ denotes any of $\epsilon_{\rm s}$, $\mu_{\rm s}$, and $1/\rho_{\rm s}$)

$$\boldsymbol{\gamma}(\alpha_{\rm s}) = \sum_{i,j=1}^3 \left(\hat{\boldsymbol{a}}_i \cdot \boldsymbol{\gamma}_{ij}\right)\hat{\boldsymbol{a}}_i\hat{\boldsymbol{a}}_j,$$

where $\hat{\boldsymbol{a}}_1$, $\hat{\boldsymbol{a}}_2$ and $\hat{\boldsymbol{a}}_3$ form an arbitrary set of linearly independent unit vectors, and

$$\boldsymbol{\gamma}_{ij} = \int_{\mathbb{R}^3} (\alpha_{\rm s}(\boldsymbol{x}) - 1)(\hat{\boldsymbol{a}}_j - \nabla\psi_j(\boldsymbol{x}))\,\mathrm{dV}_{\boldsymbol{x}}.$$

The scalar potential ψ_j is the unique solution of the static boundary value problem

$$\begin{cases} \nabla\cdot(\alpha_{\rm s}(\boldsymbol{x})\nabla\psi_j(\boldsymbol{x})) = \hat{\boldsymbol{a}}_j\cdot\nabla\alpha_{\rm s}(\boldsymbol{x}) \\ \psi_j(\boldsymbol{x}) = \mathcal{O}(x^{-2}) \text{ as } x\to\infty \end{cases} \quad \boldsymbol{x}\in\mathbb{R}^3,$$

where $x = |\boldsymbol{x}|$. For example, the homogeneous sphere of radius a yields $\boldsymbol{\gamma}(\alpha_{\rm s}) = \gamma(\alpha_{\rm s})\mathbf{I}_3$, where $\mathbf{I}_3$ denotes the unit dyadic in $\mathbb{R}^3$. The corresponding eigenvalues for the sphere are degenerate with $\gamma(\alpha_{\rm s}) = 4\pi a^3(\alpha_{\rm s}-1)/(\alpha_{\rm s}+2)$.

In particular, the left-hand sides of (A) and (EM) can be estimated from below by (σ being any of σ_{ext}, σ_{s}, and σ_{a})

$$\frac{B}{k_0} \min_{k \in K} \sigma(k; \hat{\boldsymbol{k}}) \leq \int_K \frac{\sigma(k; \hat{\boldsymbol{k}})}{k^2} \, \mathrm{d}k \leq \int_0^\infty \frac{\sigma_{\text{ext}}(k; \hat{\boldsymbol{k}})}{k^2} \, \mathrm{d}k,$$

where $K \subset [0, \infty)$ with center wave number k_0 and relative bandwidth $B = \int_K \mathrm{d}k/k_0$. Thus, the interpretation of (A) and (EM) is that there only is a limited amount of scattering and absorption available in any frequency interval, and that the associated upper bound solely depends on the static properties of V.

This conference presentation will focus on the derivation of (A) and (EM) and its many intriguing consequences on the direct scattering problems of acoustic and electromagnetic waves. In particular, the theoretical findings will be exemplified by temporal dispersive material parameters using the methods of moments and the Mie series approach.

2. REFERENCES

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