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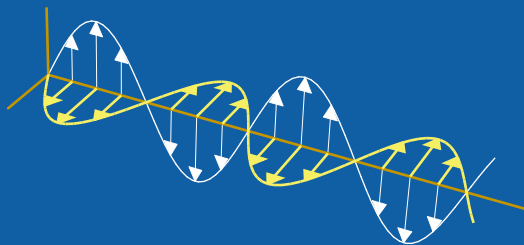
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# Time-domain Green dyadics for temporally dispersive, simple media

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## Abstract

Time-domain fundamental solutions and Green dyadics for temporally dispersive, simple media are introduced. Second forerunner approximations to the electromagnetic fields from an electric point dipole in unbounded dispersive materials are obtained. Numerical results for two frequently used material models are presented. Moreover, surface integral representations of the electromagnetic fields in dispersive media are obtained, and, finally, surface integral equations are derived for impenetrable and permeable scatterers.

## 1 Introduction

Time-harmonic fundamental solutions (Green functions) and Green dyadics for simple (linear, homogeneous, and isotropic) media are well-known concepts in electromagnetic field theory, and their importance in analyzing propagation and scattering problems is well recognized [2, 4, 8, 17]. For example, in an unbounded medium, the electric field due to any time-harmonic current distribution can be written in terms of a three-fold (spatial) convolution of the Green dyadic and the current density. From this volume representation, approximations to the far-field pattern can be derived [4]. Fundamental solution and Green dyadics can be used to obtain surface integral representations of the electric and magnetic fields in dielectric bodies as well [2, 16]. From these surface integral representations, integral equations for the tangential components of the electric and magnetic fields at the boundary can be derived [16]. These integral equations are fundamental in scattering theory.

Most media of any interest are temporally dispersive, *i.e.*, the material parameters depend on frequency. In fact, it is generally assumed that dispersion is anomalous in certain frequency bands (absorption bands) [4]. In time-harmonic scattering problems, the above technique for simple media can obviously be adopted. However, in order to study pulse propagation phenomena in these media, time-harmonic Green functions are insufficient. One way to cope with this is to introduce time-domain Green functions. The time-domain Green functions and dyadics for vacuum are well known [10]. In the present article, the time-domain fundamental solutions and Green dyadics for temporally dispersive, simple materials are defined. Furthermore, surface integral representations of the electric and magnetic fields are derived and used to obtain surface integral equations for the tangential components of the electromagnetic fields for two standard scattering problems. The complex time domain electromagnetic field is introduced to simplify the analysis. The advantages with this complex field are more evident in the analysis of propagation problems in bi-isotropic media. Green dyadics of temporally dispersive, bi-isotropic materials are discussed in a subsequent paper.

The outline of the present article is as follows. In Section 2, the notation is introduced. In Section 3, the constitutive relations of temporally dispersive, simple media are presented and the appropriate intrinsic temporal integral operators are introduced. In Section 4, the retarded fundamental solution of the dispersive wave operator is presented. In Section 5, Green dyadics for the electric and magnetic fields are defined. The Green dyadic for the complex time-domain electromagnetic

field vector is presented in Section 6. In Section 7, the explicit expressions for the electromagnetic fields from an electric point dipole in an unbounded dispersive medium are obtained and tested numerically. Surface integral representations of the complex electromagnetic field are accounted for in Section 8. In Section 9, surface integral equations for the electromagnetic field are derived for two different scattering problems. Conclusions are drawn in Section 10.

## 2 Notation and basic theory

In this article, scalars are typed in italic style, vectors in italic boldface style, and dyadics in Roman boldface style. The radius vector is written  $\mathbf{r} = \mathbf{u}_x x + \mathbf{u}_y y + \mathbf{u}_z z$ , where  $\mathbf{u}_x$ ,  $\mathbf{u}_y$ , and  $\mathbf{u}_z$  are the right-handed Cartesian basis vectors. The modulus of the radius vector is denoted by  $r = |\mathbf{r}|$ . The identity dyadic is written  $\mathbf{I} = \mathbf{u}_x \mathbf{u}_x + \mathbf{u}_y \mathbf{u}_y + \mathbf{u}_z \mathbf{u}_z$ . Furthermore, the dyadic differential operators (see [8]),

$$\begin{aligned}\nabla\nabla &= (\partial_x \mathbf{u}_x + \partial_y \mathbf{u}_y + \partial_z \mathbf{u}_z) (\partial_x \mathbf{u}_x + \partial_y \mathbf{u}_y + \partial_z \mathbf{u}_z), \\ \nabla \times \mathbf{I} &= (\partial_x \mathbf{u}_x + \partial_y \mathbf{u}_y + \partial_z \mathbf{u}_z) \times (\mathbf{u}_x \mathbf{u}_x + \mathbf{u}_y \mathbf{u}_y + \mathbf{u}_z \mathbf{u}_z),\end{aligned}$$

the first being symmetric and the second anti-symmetric, are encountered.

Time is denoted by  $t$ . Many temporal integrals appear in this paper, and, for brevity, the integration limits  $-\infty$  and  $\infty$  are generally omitted, *i.e.*,

$$\int \dots dt' = \int_{-\infty}^{\infty} \dots dt'.$$

The electric and magnetic field intensities at the space-time point  $(\mathbf{r}, t)$  are  $\mathbf{E}(\mathbf{r}, t)$  and  $\mathbf{H}(\mathbf{r}, t)$ , respectively, and the corresponding flux densities are  $\mathbf{D}(\mathbf{r}, t)$  and  $\mathbf{B}(\mathbf{r}, t)$ . The current and charge densities are  $\mathbf{J}(\mathbf{r}, t)$  and  $\rho(\mathbf{r}, t)$ , respectively. Standard notation is used for the speed of light in vacuum,  $c_0$ , the intrinsic impedance of vacuum,  $\eta_0$ , the permittivity,  $\epsilon_0$ , and the permeability,  $\mu_0$ , of vacuum.

The Maxwell equations model the dynamics of the electromagnetic fields in macroscopic media. They are

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}, t) = -\partial_t \mathbf{B}(\mathbf{r}, t), & \text{(Faraday's law),} \\ \nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) + \partial_t \mathbf{D}(\mathbf{r}, t), & \text{(Ampère-Maxwell's law),} \end{cases} \quad (2.1)$$

where the current and charge densities are connected via the equation of continuity

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \partial_t \rho(\mathbf{r}, t) = 0. \quad (2.2)$$

In order to solve the Maxwell equations, a constitutive law must be imposed on the fields. In vacuum,  $\mathbf{D} = \epsilon_0 \mathbf{E}$  and  $\mathbf{B} = \mu_0 \mathbf{H}$ . The constitutive relation used in this paper are introduced in the Section 3.

Since the magnetic flux density is solenoidal, there exists a vector potential,  $\mathbf{A}(\mathbf{r}, t)$ , such that

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t).$$

Substituting this result into Faraday's law above shows, that, in each simply connected region, there exists a scalar potential,  $V(\mathbf{r}, t)$ , such that

$$\mathbf{E}(\mathbf{r}, t) = -\nabla V(\mathbf{r}, t) - \partial_t \mathbf{A}(\mathbf{r}, t).$$

The vector and scalar potentials are not unique [15]. In vacuum, this fact can be exploited to impose supplementary conditions, such as the Lorenz gauge, on the potentials, *cf.* Section 4.

All electromagnetic fields, potentials, and source terms are assumed to vanish before a certain time  $t$ , say  $t = 0$ . Requiring initially quiescent quantities is compatible with using the temporal anti-derivative:

$$(\partial_t^{-1} f)(\mathbf{r}, t) = \int_{-\infty}^t f(\mathbf{r}, t') dt'. \quad (2.3)$$

Notice that the temporal derivative and anti-derivative commute:  $\partial_t^{-1} \partial_t = \partial_t \partial_t^{-1} = 1$ .

### 3 Constitutive relations

The constitutive relations of a temporally dispersive, simple (linear, homogeneous, and isotropic) medium can be written as [5]

$$\begin{cases} \mathbf{D}(\mathbf{r}, t) = \epsilon_0(\mathbf{E}(\mathbf{r}, t) + (\chi^e * \mathbf{E})(\mathbf{r}, t)) = \epsilon_0 [\varepsilon \mathbf{E}](\mathbf{r}, t), \\ \mathbf{B}(\mathbf{r}, t) = \mu_0(\mathbf{H}(\mathbf{r}, t) + (\chi^m * \mathbf{H})(\mathbf{r}, t)) = \mu_0 [\mu \mathbf{H}](\mathbf{r}, t), \end{cases}$$

where the time-varying functions  $\chi^e(t)$  and  $\chi^m(t)$  are the electric and magnetic susceptibility kernels of the medium, respectively, and the temporal integral operators  $\varepsilon = (1 + \chi^e *)$  and  $\mu = (1 + \chi^m *)$  are the *relative* permittivity and permeability operators, respectively. The asterisk (\*) denotes temporal convolution:

$$(\chi * \mathbf{E})(\mathbf{r}, t) = \int \chi(t - t') \mathbf{E}(\mathbf{r}, t') dt'.$$

An isotropic medium is said to be nonmagnetic if  $\chi^m(t) = 0$  (*i.e.*,  $\mu = 1$ ).

The susceptibility kernels  $\chi^e(t)$  and  $\chi^m(t)$  vanish for  $t < 0$  due to causality and, furthermore, they are assumed to be smooth and bounded for  $t > 0$ . This implies that the wave front propagates through the medium with the speed of light in vacuum,  $c_0$ .

Pulse propagation problems in temporally dispersive, simple media can be simplified by introducing the index of refraction  $\mathcal{N} = (1 + N *)$  and the relative intrinsic impedance  $\mathcal{Z} = (1 + Z *)$ , both of which being intrinsic temporal integral operators of the medium. By definition (*cf.* Ref. 6),

$$\mathcal{N}^2 = \varepsilon \mu, \quad \mathcal{Z}^2 \varepsilon = \mu, \quad (3.1)$$

or equivalently,

$$\mathcal{N} = \mathcal{Z} \varepsilon, \quad \mathcal{N} \mathcal{Z} = \mu. \quad (3.2)$$

The refractive kernel,  $N(t)$ , and the impedance kernel,  $Z(t)$ , are well-defined by these equations; in particular, these kernels vanish for  $t < 0$  and are bounded and smooth for  $t > 0$ . Specifically, the refractive kernel satisfies the nonlinear Volterra integral equation of the second kind

$$(N * N)(t) + 2N(t) = \chi^e(t) + \chi^m(t) + (\chi^e * \chi^m)(t),$$

*cf.* the first operator identity in (3.1), whereupon the impedance kernel can be obtained by solving the linear Volterra integral equation of the second kind

$$N(t) + Z(t) + (N * Z)(t) = \chi^m(t),$$

*cf.* the second identity in (3.2). In addition to the refractive index and the intrinsic impedance, it is convenient to introduce the wave-number operator

$$\mathcal{K} = c_0^{-1} \partial_t \mathcal{N} = c_0^{-1} \partial_t (1 + N *).$$

The inverses of the operators introduced above exist and are well-defined temporal integral operators. For instance, the inverse of the refractive index can be written as  $\mathcal{N}^{-1} = (1 + N_{\text{res}} *)$ , where the resolvent kernel,  $N_{\text{res}}(t)$ , satisfies the linear Volterra integral equation of the second kind

$$N(t) + N_{\text{res}}(t) + (N * N_{\text{res}})(t) = 0.$$

The resolvent kernel vanishes for  $t < 0$  and are bounded and smooth for  $t > 0$ . In the nonmagnetic case, the impedance operator  $\mathcal{Z}$  is the inverse of the refractive operator  $\mathcal{N}$ . Notice that  $\mathcal{K}^{-1}$  exists as well.

The notation  $\mathcal{K}^2 = \mathcal{K}\mathcal{K}$  ( $\mathcal{K}^{-2} = \mathcal{K}^{-1}\mathcal{K}^{-1}$ ) is also employed.

## 4 The retarded fundamental solution

The dispersive Lorenz gauge,  $\nabla \cdot \mathbf{A} + c_0^{-2} \partial_t \mathcal{N}^2 V = 0$ , relates the vector and scalar potentials in the dispersive media to each other. Using this gauge and the equation of continuity, (2.2), leads to potential representations of the electric and magnetic fields:

$$\begin{cases} \mathbf{E} = (-\mathbf{I} + \nabla \nabla \mathcal{K}^{-2}) \cdot \partial_t \mathbf{A}, \\ \mathbf{H} = \mu_0^{-1} \mu^{-1} (\nabla \times \mathbf{I}) \cdot \mathbf{A}. \end{cases} \quad (4.1)$$

The scalar and vector potentials satisfy the dispersive wave equations

$$(-\Delta + \mathcal{K}^2) V = \epsilon_0^{-1} \epsilon^{-1} \rho, \quad (-\Delta + \mathcal{K}^2) \mathbf{A} = \mu_0 \mu \mathbf{J}.$$

These equations can be solved in terms of the retarded fundamental solution,  $\mathcal{E}(\mathbf{r}; t)$ , of the dispersive wave operator

$$-\Delta + \mathcal{K}^2. \quad (4.2)$$

The following formal definition is made:

**Definition.** Let  $N(t)$  be a function of time, which vanishes for  $t < 0$  and is bounded and smooth for  $t > 0$ . A distribution in the three-dimensional space and in time,  $\mathcal{E}(\mathbf{r}; t)$  is said to be a fundamental solution of the dispersive wave operator (4.2) if it satisfies the dispersive wave equation,  $(-\Delta + \mathcal{K}^2)\mathcal{E} = \delta_{\mathbf{0}} \otimes \delta_0$ , where  $\delta_{\mathbf{0}} = \delta(\mathbf{r})$  is the Dirac measure in  $\mathbb{R}^3$ , and  $\delta_0 = \delta(t)$  the Dirac measure in time.

The retarded (causal) fundamental solution,  $\mathcal{E}(\mathbf{r}; t)$ , of the dispersive wave operator is defined in the following theorem:

**Theorem.** Let  $N(t)$  be a function of time, which vanishes for  $t < 0$  and is bounded and smooth for  $t > 0$ . Then the distribution

$$\mathcal{E}(\mathbf{r}; t) = q(r) \frac{1}{4\pi r} \left( \delta \left( t - \frac{r}{c_0} \right) + P \left( r; t - \frac{r}{c_0} \right) \right), \quad (4.3)$$

where  $q(r)$  satisfies the ordinary differential equation

$$c_0 \partial_r q(r) = -N(+0)q(r), \quad q(0) = 1, \quad (4.4)$$

and  $P(r; t)$  satisfies the integro-differential equation

$$c_0 \partial_r P(r; t) = -N'(t) - (N'(\cdot) * P(r; \cdot))(t), \quad P(0; t) = 0, \quad (4.5)$$

is a fundamental solution of the dispersive wave operator (4.2). The propagator kernel,  $P(r; t)$ , vanishes for  $t < 0$  and is bounded and smooth in each bounded time interval  $0 < t < T$ , whence  $\mathcal{E}(\mathbf{r}; t) = 0$  for  $t < r/c_0$ .

The well-known result for vacuum,  $\mathcal{E}_0(\mathbf{r}; t) = 1/(4\pi r) \delta(t - r/c_0)$ , is obtained from (4.3) by setting  $N = 0$ .

Using equations (4.4)–(4.5) and the fact that  $\partial_r \delta(t - r/c_0) = -c_0^{-1} \partial_t \delta(t - r/c_0)$ , show that the retarded fundamental solution,  $\mathcal{E}(\mathbf{r}; t)$ , satisfies the integro-differential equation

$$\partial_r (r\mathcal{E}) = -\mathcal{K}(r\mathcal{E}), \quad 4\pi \lim_{r \rightarrow 0} r\mathcal{E} = \delta_0. \quad (4.6)$$

The gradient of  $\mathcal{E}(\mathbf{r}; t)$  is then given by

$$\nabla \mathcal{E} = -\mathbf{r} \left( Pf. \frac{1}{r^3} + \frac{1}{r^2} \mathcal{K} \right) (r\mathcal{E}) = - \left( \frac{1}{r^2} + \frac{1}{r} \mathcal{K} \right) (r\mathcal{E}), \quad (4.7)$$

where Laurent Schwartz' *pseudo-fonction*  $Pf.(1/r^3)$  is defined in Appendix A. Taking the divergence of  $\nabla \mathcal{E}$ , using (4.7) and the rules of differentiation in Appendix A give

$$\Delta \mathcal{E} = \mathcal{K}^2 \mathcal{E} + \frac{4\pi}{3} (\mathbf{r} \cdot \nabla \delta_{\mathbf{0}}) r \mathcal{E}.$$

Using the limit value in (4.6) proves the theorem.

The function  $q(r)$  in (4.4) can be calculated explicitly:

$$q(r) = \exp \left( -\frac{r}{c_0} N(+0) \right). \quad (4.8)$$



A closed-form expression for the propagator kernel satisfying (4.5) cannot be obtained in general. However, it can be represented by an infinite function series:

$$P(r; t) = \sum_{k=1}^{\infty} \frac{1}{k!} \left( -\frac{r}{c_0} \right)^k ((N'*)^{k-1} N') (t). \quad (4.9)$$

Using the general identity for causal convolutions

$$t \frac{\overbrace{(f * \dots * f)}^{k \text{ functions}}}{k!} = t f * \frac{\overbrace{(f * \dots * f)}^{k-1 \text{ functions}}}{(k-1)!}, \quad k > 1,$$

which can be proved by mathematical induction, the propagator kernel is seen to satisfy the temporal Volterra integral equation of the second kind [6],

$$tP(r; t) = F(r; t) + (F(r; \cdot) * P(r; \cdot))(t), \quad F(r; t) = -t \frac{r}{c_0} N'(t). \quad (4.10)$$

This integral equation can be used in numerical computations. The initial condition becomes  $P(r; +0) = -N'(+0)r/c_0$ .

The importance of fundamental solutions of the dispersive wave operator (4.2) is analogous to the importance of fundamental solutions of differential operators with constant coefficients: if  $u(\mathbf{r}, t)$  and  $f(\mathbf{r}, t)$  are distributions in space and in time, such that  $(-\Delta + \mathcal{K}^2)u = f$ , then

$$u(\mathbf{r}, t) = \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \mathcal{E}(\mathbf{r} - \mathbf{r}'; t - t') f(\mathbf{r}', t') dt' \right) dv'.$$

Consequently, the volume representations of the vector and scalar potentials in an unbounded, temporally dispersive, simple medium become

$$\begin{cases} \mathbf{A}(\mathbf{r}, t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(\mathbf{r} - \mathbf{r}'; t - t') \mu_0 [\boldsymbol{\mu} \mathbf{J}] (\mathbf{r}', t') dt' dv', \\ V(\mathbf{r}, t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{E}(\mathbf{r} - \mathbf{r}'; t - t') \epsilon_0^{-1} [\epsilon^{-1} \rho] (\mathbf{r}', t') dt' dv'. \end{cases}$$

Formally, in view of (4.6), the time-retarded fundamental solution of the dispersive wave operator,  $\mathcal{E}(\mathbf{r}; t)$ , can be expressed in terms of the temporal integro-differential operator,

$$\mathcal{E}(\mathbf{r}; \cdot) * = \frac{1}{4\pi r} \exp(-r\mathcal{K}), \quad (4.11)$$

which is to be interpreted in terms of the McLaurin series for the exponential function.

Using the characteristic property of the exponential, the operator (4.11) can be factored in three terms: the retarded delta function,  $\delta(t - r/c_0) * = \exp(-r/c_0 \partial_t)$ , the wave-front propagator  $q(r)/(4\pi r)$ , where  $q(r)$  is given by (4.8), and the wave propagator,  $(\delta(\cdot) + P(r; \cdot)) * = \exp(-(r/c_0)N'(\cdot)*)$ . The wave propagator is a temporal integral operator interpreted in terms of the McLaurin series for the exponential, *cf.* (4.9). The wave propagator and the wave-front propagator concepts are well known from one-dimensional propagation problems involving temporally dispersive media (see, *e.g.*, [11]).

## 5 Green dyadics

The time-domain Green dyadic for the electric field,  $\mathbf{G}_E(\mathbf{r}; t)$ , is defined by

$$\mathbf{E}(\mathbf{r}, t) = \int_{\mathbb{R}^3} \int \mathbf{G}_E(\mathbf{r} - \mathbf{r}'; t - t') \cdot \mu_0 \partial_{t'} [\mu \mathbf{J}](\mathbf{r}', t') dt' dv'. \quad (5.1)$$

According to the potential representation (4.1), the Green dyadic  $\mathbf{G}_E(\mathbf{r}; t)$  becomes

$$\mathbf{G}_E = -(\mathbf{I} - \nabla \nabla \mathcal{K}^{-2}) \mathcal{E}, \quad (5.2)$$

where  $\mathbf{I}$  is the unit dyadic. This definition of the Green dyadic coincides with the one used by Van Bladel [17] (frequency domain) and, except for a multiplicative sign, with the one used by Lindell [8] (frequency domain), and by Marx and Maystre [10] (time domain, vacuum).

As in the time-harmonic analysis, the second spatial differentiation must be carried out with care in the case the field-point lies within the current distribution. For a result obtained by using Schwartz' *pseudo-functions*, see Appendix A . For a more general time-harmonic approach, based on first omitting a not necessarily spherical neighborhood of the field-point before differentiation, see Van Bladel [17].

The time-domain Green dyadic for the magnetic field,  $\mathbf{G}_H(\mathbf{r}; t)$ , is defined by

$$\mathbf{H}(\mathbf{r}, t) = \int_{\mathbb{R}^3} \int \mathbf{G}_H(\mathbf{r} - \mathbf{r}'; t - t') \cdot \mu_0 \partial_{t'} [\mu \mathbf{J}](\mathbf{r}', t') dt' dv'. \quad (5.3)$$

The Green dyadic  $\mathbf{G}_H(\mathbf{r}; t)$  is given by

$$\mathbf{G}_H = (\nabla \times \mathbf{I}) \mu_0^{-1} \partial_t^{-1} \mu^{-1} \mathcal{E} = -\eta_0^{-1} \mathcal{Z}^{-1} \left( \frac{1}{r^2} \mathcal{K}^{-1} + \frac{1}{r} \right) (\mathbf{r} \mathcal{E} \times \mathbf{I}), \quad (5.4)$$

where equation (4.7) has been used. Since  $\mathbf{r} \mathcal{E} \times \mathbf{I}$  is regular,  $\mathbf{G}_H$  has merely a weak singularity at the origin.

The Maxwell equations (2.1) show that the Green dyadics for the electric and magnetic fields are related through

$$\nabla \times \mathbf{G}_E = -\mu_0 \partial_t \mu \mathbf{G}_H, \quad \nabla \times \mathbf{G}_H = \epsilon_0 \partial_t \mathcal{E} \mathbf{G}_E + \mathbf{I} \mu_0^{-1} \partial_t^{-1} \mu^{-1} [\delta_0 \otimes \delta_0]. \quad (5.5)$$

Obviously, the dyadics  $\mathbf{G}_E(\mathbf{r}; t)$  and  $\mathbf{G}_H(\mathbf{r}; t)$  satisfy the condition of causality, *i.e.*,  $\mathbf{G}_E(\mathbf{r}; t) = \mathbf{0}$ ,  $\mathbf{G}_H(\mathbf{r}; t) = \mathbf{0}$  for  $t < 0$ . Notice that the symmetric dyadic  $\mathbf{G}_E$  is symmetric in the spatial variables and that the anti-symmetric dyadic  $\mathbf{G}_H$  is anti-symmetric in the spatial variables; therefore, the following relations hold:

$$\mathbf{G}_E(\mathbf{r} - \mathbf{r}'; t - t') = \mathbf{G}_E^T(\mathbf{r}' - \mathbf{r}; t - t'), \quad \mathbf{G}_H(\mathbf{r} - \mathbf{r}'; t - t') = \mathbf{G}_H^T(\mathbf{r}' - \mathbf{r}; t - t').$$

Recall that the transpose of a dyadic  $\mathbf{A}$  is a dyadic  $\mathbf{A}^T$  such that  $\mathbf{A}^T \cdot \mathbf{F} = \mathbf{F} \cdot \mathbf{A}$  for each vector  $\mathbf{F}$  and that  $\mathbf{A}$  is said to be symmetric if  $\mathbf{A} = \mathbf{A}^T$  and anti-symmetric if  $\mathbf{A} = -\mathbf{A}^T$ .

## 6 The complex time-dependent field

The notion of complex time-dependent fields is not new in electromagnetic field analysis. These fields have been used before under different names [1, 9, 15]. In this paper, a complex, time-dependent electromagnetic field,  $\mathbf{Q}(\mathbf{r}, t)$ , is defined by

$$\mathbf{Q} = \frac{1}{2}(\mathbf{E} - i\eta_0\mathcal{Z}\mathbf{H}), \quad (6.1)$$

where  $i$  is the imaginary unit, *cf.* Ref. 12. Note that the vector fields  $\mathbf{E}(\mathbf{r}, t)$  and  $\mathbf{H}(\mathbf{r}, t)$  are real-valued. Using this field reduces the Maxwell equations to the compact form

$$\nabla \times \mathbf{Q} = -i\mathcal{K}\mathbf{Q} - \frac{1}{2}i\eta_0\mathcal{Z}\mathbf{J}. \quad (6.2)$$

Observe, that  $\mathbf{E} = \mathbf{Q} + \mathbf{Q}^*$  and  $\eta_0\mathbf{H} = i\mathcal{Z}^{-1}(\mathbf{Q} - \mathbf{Q}^*)$ , where  $\mathbf{Q}^*$  is the complex conjugate of  $\mathbf{Q}$ .

In analogy with (5.1) and (5.3), the time-domain Green dyadic for the complex electromagnetic field,  $\mathbf{G}_Q(\mathbf{r}; t)$ , is defined by

$$\mathbf{Q}(\mathbf{r}, t) = \int_{\mathbb{R}^3} \int \mathbf{G}_Q(\mathbf{r} - \mathbf{r}'; t - t') \cdot \mu_0 \partial_{t'} [\mu \mathbf{J}](\mathbf{r}', t') dt' dv', \quad (6.3)$$

thus, *cf.* (6.1),

$$\mathbf{G}_Q = \frac{1}{2}(\mathbf{G}_E - i\eta_0\mathcal{Z}\mathbf{G}_H) = -\frac{1}{2}(\mathbf{I} - \nabla\nabla\mathcal{K}^{-2} + i(\nabla \times \mathbf{I})\mathcal{K}^{-1})\mathcal{E}. \quad (6.4)$$

Observe, that,  $\mathbf{G}_E = \mathbf{G}_Q + \mathbf{G}_Q^*$ , and,  $\eta_0\mathbf{G}_H = i\mathcal{Z}^{-1}(\mathbf{G}_Q - \mathbf{G}_Q^*)$ , where  $\mathbf{G}_Q^*$  is the complex conjugate of  $\mathbf{G}_Q$ .

The differential equations (5.5) show that the Green dyadic for the complex electromagnetic field satisfies the integro-differential equation

$$2[\nabla \times \mathbf{I} + i\mathbf{I}\mathcal{K}] \cdot \mathbf{G}_Q = -i\mathbf{I}\mathcal{K}^{-1}[\delta_0 \otimes \delta_0]. \quad (6.5)$$

Obviously, the dyadic  $\mathbf{G}_Q(\mathbf{r}; t)$  is causal:  $\mathbf{G}_Q(\mathbf{r}; t) = \mathbf{0}$  for  $t < 0$ . Notice that

$$\mathbf{G}_Q(\mathbf{r} - \mathbf{r}'; t - t') = \mathbf{G}_Q^T(\mathbf{r}' - \mathbf{r}; t - t'). \quad (6.6)$$

In Appendix A, the Green dyadic (6.4) is given explicitly in terms of Schwartz' *pseudo-fonctions*.

## 7 Example

### 7.1 Point dipole in an unbounded dispersive medium

An electric point dipole concentrated at the origin which is flashed on and off at  $t = 0$  is characterized by the dipole moment  $\mathbf{p}\delta_0$ , where  $\mathbf{p}$  is a constant vector. For

this source distribution, the charge density is  $\rho = -(\mathbf{p} \cdot \nabla)(\delta_0 \otimes \delta_0)$ . According to the equation of continuity (2.2), the current density becomes  $\mathbf{J} = \mathbf{p} \partial_t (\delta_0 \otimes \delta_0)$ . Straightforward calculations using equations (6.3) and (A.1) lead to the expression

$$\begin{aligned} 2\epsilon_0 \mathbf{Q} = & -\frac{1}{3} \mathbf{p} \epsilon^{-1} [\delta_0 \otimes \delta_0] + (\mathbf{u}_r \mathbf{u}_r - \mathbf{I}) \cdot \mathbf{p} c_0^{-2} \partial_t^2 \mu \mathcal{E} \\ & + (3\mathbf{u}_r \mathbf{u}_r - \mathbf{I}) \cdot \mathbf{p} \left( Pf. \left( \frac{1}{r^3} \right) c_0 \partial_t^{-1} \epsilon^{-1} + \frac{1}{r^2} \mathcal{Z} \right) c_0^{-1} \partial_t (r \mathcal{E}) \\ & + i \left( \frac{1}{r^2} + \frac{1}{r} \mathcal{K} \right) c_0^{-1} \partial_t \mathcal{Z} (r \mathcal{E} \times \mathbf{p}) \end{aligned}$$

for the complex time-dependent electromagnetic field due to the electric point dipole.

For the electric and magnetic fields, one obtains

$$\begin{aligned} \epsilon_0 \mathbf{E} = & -\frac{1}{3} \mathbf{p} (1 + \chi^{e*})^{-1} [\delta_0 \otimes \delta_0] + (\mathbf{u}_r \mathbf{u}_r - \mathbf{I}) \cdot \mathbf{p} c_0^{-2} \partial_t^2 (1 + \chi^{m*}) \mathcal{E} \\ & + (3\mathbf{u}_r \mathbf{u}_r - \mathbf{I}) \cdot \mathbf{p} \left( Pf. \left( \frac{1}{r^3} \right) (1 + \chi^{e*})^{-1} (r \mathcal{E}) + \frac{1}{r^2} c_0^{-1} \partial_t (1 + Z^*) (r \mathcal{E}) \right) \end{aligned} \quad (7.1)$$

and

$$\mathbf{H} = \left( \frac{1}{r^2} + \frac{1}{r} c_0^{-1} \partial_t (1 + N^*) \right) \partial_t (\mathbf{p} \times r \mathcal{E}). \quad (7.2)$$

If  $\mathbf{p} = \mathbf{u}_z p$ , there are only three nonvanishing field components:

$$\mathbf{E} = \mathbf{u}_r E_r + \mathbf{u}_\theta E_\theta, \quad \mathbf{H} = \mathbf{u}_\phi H_\phi, \quad (7.3)$$

where  $\mathbf{u}_r$ ,  $\mathbf{u}_\theta$ , and  $\mathbf{u}_\phi$  are the basis vectors in spherical coordinates. Numerically, the fields  $\mathbf{E}$  and  $\mathbf{H}$  can be obtained by first solving the integral equation (4.10) and then performing all the convolutions in (7.1)–(7.2). This procedure is time and memory consuming.

Using the technique introduced in Ref. 6 gives an approximation to the dipole fields with respect to the slowly varying components (second forerunner approximations). Applying this technique, the representation (4.11) is approximated by

$$\mathcal{E}(\mathbf{r}; \cdot)^* \approx \frac{1}{4\pi r} \exp \left( -\frac{r}{c_0} \left( (1+n_1) \partial_t + n_2 \partial_t^2 + n_3 \partial_t^3 \right) \right) = \tilde{\mathcal{E}}(\mathbf{r}; \cdot)^*, \quad (7.4)$$

where

$$\begin{cases} \tilde{\mathcal{E}}(\mathbf{r}; t) = \frac{1}{4\pi r} \exp \left( \frac{n_2^2}{27n_3^2} \frac{r}{c_0} - \frac{n_2}{3n_3} (t - t_1(r)) \right) \frac{\text{Ai}(\text{sign}(n_3)(t - t_1(r))/t_3(r))}{t_3(r)}, \\ n_m = \frac{(-1)^{m-1}}{(m-1)!} \int_0^\infty t^{m-1} N(t) dt, \quad m = 1, 2, 3, \\ t_1(r) = \left( n_1 + 1 - \frac{n_2^2}{3n_3} \right) \frac{r}{c_0}, \quad t_3(r) = \left( \frac{3|n_3|r}{c_0} \right)^{\frac{1}{3}} \end{cases}$$

and  $\text{Ai}(x)$  is the Airy function of the first kind. Here, it is assumed that the refractive coefficients  $n_m$  exist and are finite. To get asymptotic expressions for the dipole fields, approximate the convolution operators  $(1 + Z^*)$ ,  $(1 + \chi^m)^*$ , and  $(1 + \chi_{\text{res}}^e)^* = (1 + \chi^e)^{-1}$  in (7.1)–(7.2) by the first three terms in their series representations (see [6]), *e.g.*,

$$(1 + Z^*) \approx (1 + z_1) + z_2 \partial_t + z_3 \partial_t^2,$$

where the moments are

$$z_m = \frac{(-1)^{m-1}}{(m-1)!} \int_0^\infty t^{m-1} Z(t) dt, \quad m = 1, 2, 3.$$

Substituting these approximations and (7.4) into (7.1)–(7.2) and using the Airy equation give approximations to the dipole fields in terms of the Airy function and its first derivative. The result for the nonmagnetic case ( $\chi^m = 0$ ) is given explicitly in Appendix B. These expressions contain only algebraic combinations of the known quantities, the well-known Airy function, and its first derivative, which are standard functions in, *e.g.*, MATLAB 5. Consequently, no large computer capacities are needed to calculate the asymptotic fields.

Under certain conditions on the susceptibility kernels, one can get better approximations to the dipole fields by using more terms in the series decompositions of convolution operators in (7.4) and (7.1)–(7.2). The result is then expressed in terms of hyper-Airy functions [6].

Notice that this technique cannot be used to obtain the first precursor (the wave-front behavior) of the fields.

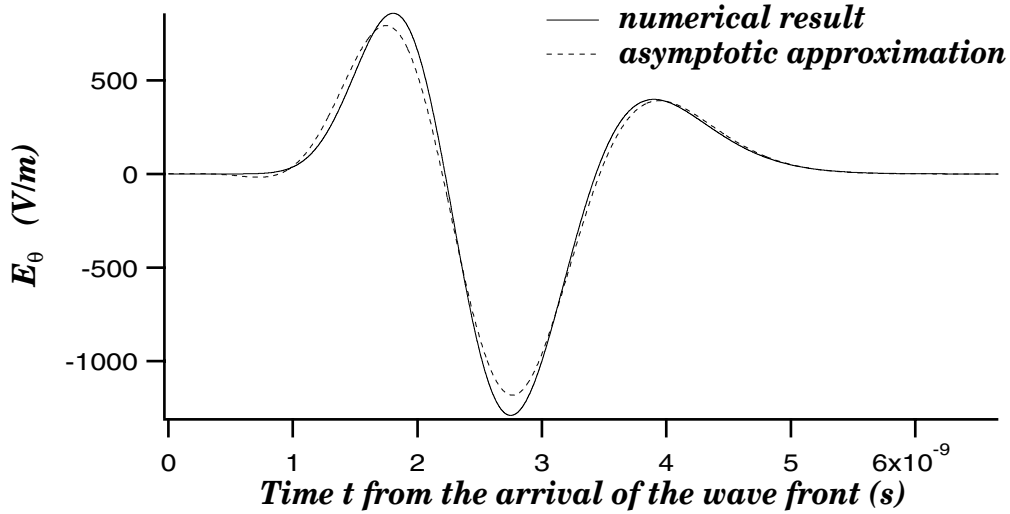
## 7.2 Numerical calculations

In this subsection, the asymptotic approximations to the dipole fields (7.3) given in Appendix B are specified for two frequently used material models. The results are compared to the numerical solution obtained by first solving the integral equation (4.10) and then performing all the convolutions in (7.1) and (7.2).

The first material model to be discussed is the Debye model [7]. The susceptibility kernels are given by

$$\chi^e(t) = H(t)\alpha \exp(-\beta t), \quad \chi^m(t) = 0,$$

where  $\alpha$  is a frequency and  $1/\beta$  the relaxation time. The Debye model is a good approximation for polar liquids (*e.g.*, water, alcohols) at microwave frequencies. For the Debye model, all the coefficients in the asymptotic expressions can be calculated



**Figure 1:** The field  $E_\theta$  from a dipole characterized by  $\mathbf{p} = \mathbf{u}_z 10^{-17} \text{ C} \cdot \text{m} \cdot \text{s}$  in a Debye medium at a distance  $r = 1 \text{ m}$  from the dipole at the observation angle  $\theta = \pi/4$ . The medium is characterized by the parameters  $\alpha = 3 \cdot 10^{10} \text{ s}^{-1}$ ,  $\beta = 1.2 \cdot 10^{10} \text{ s}^{-1}$ .

analytically. They are (*cf.* Ref. 6)

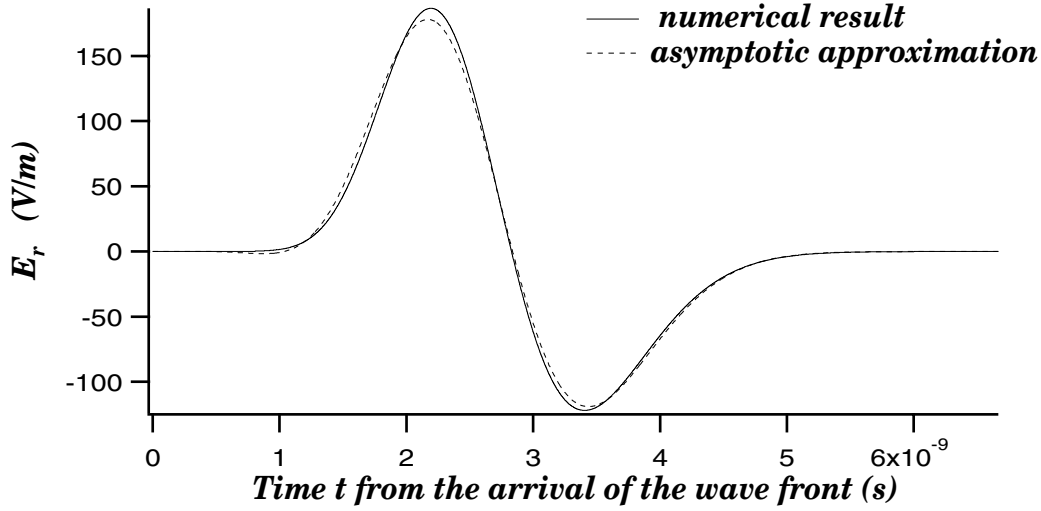
$$\begin{aligned} \chi_1 &= \frac{\alpha}{\beta}, & \chi_2 &= -\frac{\alpha}{\beta^2}, & \chi_3 &= \frac{\alpha}{\beta^3}, \\ \chi_{\text{res1}} &= -\frac{\chi_1}{1 + \chi_1}, & \chi_{\text{res2}} &= -\frac{\chi_2(1 + \chi_{\text{res1}})}{1 + \chi_1}, & \chi_{\text{res3}} &= -\frac{(1 + \chi_{\text{res1}})\chi_3 + \chi_{\text{res2}}\chi_2}{1 + \chi_1}, \\ n_1 &= \sqrt{1 + \frac{\alpha}{\beta}} - 1, & n_2 &= -\frac{\alpha}{2\beta^2 \sqrt{1 + \frac{\alpha}{\beta}}}, & n_3 &= \frac{4\beta\alpha + 3\alpha^2}{8\beta^4(1 + \alpha/\beta)^{3/2}}, \\ z_1 &= -\frac{n_1}{1 + n_1}, & z_2 &= -\frac{n_2(1 + z_1)}{1 + n_1}, & z_3 &= -\frac{(1 + z_1)n_3 + z_2n_2}{1 + n_1}. \end{aligned}$$

In Figures 1–3, the numerical results as well as the asymptotic approximations to the dipole fields in a Debye medium characterized by the parameters  $\alpha = 3 \cdot 10^{10} \text{ s}^{-1}$  and  $\beta = 1.2 \cdot 10^{10} \text{ s}^{-1}$  at a distance of 1 m from the dipole at the observation angle  $\theta = \pi/4$  are presented. The dipole moment is  $\mathbf{u}_z 10^{-17} \delta(t) \text{ C} \cdot \text{m}$ . The Debye model subjected to these parameters is a good approximation to the susceptibility function for water. The figures show good agreement between the numerical results and the approximations.

The second example is the single resonance Lorentz model [7]. The susceptibility kernels are given by

$$\chi^e(t) = H(t) \frac{\omega_p^2}{\nu_0} \sin(\nu_0 t) \exp\left(-\frac{\nu t}{2}\right), \quad \chi^m(t) = 0,$$

where  $\omega_0$ ,  $\omega_p$ , and  $\nu$  are the harmonic, plasma, and collision frequencies of the medium, respectively, and  $\nu_0 = \sqrt{\omega_0^2 - \nu^2/4}$ . The Lorentz model is often used for



**Figure 2:** The field  $E_r$  from a dipole in a Debye medium. For details see caption of Fig. 1.

solids at infra-red and optical frequencies. For this model, the coefficients in the asymptotic expressions are (*cf.* Ref. 6)

$$\chi_1 = \frac{\omega_p^2}{\omega_0^2}, \quad \chi_2 = -\frac{\nu\omega_p^2}{\omega_0^4}, \quad \chi_3 = -\frac{\omega_p^2(\omega_0^2 - \nu^2)}{\omega_0^6},$$

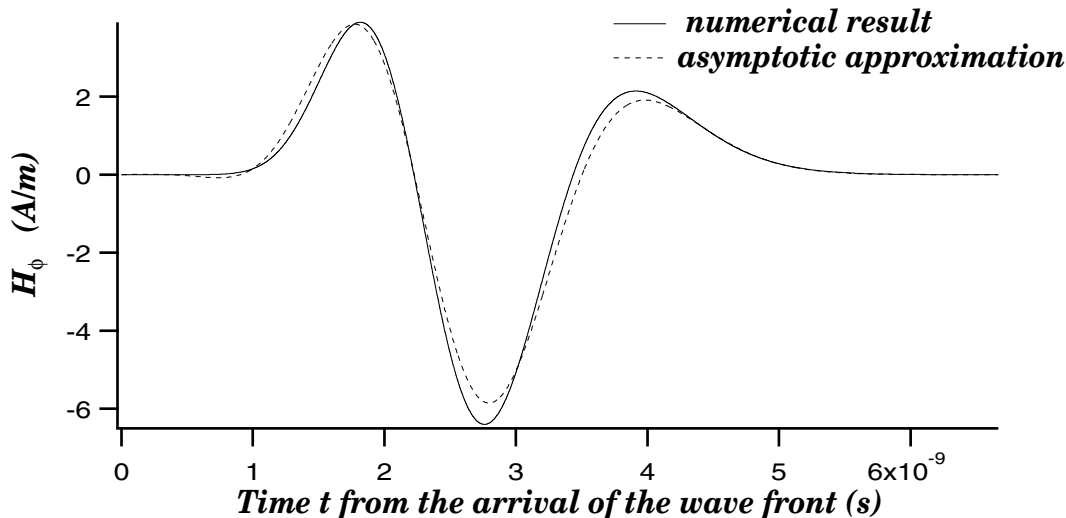
$$\chi_{\text{res1}} = -\frac{\chi_1}{1 + \chi_1}, \quad \chi_{\text{res2}} = -\frac{\chi_2(1 + \chi_{\text{res1}})}{1 + \chi_1}, \quad \chi_{\text{res3}} = -\frac{(1 + \chi_{\text{res1}})\chi_3 + \chi_{\text{res2}}\chi_2}{1 + \chi_1},$$

$$n_1 = \sqrt{1 + \frac{\omega_p^2}{\omega_0^2}} - 1, \quad n_2 = \frac{-\nu\omega_p^2}{2\omega_0^3\sqrt{\omega_0^2 + \omega_p^2}}, \quad n_3 = \frac{-1}{2\omega_0^5\sqrt{\omega_0^2 + \omega_p^2}} \left( \omega_p^2(\omega_0^2 - \nu^2) + \frac{\nu^2\omega_p^4}{4(\omega_0^2 + \omega_p^2)} \right),$$

$$z_1 = -\frac{n_1}{1 + n_1}, \quad z_2 = -\frac{n_2(1 + z_1)}{1 + n_1}, \quad z_3 = -\frac{(1 + z_1)n_3 + z_2n_2}{1 + n_1}.$$

The numerical results and the asymptotic approximations to the dipole fields in a single resonance Lorentz medium characterized by the parameters  $\omega_p = \sqrt{20} \cdot 10^{16} \text{ s}^{-1}$ ,  $\omega_0 = 4 \cdot 10^{16} \text{ s}^{-1}$  and  $\nu = 56 \cdot 10^{14} \text{ s}^{-1}$  at a distance of  $10^{-6} \text{ m}$  from the dipole at the observation angle  $\theta = \pi/4$  are displayed in Figures 5–7. The dipole moment is  $\mathbf{u}_z 10^{-17} \delta(t) \text{ C} \cdot \text{m}$ . The parameters are taken from Karlsson and Rikte [6]. The figures show that even in this much more dynamical case the first few oscillations of the second precursor (the slowly varying components) are correctly approximated. As was already mentioned, it is impossible to reconstruct the fast oscillations (the first precursor) using the introduced technique.

Figures 4 and 8 illustrate the propagation of the electromagnetic pulse due to a dipole in a Debye and a Lorentz medium, respectively. Only the asymptotic approximations with respect to the slowly varying components can be calculated at large propagation depths. Note that the time  $t$  in all the figures denotes the



**Figure 3:** The field  $H_\phi$  from a dipole in a Debye medium. For details see caption of Fig. 1.

wave-front time, *i.e.*,  $t = 0$  at  $\mathbf{r} = \mathbf{r}_0$  when the wave front arrives at the point  $\mathbf{r} = \mathbf{r}_0$ .

## 8 Surface integral representations

In this section, the Green dyadic  $\mathbf{G}_Q(\mathbf{r}; t)$  is used to obtain surface integral representation of the complex time-dependent electromagnetic field  $\mathbf{Q}(\mathbf{r}, t)$  in a temporally dispersive simple medium. The described method is well-known in time-harmonic investigations, but seems to be new in time-domain analysis of pulse propagation in dispersive materials.

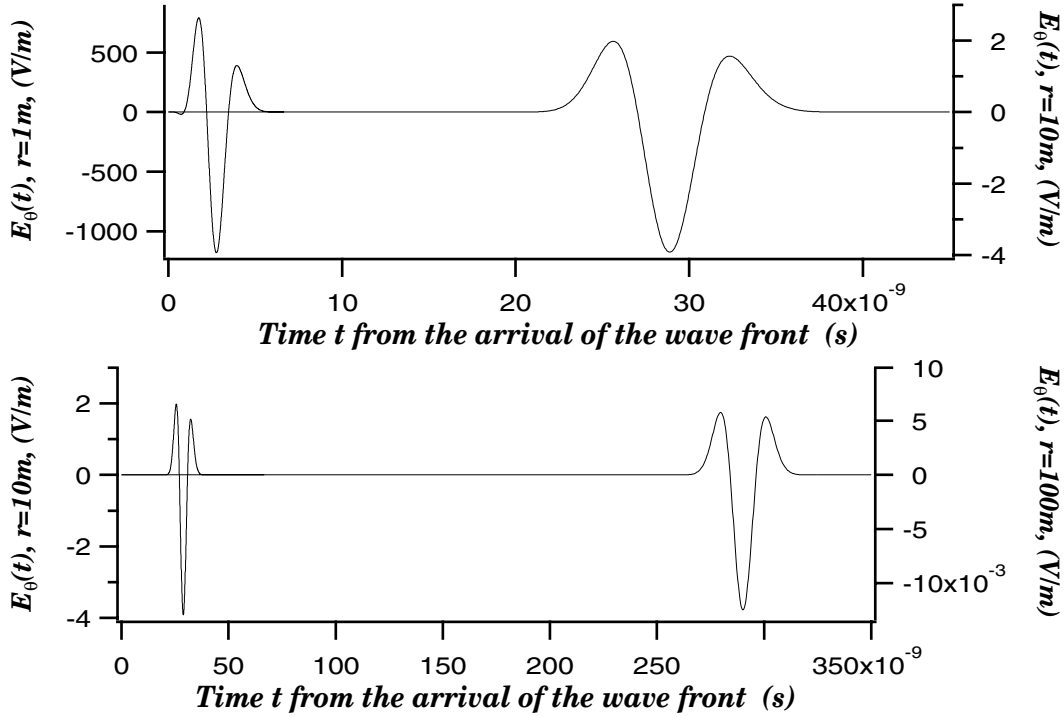
Let  $V_- \subset \mathbb{R}^3$  denote a bounded open domain,  $S = \partial V_-$  its regular surface, and  $V_+ = \mathbb{R}^3 \setminus \overline{V_-}$ . In the rest of this paper,  $\mathbf{u}_n = \mathbf{u}_n(\mathbf{r})$  denotes the outward (with respect to  $V_-$ ) unit normal vector to  $S$  at  $\mathbf{r}$  (the notation  $\mathbf{u}'_n = \mathbf{u}_n(\mathbf{r}')$  is used as well). The following limit values of the complex field  $\mathbf{Q}(\mathbf{r}, t)$  will be needed:

$$\mathbf{Q}_\pm(\mathbf{r}, t) = \lim_{V_\pm \ni \mathbf{r}' \rightarrow \mathbf{r}} \mathbf{Q}(\mathbf{r}', t), \quad \mathbf{r} \in S.$$

In this paragraph, the bounded domain  $V_-$  is assumed to be filled with a temporally dispersive, simple (linear, homogeneous, and isotropic) medium, whereas the medium filling  $V_+$  is arbitrary. The aim is to express the complex field  $\mathbf{Q}(\mathbf{r}, t)$ ,  $\mathbf{r} \in V_-$ , in terms of its tangential components at the boundary,  $\mathbf{u}_n \times \mathbf{Q}_-(\mathbf{r}, t)$ ,  $\mathbf{r} \in S$ , and the current density in the volume,  $\mathbf{J}(\mathbf{r}, t)$ ,  $\mathbf{r} \in V_-$ . Substituting equations (6.2) and (6.5) into the rule

$$\begin{aligned} \nabla' \cdot (\mathbf{Q}(\mathbf{r}', t') \times \mathbf{G}_Q(\mathbf{r}' - \mathbf{r}; t - t')) \\ = (\nabla' \times \mathbf{Q}(\mathbf{r}', t')) \cdot \mathbf{G}_Q(\mathbf{r}' - \mathbf{r}; t - t') - \mathbf{Q}(\mathbf{r}', t') \cdot (\nabla' \times \mathbf{G}_Q(\mathbf{r}' - \mathbf{r}; t - t')) \end{aligned}$$





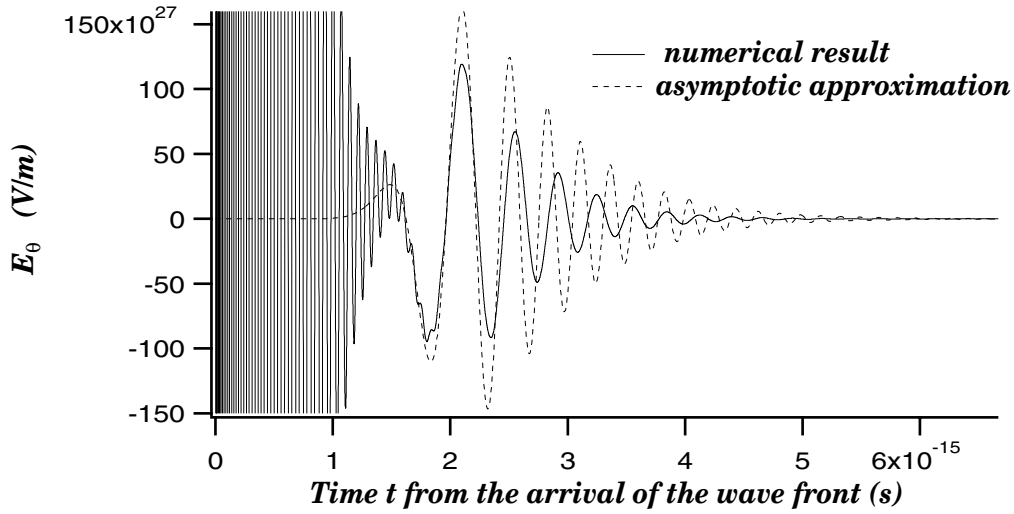
**Figure 4:** Propagation of the  $\theta$ -component of the field  $\mathbf{E}$  from a dipole characterized by  $\mathbf{p} = \mathbf{u}_z 10^{-17} \text{ C} \cdot \text{m} \cdot \text{s}$  in a Debye medium ( $\alpha = 3 \cdot 10^{10} \text{ s}^{-1}$ ,  $\beta = 1.2 \cdot 10^{10} \text{ s}^{-1}$ ).  $E_\theta$  is presented for three propagation distances:  $r = 1 \text{ m}$ ,  $r = 10 \text{ m}$ , and  $r = 100 \text{ m}$  ( $\theta = \pi/4$ ). Note that different scales are used for different distances. The fields  $E_r$  and  $H_\phi$  change in a similar way.

gives

$$\begin{aligned}
 \nabla' \cdot (\mathbf{Q}(\mathbf{r}', t') \times \mathbf{G}_Q(\mathbf{r}' - \mathbf{r}; t - t')) &= \frac{1}{2} i \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') [\mathcal{K}^{-1} \mathbf{Q}](\mathbf{r}, t) \\
 - \frac{1}{2} i \eta_0 [\mathcal{Z} \mathbf{J}](\mathbf{r}', t') \cdot \mathbf{G}_Q(\mathbf{r}' - \mathbf{r}; t - t') &- i [\mathcal{K} \mathbf{Q}](\mathbf{r}', t') \cdot \mathbf{G}_Q(\mathbf{r}' - \mathbf{r}; t - t') \\
 + i \mathbf{Q}(\mathbf{r}', t') \cdot [\mathcal{K} \mathbf{G}_Q](\mathbf{r}' - \mathbf{r}; t - t'). &
 \end{aligned} \tag{8.1}$$

Since the associative and commutative laws hold for causal convolutions, the last two terms in the right member cancel upon integration with respect to  $t'$ . Upon integrating the result over the bounded volume  $V_-$  as well, Gauss theorem for dyadics can be applied:  $\oint_S \mathbf{u}_n \cdot \mathbf{A} dS = \int_{V_-} \nabla \cdot \mathbf{A} dv$ . Using the relation (6.6) and the equality

$$(\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) \cdot \mathbf{G}_Q(\mathbf{r}' - \mathbf{r}; t - t') = \mathbf{u}'_n \cdot (\mathbf{Q}_-(\mathbf{r}', t') \times \mathbf{G}_Q(\mathbf{r}' - \mathbf{r}; t - t')),$$



**Figure 5:** The field  $E_\theta$  from a dipole characterized by  $\mathbf{p} = \mathbf{u}_z 10^{-17} \text{ C} \cdot \text{m} \cdot \text{s}$  in a Lorentz medium at a distance  $r = 10^{-6} \text{ m}$  from the dipole at the observation angle  $\theta = \pi/4$ . The medium is characterized by the parameters  $\omega_p = \sqrt{20} \cdot 10^{16} \text{ s}^{-1}$ ,  $\omega_0 = 4 \cdot 10^{16} \text{ s}^{-1}$  and  $\nu = 56 \cdot 10^{14} \text{ s}^{-1}$ .

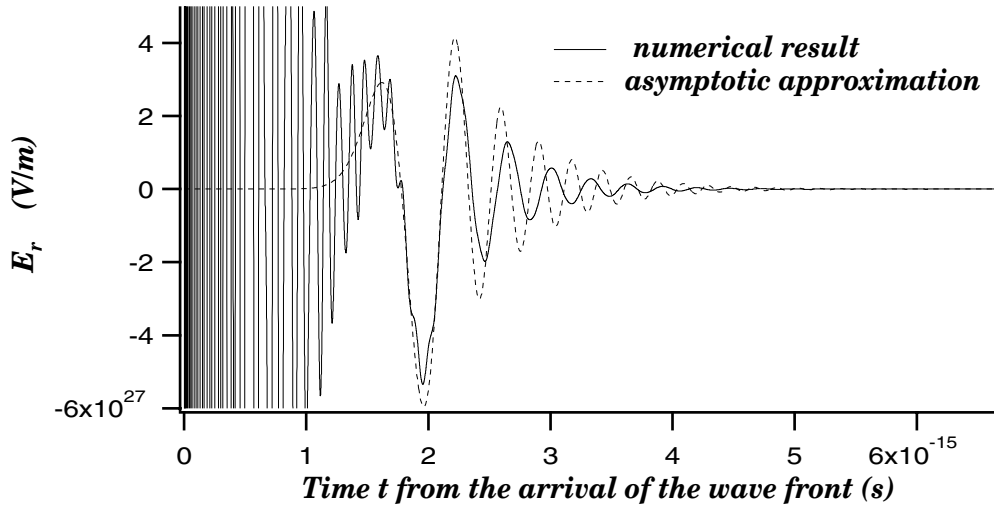
yields the final result, which can be referred to as Huygens' principle:

$$\begin{aligned} \oint_S \int \mathbf{G}_Q(\mathbf{r}-\mathbf{r}'; t-t') \cdot (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dt' dS' &= \frac{1}{2} i [\mathcal{K}^{-1} \mathbf{Q}] (\mathbf{r}, t) \\ - \frac{1}{2} i \int_{V_-} \int \mathbf{G}_Q(\mathbf{r}-\mathbf{r}'; t-t') \cdot \eta_0 [\mathcal{Z} \mathbf{J}] (\mathbf{r}', t') dt' dv', \quad \mathbf{r} \in V_-. \end{aligned} \quad (8.2)$$

The first term in the right member does not arise if the field point  $\mathbf{r}$  is chosen exterior to the volume  $V_-$ . The second term in the right member vanishes if the volume  $V_-$  is source-free. Decomposing the complex vector field into its real and imaginary parts yields the surface integral representations of the electric and magnetic fields.

Suppose now that the unbounded domain  $V_+$  consists of a temporally dispersive, simple medium. The medium filling  $V_-$  is arbitrary. Furthermore, assume that the sources are localized in space. To express the field  $\mathbf{Q}(\mathbf{r}, t)$ ,  $\mathbf{r} \in V_+$  in terms of the tangential field  $\mathbf{u}_n \times \mathbf{Q}_+(\mathbf{r}, t)$ ,  $\mathbf{r} \in S$  and the current density  $\mathbf{J}(\mathbf{r}, t)$ ,  $\mathbf{r} \in V_+$ , introduce an extra surface, say a sphere  $S_R$  of radius  $R$ , containing the sources and the scatterer (domain  $V_-$ ) and apply the above method to the finite volume between  $S$  and  $S_R$ . For every finite time interval  $[0, T]$ ,  $R$  can be chosen large enough so that  $\mathbf{Q}(\mathbf{r}, t) = \mathbf{0}$  and  $\mathbf{J}(\mathbf{r}, t) = \mathbf{0}$  for all  $\mathbf{r}$  outside  $S_R$  and  $t \in [0, T]$ , as a consequence of the finite speed of the wave front in dispersive materials [13]. This means that the surface integral over  $S_R$  is zero and the following integral representation is valid:

$$\begin{aligned} \oint_S \int \mathbf{G}_Q(\mathbf{r}-\mathbf{r}'; t-t') \cdot (\mathbf{u}'_n \times \mathbf{Q}_+(\mathbf{r}', t')) dt' dS' &= -\frac{1}{2} i [\mathcal{K}^{-1} \mathbf{Q}] (\mathbf{r}, t) \\ + \frac{1}{2} i \int_{V_+} \int \mathbf{G}_Q(\mathbf{r}-\mathbf{r}'; t-t') \cdot \eta_0 [\mathcal{Z} \mathbf{J}] (\mathbf{r}', t') dt' dv', \quad \mathbf{r} \in V_+. \end{aligned} \quad (8.3)$$



**Figure 6:** The field  $E_r$  from a dipole in a Lorentz medium. For details see caption of Fig. 5.

## 9 Surface integral equations

In this section, the aim is to obtain surface integral equations for the tangential components of the fields. The materials in both domains,  $V_+$  and  $V_-$ , then have to be specified. The analysis is similar to the fixed-frequency technique used by Ström [16]. First, by going to the limit  $\mathbf{r} \rightarrow S_{\pm}$  (i.e.,  $V_{\pm} \ni \mathbf{r} \rightarrow S$ ) in (8.2) and (8.3), surface integral *relations* for the complex fields  $\mathbf{Q}_{\pm}(\mathbf{r}, t)$  are obtained. Then, using the boundary conditions, surface integral equations can be derived.

Equation (8.2) can be written as

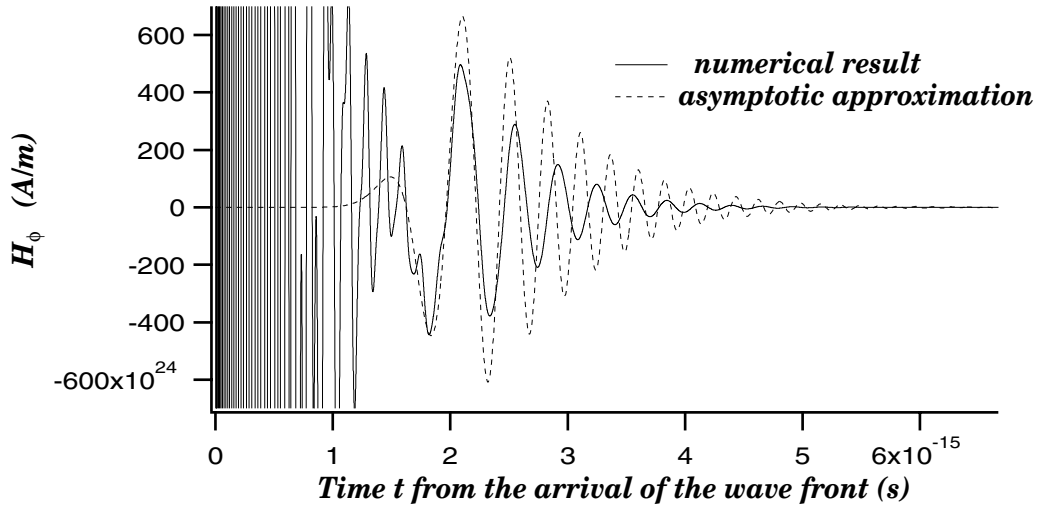
$$\begin{aligned} \frac{i}{2} [\mathcal{K}^{-1} \mathbf{Q}] (\mathbf{r}, t) \Big|_{\mathbf{0}} &= \frac{i}{2} [\mathcal{K}^{-1} \mathbf{Q}_i] (\mathbf{r}, t) \\ &+ \oint_S \int \mathbf{G}_Q(\mathbf{r} - \mathbf{r}'; t - t') \cdot (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dt' dS', \quad \begin{cases} \mathbf{r} \in V_- \\ \mathbf{r} \in V_+, \end{cases} \end{aligned} \quad (9.1)$$

where the source term

$$\mathbf{Q}_i(\mathbf{r}, t) = \int_{V_-} \int \mathbf{G}_Q(\mathbf{r} - \mathbf{r}'; t - t') \cdot \mu_0 \partial_{t'} [\mu \mathbf{J}] (\mathbf{r}', t') dt' dv' = \frac{1}{2} (\mathbf{E}_i(\mathbf{r}, t) - i\eta_0 [\mathcal{Z} \mathbf{H}_i] (\mathbf{r}, t)) \quad (9.2)$$

is supposed to be known. From the Gauss surface divergence theorem, it follows that

$$\begin{aligned} \oint_S (\nabla \nabla \mathcal{E}(\mathbf{r} - \mathbf{r}'; t - t')) \cdot (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dS' \\ = \nabla \oint_S (\mathcal{E}(\mathbf{r} - \mathbf{r}'; t - t')) \nabla'_S \cdot (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dS', \end{aligned} \quad (9.3)$$



**Figure 7:** The field  $H_\phi$  from a dipole in a Lorentz medium. For details see caption of Fig. 5.

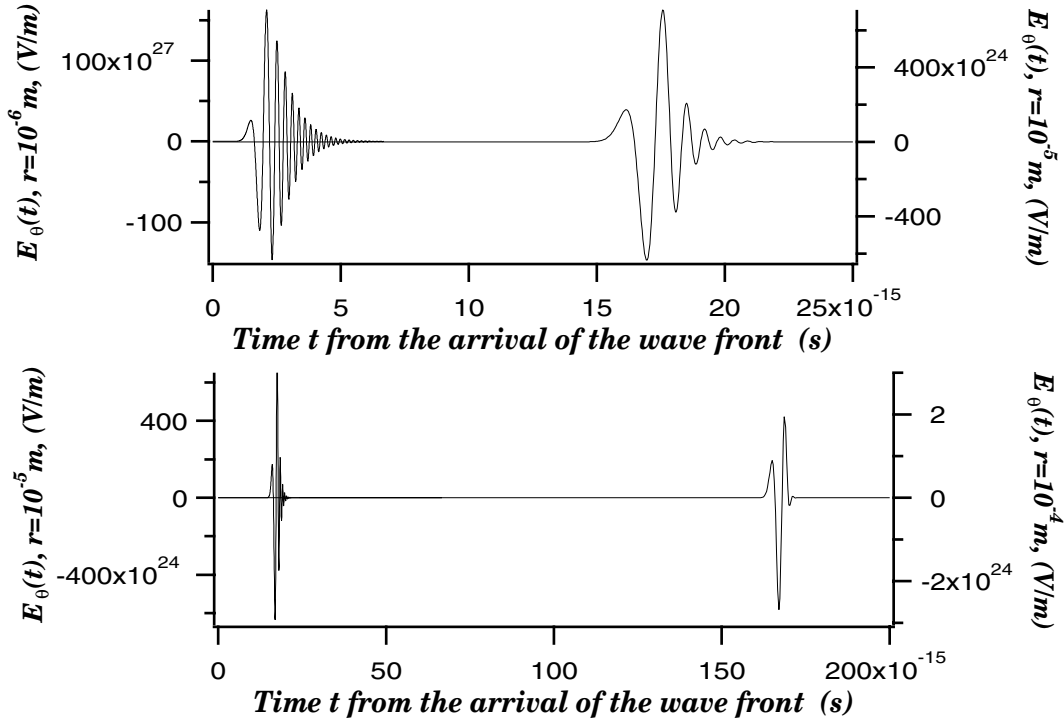
where  $\nabla_S \cdot$  is the surface divergence [2]. Now one can write the surface integral on the right-hand side of (9.1) as

$$\begin{aligned}
& \oint_S \int \mathbf{G}_Q(\mathbf{r}-\mathbf{r}'; t-t') \cdot (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dt' dS' \\
&= -\frac{1}{2} \oint_S \int \mathcal{E}(\mathbf{r}-\mathbf{r}'; t-t') \mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t') dt' dS' \\
&+ \frac{1}{2} \nabla \oint_S \int [\mathcal{K}^{-2} \mathcal{E}](\mathbf{r}-\mathbf{r}'; t-t') \nabla'_S \cdot (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dt' dS' \\
&- \frac{i}{2} \nabla \times \oint_S \int [\mathcal{K}^{-1} \mathcal{E}](\mathbf{r}-\mathbf{r}'; t-t') (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dt' dS', \quad \mathbf{r} \notin S.
\end{aligned} \tag{9.4}$$

The definition of the surface divergence  $\nabla_S \cdot$  can be found in [2]. In the limit  $\mathbf{r} \rightarrow S_\pm$ , the representation (9.1) transforms into the surface integral relation for the complex field  $\mathbf{Q}_-(\mathbf{r}, t)$ . Caution must be exercised handling the terms containing the spatial derivatives of  $\mathcal{E}(\mathbf{r}; t)$  because they are discontinuous across  $S$ . Straightforward generalization of the results in [16] (see also [3, Appendix C]) gives

$$\begin{aligned}
& \nabla \oint_S \int \mathcal{E}(\mathbf{r}-\mathbf{r}'; t-t') f(\mathbf{r}', t') dt' dS' \\
&= \oint_S \int \nabla \mathcal{E}(\mathbf{r}-\mathbf{r}'; t-t') f(\mathbf{r}', t') dt' dS' \pm \frac{1}{2} \mathbf{u}_n f(\mathbf{r}, t), \quad \mathbf{r} \rightarrow S_\pm, \\
& \nabla \times \oint_S \int \mathcal{E}(\mathbf{r}-\mathbf{r}'; t-t') \mathbf{F}(\mathbf{r}', t') dt' dS' \\
&= \oint_S \int (\nabla \mathcal{E}(\mathbf{r}-\mathbf{r}'; t-t')) \times \mathbf{F}(\mathbf{r}', t') dt' dS' \pm \frac{1}{2} \mathbf{u}_n \times \mathbf{F}(\mathbf{r}, t), \quad \mathbf{r} \rightarrow S_\pm
\end{aligned}$$

for any sufficiently regular scalar field  $f(\mathbf{r}, t)$  and vector field  $\mathbf{F}(\mathbf{r}, t)$ . In these expressions, all integrals on the right-hand sides exist as principal value integrals.



**Figure 8:** Propagation of the  $\theta$ -component of the field  $\mathbf{E}$  from a dipole characterized by  $\mathbf{p} = \mathbf{u}_z 10^{-17} \text{ C} \cdot \text{m} \cdot \text{s}$  in a Lorentz medium ( $\omega_p = \sqrt{20} \cdot 10^{16} \text{ s}^{-1}$ ,  $\omega_0 = 4 \cdot 10^{16} \text{ s}^{-1}$ ,  $\nu = 56 \cdot 10^{14} \text{ s}^{-1}$ ).  $E_\theta$  is presented for three propagation distances:  $r = 10^{-6} \text{ m}$ ,  $r = 10^{-5} \text{ m}$ , and  $r = 10^{-4} \text{ m}$  ( $\theta = \pi/4$ ). Note that different scales are used for different distances. The fields  $E_r$  and  $H_\phi$  change in a similar way.

Using (9.4) and the jump relations above, one obtains for  $\mathbf{r} \in S$

$$\begin{aligned} \frac{i}{2} \left[ \mathcal{K}^{-1} \mathbf{Q}_- \right]_0(\mathbf{r}, t) \Big\} &= \frac{i}{2} \left[ \mathcal{K}^{-1} \mathbf{Q}_i \right](\mathbf{r}, t) + \oint_S \int \mathbf{G}_Q(\mathbf{r} - \mathbf{r}'; t - t') \cdot (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dt' dS' \\ &\pm \frac{1}{4} \mathbf{u}_n \nabla_S \cdot (\mathbf{u}_n \times [\mathcal{K}^{-2} \mathbf{Q}_-](\mathbf{r}, t)) \mp \frac{i}{4} \mathbf{u}_n \times (\mathbf{u}_n \times [\mathcal{K}^{-1} \mathbf{Q}_-](\mathbf{r}, t)). \end{aligned} \quad (9.5)$$

The second term on the right-hand side of (9.5) is interpreted as

$$\begin{aligned} &\oint_S \int \mathbf{G}_Q(\mathbf{r} - \mathbf{r}'; t - t') \cdot (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dt' dS' \\ &= -\frac{1}{2} \oint_S \int \mathcal{E}(\mathbf{r} - \mathbf{r}'; t - t') \mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t') dt' dS' \\ &+ \frac{1}{2} \oint_S \int \nabla([\mathcal{K}^{-2} \mathcal{E}](\mathbf{r} - \mathbf{r}'; t - t')) \nabla'_S \cdot (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dt' dS' \\ &- \frac{i}{2} \oint_S \int \nabla([\mathcal{K}^{-1} \mathcal{E}](\mathbf{r} - \mathbf{r}'; t - t')) \times (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dt' dS', \quad \mathbf{r} \in S, \end{aligned} \quad (9.6)$$

where the integrals exist as principal value integrals. Using the Maxwell equations (6.2) and the fact that  $\nabla_S \cdot (\mathbf{u}_n \times \mathbf{Q}_-) = -\mathbf{u}_n \cdot (\nabla \times \mathbf{Q}_-)$ , both equations (9.5) reduce to ( $\mathbf{r} \in S$ )

$$\frac{i}{4} [\mathcal{K}^{-1} \mathbf{Q}_-] (\mathbf{r}, t) = \frac{i}{2} [\mathcal{K}^{-1} \mathbf{Q}_i] (\mathbf{r}, t) + \oint_S \int \mathbf{G}_Q(\mathbf{r} - \mathbf{r}'; t - t') \cdot (\mathbf{u}'_n \times \mathbf{Q}_-(\mathbf{r}', t')) dt' dS', \quad (9.7)$$

where the surface integral term is given by (9.6).

The integral relation based on the equation (8.3) can be derived in the same way. The result is ( $\mathbf{r} \in S$ )

$$\frac{i}{4} [\mathcal{K}^{-1} \mathbf{Q}_+] (\mathbf{r}, t) = \frac{i}{2} [\mathcal{K}^{-1} \mathbf{Q}_i] (\mathbf{r}, t) - \oint_S \int \mathbf{G}_Q(\mathbf{r} - \mathbf{r}'; t - t') \cdot (\mathbf{u}'_n \times \mathbf{Q}_+(\mathbf{r}', t')) dt' dS', \quad (9.8)$$

where the source term  $\mathbf{Q}_i(\mathbf{r}, t)$  now is given by (9.2) with the spatial integration over the domain  $V_+$  instead of  $V_-$ .

Notice, that in (9.7), all the parameters (the operators  $\mathcal{N}$ ,  $\mathcal{Z}$ ,  $\varepsilon$  and  $\mu$ ) correspond to the domain  $V_-$ , whereas in (9.8), the parameters describe the domain  $V_+$ . For the case when both domains  $V_{\pm}$  are temporally dispersive, the intrinsic integral operators  $\mathcal{N}$ ,  $\mathcal{Z}$ ,  $\varepsilon$ , and  $\mu$  as well as the dispersive fundamental solutions,  $\mathcal{E}(\mathbf{r}; t)$  in the domains  $V_{\pm}$  are endowed with the subscripts  $\pm$ , respectively.

Two standard scattering problems are now discussed:

- the surface  $S$  bounds a perfectly conducting scatterer;
- the surface  $S$  is an interface between two dispersive materials.

## 9.1 Perfectly conducting scatterer

In this subsection,  $V_-$  is a perfect conductor and  $V_+$  a temporally dispersive medium. The boundary condition on the surface  $S$  is  $\mathbf{u}_n \times \mathbf{E}_+(\mathbf{r}, t) = \mathbf{0}$ . Taking the cross product of both members of (9.8) with  $\mathbf{u}_n$  and using the boundary condition give the following integral equation for the surface current density  $\mathbf{J}_S^e(\mathbf{r}, t) := \mathbf{u}_n \times \mathbf{H}_+(\mathbf{r}, t)$ ,  $\mathbf{r} \in S$ :

$$\mathbf{J}_S^e(\mathbf{r}, t) = \frac{i4}{\eta_0} \mathcal{Z}^{-1} (\mathbf{u}_n \times \mathbf{Q}_i(\mathbf{r}, t) + \mathbf{u}_n \times \oint_S \int \mathbf{G}_Q(\mathbf{r} - \mathbf{r}'; t - t') \cdot \mu_0 \partial_{t'} [\mu \mathbf{J}_S^e] (\mathbf{r}', t') dt' dS').$$

Separating this equation into its real and imaginary parts gives two alternative integral equations for the surface current density. Explicitly, ( $\mathbf{r} \in S$ )

$$\begin{aligned} \mathbf{J}_S^e(\mathbf{r}, t) &= 2\mathbf{u}_n \times \mathbf{H}_i(\mathbf{r}, t) + 2\mathbf{u}_n \times \oint_S \int \nabla \mathcal{E}(\mathbf{r} - \mathbf{r}'; t - t') \times \mathbf{J}_S^e(\mathbf{r}', t') dt' dS', \\ \mathbf{0} &= \mathbf{u}_n \times \mathbf{E}_i(\mathbf{r}, t) - \frac{1}{c_0} \mathbf{u}_n \times \oint_S \int \partial_t [\mu \mathcal{E}] (\mathbf{r} - \mathbf{r}'; t - t') \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS' \\ &\quad + c_0 \mathbf{u}_n \times \oint_S \int \nabla \partial_t^{-1} [\varepsilon^{-1} \mathcal{E}] (\mathbf{r} - \mathbf{r}'; t - t') \nabla'_S \cdot \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS', \end{aligned}$$

where the surface integrals exist as principal value integrals. Notice that the first of these integral equations is of the second kind, while the second is of the first kind. These equations can be used for numerical calculations.

The cavity problem ( $V_-$  is a temporally dispersive medium and  $V_+$  is a perfect conductor) can be treated analogously.

## 9.2 Permeable scatterer

If the surface  $S$  is an interface between two different temporally dispersive materials, then the boundary conditions are

$$\begin{aligned}\mathbf{u}_n \times \mathbf{E}_+(\mathbf{r}, t) &= \mathbf{u}_n \times \mathbf{E}_-(\mathbf{r}, t) =: \mathbf{J}_S^m(\mathbf{r}, t) \\ \mathbf{u}_n \times \mathbf{H}_+(\mathbf{r}, t) &= \mathbf{u}_n \times \mathbf{H}_-(\mathbf{r}, t) =: \mathbf{J}_S^e(\mathbf{r}, t).\end{aligned}$$

Suppose that the bounded domain  $V_-$  is source-free. Taking the cross product of the left- and right-hand sides of equations (9.7)–(9.8) with  $\mathbf{u}_n$  and using the boundary conditions give four integral equations for the surface fields  $\mathbf{J}_S^m(\mathbf{r}, t)$  and  $\mathbf{J}_S^e(\mathbf{r}, t)$ :

$$\begin{aligned}\eta_0 \mathbf{J}_S^e(\mathbf{r}, t) &= 2\mathbf{u}_n \times \eta_0 \mathbf{H}_i(\mathbf{r}, t) \\ &+ \frac{2}{c_0} \mathbf{u}_n \times \oint_S \int \partial_t [\varepsilon_+ \mathcal{E}_+] (\mathbf{r} - \mathbf{r}'; t - t') \mathbf{J}_S^m(\mathbf{r}', t') dt' dS' \\ &- 2c_0 \mathbf{u}_n \times \oint_S \int \nabla \partial_t^{-1} [\mu_+^{-1} \mathcal{E}_+] (\mathbf{r} - \mathbf{r}'; t - t') \nabla'_S \cdot \mathbf{J}_S^m(\mathbf{r}', t') dt' dS' \\ &+ 2\mathbf{u}_n \times \oint_S \int \nabla \mathcal{E}_+(\mathbf{r} - \mathbf{r}'; t - t') \times \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS',\end{aligned}\tag{9.9}$$

$$\begin{aligned}\mathbf{J}_S^m(\mathbf{r}, t) &= 2\mathbf{u}_n \times \mathbf{E}_i(\mathbf{r}, t) \\ &- \frac{2}{c_0} \mathbf{u}_n \times \oint_S \int \partial_t [\mu_+ \mathcal{E}_+] (\mathbf{r} - \mathbf{r}'; t - t') \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS' \\ &+ 2c_0 \mathbf{u}_n \times \oint_S \int \nabla \partial_t^{-1} [\varepsilon_+^{-1} \mathcal{E}_+] (\mathbf{r} - \mathbf{r}'; t - t') \nabla'_S \cdot \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS' \\ &+ 2\mathbf{u}_n \times \oint_S \int \nabla \mathcal{E}_+(\mathbf{r} - \mathbf{r}'; t - t') \times \mathbf{J}_S^m(\mathbf{r}', t') dt' dS',\end{aligned}\tag{9.10}$$

$$\begin{aligned}\eta_0 \mathbf{J}_S^e(\mathbf{r}, t) &= -\frac{2}{c_0} \mathbf{u}_n \times \oint_S \int \partial_t [\varepsilon_- \mathcal{E}_-] (\mathbf{r} - \mathbf{r}'; t - t') \mathbf{J}_S^m(\mathbf{r}', t') dt' dS' \\ &+ 2c_0 \mathbf{u}_n \times \oint_S \int \nabla \partial_t^{-1} [\mu_-^{-1} \mathcal{E}_-] (\mathbf{r} - \mathbf{r}'; t - t') \nabla'_S \cdot \mathbf{J}_S^m(\mathbf{r}', t') dt' dS' \\ &- 2\mathbf{u}_n \times \oint_S \int \nabla \mathcal{E}_-(\mathbf{r} - \mathbf{r}'; t - t') \times \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS',\end{aligned}\tag{9.11}$$

$$\begin{aligned}
\mathbf{J}_S^m(\mathbf{r}, t) &= \frac{2}{c_0} \mathbf{u}_n \times \oint_S \int \partial_t [\mu_- \mathcal{E}_-](\mathbf{r} - \mathbf{r}'; t - t') \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS' \\
&\quad - 2c_0 \mathbf{u}_n \times \oint_S \int \nabla \partial_t^{-1} [\varepsilon_-^{-1} \mathcal{E}_-](\mathbf{r} - \mathbf{r}'; t - t') \nabla'_S \cdot \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS' \\
&\quad - 2\mathbf{u}_n \times \oint_S \int \nabla \mathcal{E}_-(\mathbf{r} - \mathbf{r}'; t - t') \times \mathbf{J}_S^m(\mathbf{r}', t') dt' dS',
\end{aligned} \tag{9.12}$$

where the surface integrals exist as principal value integrals. Since the surface divergences  $\nabla_S \cdot \mathbf{J}_S^e$  and  $\nabla_S \cdot \mathbf{J}_S^m$  enter these equations, they are unattractive from a numerical point of view. Moving the derivative from the surface fields to the  $\nabla \mathcal{E}$ -terms (integration by parts) does not reduce this inconvenience because then the highly singular second space derivatives of the kernel  $\mathcal{E}(\mathbf{r} - \mathbf{r}'; t - t')$  have to be dealt with. However, in the case when both materials have the same value of  $N(0)$  ( $N_+(0) = N_-(0)$ ), one can combine equations (9.9), (9.11) and equations (9.10), (9.12) so that neither surface divergences nor highly singular terms appear in the resulting equations.

Operating with  $\mu_+$  on both sides of (9.9) and with  $\mu_-$  on both sides of (9.11) and adding the results give

$$\begin{aligned}
[(\mu_+ + \mu_-) \eta_0 \mathbf{J}_S^e](\mathbf{r}, t) &= 2\mathbf{u}_n \times [\mu_+ \eta_0 \mathbf{H}_i](\mathbf{r}, t) \\
&\quad + \frac{2}{c_0} \mathbf{u}_n \times \oint_S \int \partial_t [\mathcal{N}_+^2 \mathcal{E}_+ - \mathcal{N}_-^2 \mathcal{E}_-](\mathbf{r} - \mathbf{r}'; t - t') \mathbf{J}_S^m(\mathbf{r}', t') dt' dS' \\
&\quad - 2c_0 \mathbf{u}_n \times \oint_S \int (\nabla \nabla \partial_t^{-1} ([\mathcal{E}_+ - \mathcal{E}_-](\mathbf{r} - \mathbf{r}'; t - t'))) \cdot \mathbf{J}_S^m(\mathbf{r}', t') dt' dS' \\
&\quad + 2\mathbf{u}_n \times \oint_S \int \nabla ([\mu_+ \mathcal{E}_+ - \mu_- \mathcal{E}_-](\mathbf{r} - \mathbf{r}'; t - t')) \times \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS',
\end{aligned} \tag{9.13}$$

where integration by parts was used to obtain the third term on the right-hand side. Similarly, from the equations (9.10) and (9.12), it follows that

$$\begin{aligned}
[\varepsilon_+ + \varepsilon_-] \mathbf{J}_S^m(\mathbf{r}, t) &= 2\mathbf{u}_n \times [\varepsilon_+ \mathbf{E}_i](\mathbf{r}, t) \\
&\quad - \frac{2}{c_0} \mathbf{u}_n \times \oint_S \int \partial_t [\mathcal{N}_+^2 \mathcal{E}_+ - \mathcal{N}_-^2 \mathcal{E}_-](\mathbf{r} - \mathbf{r}'; t - t') \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS' \\
&\quad + 2c_0 \mathbf{u}_n \times \oint_S \int \nabla \nabla \partial_t^{-1} ([\mathcal{E}_+ - \mathcal{E}_-](\mathbf{r} - \mathbf{r}'; t - t')) \cdot \eta_0 \mathbf{J}_S^e(\mathbf{r}', t') dt' dS' \\
&\quad + 2\mathbf{u}_n \times \oint_S \int \nabla ([\varepsilon_+ \mathcal{E}_+ - \varepsilon_- \mathcal{E}_-](\mathbf{r} - \mathbf{r}'; t - t')) \times \mathbf{J}_S^m(\mathbf{r}', t') dt' dS'.
\end{aligned} \tag{9.14}$$

As before, the surface integrals in equations (9.13)–(9.14) exist as principal value integrals. Recall that  $\mathcal{E}_\pm(\mathbf{r}, t) = q_\pm(r)/(4\pi r) (\delta(t - r/c_0) + P_\pm(r; t - r/c_0))$ ,  $P(0; t) = 0$ . If  $N_+(0) = N_-(0)$ , then  $q_+(r) = q_-(r)$  and the strongest singularities in  $\nabla \nabla$ -term disappear. Equations (9.13)–(9.14) are a pair of coupled integral equations of the second kind in the variables  $\mathbf{J}_S^e(\mathbf{r}, t)$  and  $\mathbf{J}_S^m(\mathbf{r}, t)$ . In principle, they can be used for numerical calculations.

By Fourier transformation with respect to time the results in Ref. 16 can be obtained.



## 10 Conclusion

This paper concerns electromagnetic pulse propagation in temporally dispersive, simple media. The analysis is performed in the time domain. Specifically, Green dyadics are introduced in terms of the retarded fundamental solution of the dispersive wave operator. The Green dyadics are used to obtain the surface integral representations for time-varying electromagnetic fields. Surface integral equations for the tangential components of the fields on the boundary of an impenetrable scatterer and on the interface between two dispersive materials are obtained.

The use of the complex time-dependent electromagnetic field simplifies the analysis of propagation problems in isotropic materials. This approach is particularly advantageous in the analysis of bi-isotropic media. Green dyadics for bi-isotropic media will be discussed in a subsequent paper.

The numerical example of an electric point dipole shows that the approximation technique in Section 7 can successfully be used to obtain the electromagnetic fields at arbitrary distances. General time-dependent sources can be treated in the same manner. The region, where one switches from the numerical values of fields to the asymptotic expressions (second forerunner approximations), depends on the medium and the source term of the problem. The described methods can be used to determine fields from, *e.g.*, microwave sources (antennas), and short-pulse optical sources in dispersive media.

The surface integral equations derived in Section 9, can be applied to scattering problems in dispersive media. It is conjectured that they can be solved numerically (using, *e.g.*, method of moments). Once the surface integral equations in Section 9 have been solved, the surface integral representations in Section 8 can be used to obtain the field vectors in the entire domain of interest.

As was already mentioned, a possible way to generalize the present results is to analyze Green dyadics in more complicated materials.

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## Appendix A Formulae for the Green dyadics

In this appendix, the Green dyadics for the electric field,  $\mathbf{G}_E$ , and for the complex electromagnetic field,  $\mathbf{G}_Q$ , are derived using Schwartz' *pseudo-fonctions* in  $\mathbb{R}^3$ ,  $Pf.(1/r^k)$ ,  $k = 3, 4, 5$ , see [14]. The Green dyadic for the magnetic field,  $\mathbf{G}_H$ , has been given already, *cf.* equation (5.4).

Schwartz' *pseudo-fonctions* represent so called finite parts of certain divergent integrals with spherically symmetric integrands. The finite parts can be identified by first omitting a small spherical region with radius  $\epsilon$  about the origin. The definitions

are:

$$\begin{aligned} \langle Pf. \left( \frac{1}{r^3} \right), \phi \rangle &= \lim_{\epsilon \rightarrow 0} \left( \int_{r \geq \epsilon} \frac{\phi(\mathbf{r})}{r^3} dv + 4\pi\phi(\mathbf{0}) \ln \epsilon \right), \\ \langle Pf. \left( \frac{1}{r^4} \right), \phi \rangle &= \lim_{\epsilon \rightarrow 0} \left( \int_{r \geq \epsilon} \frac{\phi(\mathbf{r})}{r^4} dv - 4\pi\phi(\mathbf{0}) \frac{1}{\epsilon} \right), \\ \langle Pf. \left( \frac{1}{r^5} \right), \phi \rangle &= \lim_{\epsilon \rightarrow 0} \left( \int_{r \geq \epsilon} \frac{\phi(\mathbf{r})}{r^5} dv - 2\pi\phi(\mathbf{0}) \frac{1}{\epsilon^2} + \frac{2\pi}{3} \Delta\phi(\mathbf{0}) \ln \epsilon \right), \end{aligned}$$

for all test functions  $\phi$ . These limits exist; for instance, integration by parts shows that

$$\langle Pf. \left( \frac{1}{r^3} \right), \phi \rangle = \int \ln r \nabla \left( \frac{1}{r} \right) \cdot \nabla \phi(\mathbf{r}) dv,$$

which is finite for all test functions  $\phi$ .

Using these definitions yields the elementary rules of differentiation

$$\begin{aligned} \nabla \left( \frac{1}{r} \right) &= -\mathbf{r} Pf. \left( \frac{1}{r^3} \right), \\ \nabla \left( \frac{1}{r^2} \right) &= -2\mathbf{r} Pf. \left( \frac{1}{r^4} \right), \\ \nabla \left( Pf. \left( \frac{1}{r^3} \right) \right) &= -3\mathbf{r} Pf. \left( \frac{1}{r^5} \right) - \frac{4\pi}{3} \nabla \delta(\mathbf{r}), \\ r^2 Pf. \left( \frac{1}{r^5} \right) &= Pf. \left( \frac{1}{r^3} \right), \\ r Pf. \left( \frac{1}{r^3} \right) &= r^2 Pf. \left( \frac{1}{r^4} \right) = \frac{1}{r^2}. \end{aligned}$$

Recall that the product of two distributions is not allowed in general; however, the product of  $r$  and  $Pf.(1/r^3)$  is well-defined ( $= 1/r^2$ ).

Use of the above rules, equations (4.6) and (4.7), and the dyadic rules

$$\nabla(\phi \mathbf{A}) = \phi \nabla \mathbf{A} + (\nabla \phi) \mathbf{A}, \quad \nabla \mathbf{r} = \mathbf{I}$$

leads to an explicit expression for the Green dyadic for the electric field (5.2):

$$\mathbf{G}_E = -\frac{1}{3} \mathbf{I} \mathcal{K}^{-2} [\delta_0 \otimes \delta_0] + (\mathbf{u}_r \mathbf{u}_r - \mathbf{I}) \mathcal{E} + (3\mathbf{u}_r \mathbf{u}_r - \mathbf{I}) \left( Pf. \left( \frac{1}{r^3} \right) \mathcal{K}^{-2} + \frac{1}{r^2} \mathcal{K}^{-1} \right) r \mathcal{E},$$

where  $\mathbf{u}_r = \mathbf{r}/r$  is the unit vector in the direction of  $\mathbf{r}$ . By Fourier transformation with respect to time, the well-known fixed frequency result is found [17]. Similarly, the well-known time-domain result for vacuum is obtained by setting  $\mathcal{K} = c_0^{-1} \partial_t$  (*cf.* Ref. 10)

$$\mathbf{G}_E(\mathbf{r}; t) = -\frac{c_0^2}{3} \mathbf{I} \delta(\mathbf{r}) t H(t) + (\mathbf{u}_r \mathbf{u}_r - \mathbf{I}) \frac{\delta(t - r/c_0)}{4\pi r} + (3\mathbf{u}_r \mathbf{u}_r - \mathbf{I}) \frac{c_0^2}{4\pi} t H\left(t - \frac{r}{c_0}\right) Pf. \left( \frac{1}{r^3} \right),$$

where  $H(t)$  is the temporal Heaviside step.

The Green dyadic for the complex electromagnetic field (6.4) becomes

$$\begin{aligned}
2\mathbf{G}_Q = & -\frac{1}{3}\mathbf{I}\mathcal{K}^{-2}[\delta_0 \otimes \delta_0] + (\mathbf{u}_r \mathbf{u}_r - \mathbf{I})\mathcal{E} + (3\mathbf{u}_r \mathbf{u}_r - \mathbf{I}) \left( Pf. \left( \frac{1}{r^3} \right) \mathcal{K}^{-2} + \frac{1}{r^2} \mathcal{K}^{-1} \right) r\mathcal{E} \\
& + i \left( \frac{1}{r^2} \mathcal{K}^{-1} + \frac{1}{r} \right) (r\mathcal{E} \times \mathbf{I}).
\end{aligned} \tag{A.1}$$

## Appendix B Explicit expressions for the asymptotic approximations of the dipole fields

In terms of the moments defined in Section 7, second forerunner approximations to the electric and magnetic fields (7.3) from an electric point dipole  $\mathbf{p} = \mathbf{u}_z p$  are given explicitly by the following expressions,  $r \neq 0$ :

$$\begin{aligned}
E_r(r, t) \approx & \frac{2 \cos \theta p \exp(a(r) - b(t - t_1(r)))}{4\pi\epsilon_0 t_3(r)} \\
& \times \left\{ \text{Ai}(s(r)(t - t_1(r))) \left[ \frac{(1 + \chi_{\text{res}1}^e) - \chi_{\text{res}2}^e b + \chi_{\text{res}3}^e (b^2 + s^3(r)(t - t_1(r)))}{r^3} \right. \right. \\
& + \left. \left. \frac{-(1 + z_1)b + z_2(b^2 + s^3(r)(t - t_1(r))) + z_3(s^3(r) - b^3 - 3bs^3(r)(t - t_1(r)))}{c_0 r^2} \right] \right. \\
& + \text{Ai}'(s(r)(t - t_1(r))) \left[ \frac{\chi_{\text{res}2}^e s(r) - 2\chi_{\text{res}3}^e bs(r)}{r^3} \right. \\
& + \left. \left. \frac{(1 + z_1)s(r) - 2z_2 bs(r) + z_3(3b^2 s(r) + s^4(r)(t - t_1(r)))}{c_0 r^2} \right] \right\},
\end{aligned}$$

$$\begin{aligned}
E_\theta(r, t) \approx & \frac{\sin \theta p \exp(a(r) - b(t - t_1(r)))}{4\pi\epsilon_0 t_3(r)} \\
& \times \left\{ \text{Ai}(s(r)(t - t_1(r))) \left[ \frac{b^2 + s^3(r)(t - t_1(r))}{rc_0^2} + \frac{(1 + \chi_{\text{res}1}^e) - \chi_{\text{res}2}^e b + \chi_{\text{res}3}^e (b^2 + s^3(r)(t - t_1(r)))}{r^3} \right. \right. \\
& + \left. \left. \frac{-(1 + z_1)b + z_2(b^2 + s^3(r)(t - t_1(r))) + z_3(s^3(r) - b^3 - 3bs^3(r)(t - t_1(r)))}{c_0 r^2} \right] \right. \\
& + \text{Ai}'(s(r)(t - t_1(r))) \left[ \frac{-2bs(r)}{rc_0^2} + \frac{\chi_{\text{res}2}^e s(r) - 2\chi_{\text{res}3}^e bs(r)}{r^3} \right. \\
& + \left. \left. \frac{(1 + z_1)s(r) - 2z_2 bs(r) + z_3(3b^2 s(r) + s^4(r)(t - t_1(r)))}{c_0 r^2} \right] \right\},
\end{aligned}$$

$$\begin{aligned}
H_\phi(r, t) \approx & \frac{\sin \theta p \exp(a(r) - b(t - t_1(r)))}{4\pi t_3(r)} \\
& \times \left\{ \text{Ai}(s(r)(t - t_1(r))) \left[ \frac{-b}{r^2} + \frac{(1 + n_1)(b^2 + s^3(r)(t - t_1(r)))}{c_0 r} \right. \right. \\
& + \left. \left. \frac{n_2(s^3(r) - b^3 - 3bs^3(r)(t - t_1(r))) + n_3(b^4 + 6b^2s^3(r)(t - t_1(r)) - 4bs^3(r) + s^6(r)(t - t_1(r))^2)}{c_0 r} \right] \right. \\
& + \text{Ai}'(s(r)(t - t_1(r))) \left[ \frac{s}{r^2} + \frac{-2(1 + n_1)bs(r) + n_2(3b^2s(r) + s^4(r)(t - t_1(r)))}{c_0 r} \right. \\
& \left. \left. + \frac{n_3(2s^4(r) - 4b^3s(r) - 4bs^4(r)(t - t_1(r)))}{c_0 r} \right] \right\},
\end{aligned}$$

where

$$a(r) = \frac{n_2^3 r}{27n_3^2 c_0}, \quad b = \frac{n_2}{3n_3}, \quad s(r) = \frac{\text{sign}(n_3)}{t_3(r)}.$$

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