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J.F. Vasconcelos, A. Rantzer, C. Silvestre and P. Oliveira

Abstract— **Lyapunov methods and density functions provide dual characterizations of the solutions of a nonlinear dynamic system. This work exploits the idea of combining both techniques, to yield stability results that are valid for almost all the solutions of the system. Based on the combination of Lyapunov and density functions, analysis methods are proposed for the derivation of almost input-to-state stability, and of almost global stability in nonlinear systems. The techniques are illustrated for an inertial attitude observer, where angular velocity readings are corrupted by non-idealities.**

I. INTRODUCTION

Global stability is usually a highly desirable property in control and estimation algorithms. However, topological obstacles arise due to the fact that a smooth vector field can have a global attractor only if the state space is homeomorphic to $\mathbb{R}^n$ [5]. As a consequence, continuous state feedback on smooth manifolds will always produce some trajectories that do not converge to the origin [2], [9]. Due to the presence of unstable manifolds, stability analysis using Lyapunov's second theorem is more complex.

New analysis tools have been brought forward by adopting the milder notion of almost global stability [1], [12]. In this framework, an equilibrium is "almost globally stable" if stability is satisfied for all initial states outside a set of zero measure. A dual to the Lyapunov second method is developed in [12], [13], based on density functions, that represent the stationary density of a substance that flows along the system trajectories [10], [11], [12]. Almost global stability is obtained by verifying that, for a time-invariant density function, particles are generated almost everywhere and hence must flow to a sink, located at the origin.

A similar approach has been adopted for the analysis of input-to-state stability (ISS), that has been extensively developed in recent years, as presented in the comprehensive survey [16]. The limitations to global stability on non-Euclidean spaces, and the fact that global stability is a necessary condition for ISS, motivate the relaxation to almost ISS proposed in [1]. The notions of robust and weakly almost ISS are proposed, and stability results using density functions are derived. More important, it is suggested that a combination of Lyapunov methods with density function results, may be the right technique for proving almost ISS in general. Surprisingly enough, this enriching insight seems to have gone unnoticed in the subsequent literature.

This work develops the idea of combining Lyapunov and density functions, for the stability analysis of nonlinear systems. Results are formulated for the analysis of almost global asymptotic stability, and of almost ISS of the origin.

In the proposed analysis techniques, the Lyapunov function is adopted to characterize the system trajectories, however the resulting analysis is limited by the existence of unstable manifolds. Density functions are used to resolve for the regions where the Lyapunov method is inconclusive, yielding sufficient conditions for instability of undesirable equilibrium points, and for convergence of almost all solutions to the region where stability is guaranteed by the Lyapunov function. The techniques are illustrated for the error dynamics of an attitude observer defined on $\mathrm{SO}(3)$, where angular velocity readings are corrupted by non-idealities, such as bounded measurement noise and unknown bias.

This work is organized as follows. Section II describes the attitude observer, adopted to illustrate the combination of Lyapunov and density function methods. The derivation of almost ISS for nonlinear systems, using the combination of Lyapunov and density functions, is discussed in Section III. The approach is applied to demonstrate the stability of the attitude observer in the presence of inertial sensor noise. A new result for almost global stability of nonlinear systems is presented in Section IV, and is illustrated for a nonlinear system, motivated by the attitude observer dynamics subject to inertial sensor bias. Concluding remarks and future work are discussed in Section V.

NOMENCLATURE

The notation adopted is fairly standard. The set of $n \times m$ matrices with real entries is denoted as $\mathrm{M}(n,m)$ and $\mathrm{M}(n) := \mathrm{M}(n,n)$. The set of special orthogonal matrices is denoted as $\mathrm{SO}(n) := \{\mathbf{R} \in \mathrm{M}(n) : \mathbf{R}'\mathbf{R} = \mathbf{I}, \det(\mathbf{R}) = 1\}$. The nominal, the measured, and the estimated quantity $\mathbf{s}$ are denoted by $\bar{\mathbf{s}}$, $\mathbf{s}_r$ and $\hat{\mathbf{s}}$, respectively, and $\|\mathbf{s}\|$ denotes the Frobenius norm. The operator $(\mathbf{s})_\times$ produces the skew symmetric matrix defined by the vector $\mathbf{s} \in \mathbb{R}^3$ such that $(\mathbf{s})_\times \mathbf{b} = \mathbf{s} \times \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^3$, and $(\cdot)_\otimes$ is the unskew operator such that $\left((\mathbf{s})_\times\right)_\otimes = \mathbf{s}$. The time dependence of the variables will be omitted in general, but otherwise denoted for the sake of clarity.

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II. ATTITUDE OBSERVER

This section introduces the attitude observer, that is adopted to illustrate the stability analysis techniques proposed in the paper. The observer estimates the attitude of a rigid body with respect to a fixed inertial frame, by merging angular velocity measurements, with vectors observations obtained in body coordinates. The detailed formulation of the observer is presented in [17], and similar observers can be found in [6] and [8].

The rigid body kinematics are described by

$$\dot{\bar{\mathcal{R}}} = \bar{\mathcal{R}}\left(\bar{\boldsymbol{\omega}}\right)_{\times},$$

where $\bar{\mathcal{R}}$ is the rotation matrix from body frame to the inertial frame coordinates, and $\bar{\boldsymbol{\omega}}$ is the body angular velocity expressed in body coordinates.

The body angular velocity is measured by a rate gyro sensor triad, and the measurement model is

$$\boldsymbol{\omega}_r = \bar{\boldsymbol{\omega}} + \mathbf{u}_\omega, \tag{1}$$

where $\mathbf{u}_\omega$ is a measurement disturbance.

The vector observations are a function of the rigid body's attitude. The vectors coordinates are known and time-invariant in inertial frame, e.g. Earth's magnetic and gravitational fields, and measured in body coordinates by onboard sensors such as magnetometers and pendulums, among others. The vector readings are introduced in the observer by means of a conveniently defined linear coordinate transformation [17], and the transformed vector measurements are described by

$$\mathbf{X}_r = \bar{\mathcal{R}}'\,{}^I\bar{\mathbf{X}}, \tag{2}$$

where $\mathbf{X}_r := \begin{bmatrix}\mathbf{x}_{r\,1} & \dots & \mathbf{x}_{r\,n}\end{bmatrix}$, ${}^I\bar{\mathbf{X}} := \begin{bmatrix}{}^I\bar{\mathbf{x}}_1 & \dots & {}^I\bar{\mathbf{x}}_n\end{bmatrix}$, $\mathbf{X}_r, {}^I\bar{\mathbf{X}} \in \mathrm{M}(3,n)$, the leading superscript I denotes inertial coordinates, $i = 1..n$ is the vector index, and n is the number of vector measuring sensors. In this work, the vector transformation is defined such that ${}^I\bar{\mathbf{X}}\,{}^I\bar{\mathbf{X}}' = \mathbf{I}$, to shape uniformly the directionality of the vector readings. Also, it is assumed that there are at least two noncollinear ${}^I\bar{\mathbf{x}}_i$, so that all rotational degrees of freedom are observable, see [17] for a detailed characterization of the observer.

The attitude kinematics of the observer are given by

$$\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}}\left(\hat{\boldsymbol{\omega}}\right)_{\times}, \tag{3}$$

where $\hat{\mathcal{R}}$ is the estimated attitude and $\hat{\boldsymbol{\omega}}$ is the feedback term. The attitude estimate dynamics (3) are stabilized by exploiting the non-ideal angular velocity measurements (1) and the vector observations (2) in the feedback term $\hat{\boldsymbol{\omega}}$.

In this work, the combination of Lyapunov and density functions is illustrated in the stability analysis of the attitude observer for the cases where $\mathbf{u}_\omega$ is *i)* an unmodeled, bounded sensor disturbance, and *ii)* an unknown but constant sensor bias.

III. STABILITY IN THE PRESENCE OF UNMODELED INPUTS

This section discusses and formulates the combination of Lyapunov and density function techniques for the analysis of input-to-state stability, in the presence of unknown inputs. The proposed method is illustrated by analyzing the stability of the attitude observer, in the case where the inertial sensor reading is corrupted by a bounded disturbance.

A. Almost ISS using Lyapunov and Density Functions

This section studies the input-to-state stability of systems in the form

$$\dot{x} = f(x,u), \tag{4}$$

where $x \in M$ is the state, M is a smooth manifold, and $f : M \times U \to TM$, is a locally Lipschitz manifold map which satisfies $f(x,u) \in T_xM$, for all $x \in M$ and all $u \in U \subset \mathbb{R}^m$. The limitations to global stability on non-Euclidean spaces motivate the relaxation of the classical notion of ISS [16] to that proposed in [1]. Denoted as almost ISS, it allows for each input u to destabilize a zero measure set of trajectories, outside of which all trajectories converge to a neighborhood of the origin.

Definition 1 (Almost ISS, [1]): The system (4) is almost ISS with respect to the origin, denoted as 0_M, if 0_M is locally asymptotically stable and

$$\forall u \in U\ \forall \text{a.a.}\, x(t_0) \in M\ \ \limsup_{t\to\infty} |x(t)| \leq \gamma(\|u\|_\infty), \tag{5}$$

where γ is a class $\mathcal{K}$ function and $|\cdot|$ is the distance to the origin.

In this work, a method to derive almost ISS is obtained by combining the properties of Lyapunov and density functions. The adopted methodology has been sketched in [1], where it is motivated by means of examples, however it seems to have been unnoticed in subsequent literature. This section provides a contribution to the concept of combining Lyapunov and density functions, by formulating the technique in explicit mathematical statements, and characterizing the stability result as the combination of two ISS properties, introduced in the following.

Definition 2 (Local ISS): A system (4) is locally ISS with respect to 0_M, if 0_M is locally asymptotically stable and there exists $r > 0$ such that

$$\forall u \in U\ \forall\, |x(t_0)| \leq r\ \ \limsup_{t\to\infty} |x(t)| \leq \gamma_1(\|u\|_\infty), \tag{6}$$

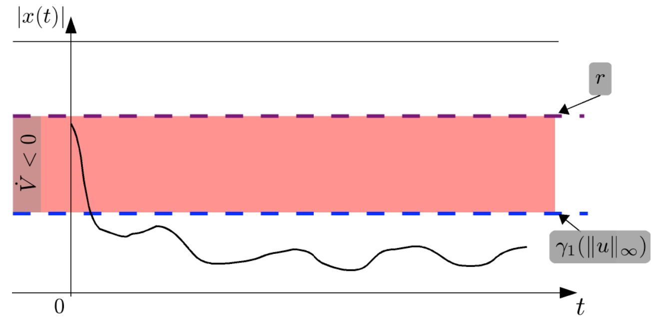
where γ_1 is a class $\mathcal{K}$ function.

Definition 3 (Weakly almost ISS, [1]): A system (4) is weakly almost ISS with respect to 0_M, if 0_M is locally asymptotically stable and

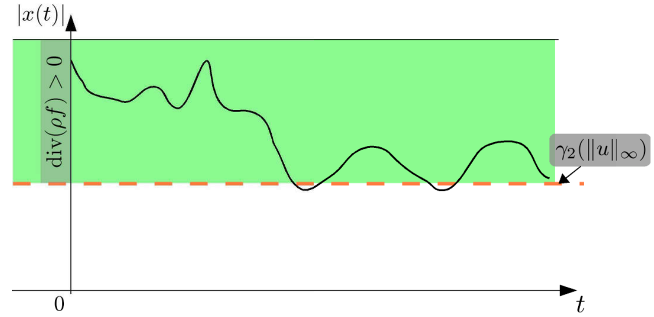
$$\forall u \in U\ \forall \text{a.a.}\, x(t_0) \in M\ \ \liminf_{t\to\infty} |x(t)| \leq \gamma_2(\|u\|_\infty), \tag{7}$$

where γ_2 is a class $\mathcal{K}$ function.

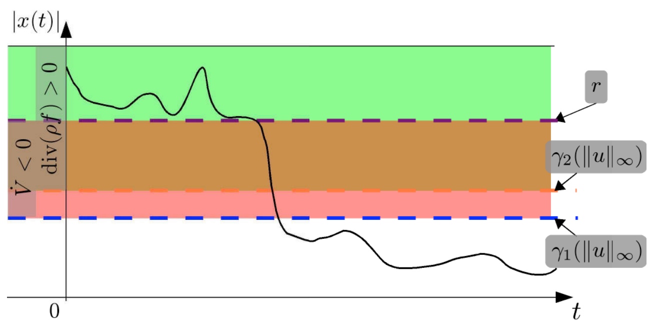
Provided that these ISS properties are verified, the main result of this section shows that almost ISS is attained.



(a) Local ISS using Lyapunov function analysis: solutions emanating below the bound r converge to the region bounded by $\gamma_1(\|u\|_\infty)$.



(b) Weakly almost ISS using density function analysis: the $\lim\inf$ property of almost all solutions satisfies the bound $\gamma_2(\|u\|_\infty)$.



(c) Almost ISS using Lyapunov and density functions analysis: by the $\lim\inf$ property, almost all trajectories enter the region below the bound r, and converge to the region bounded by $\gamma_1(\|u\|_\infty)$.

Fig. 1. Combination of Lyapunov and density functions for almost ISS of the origin.

Lemma 1 (Almost ISS): Assume that the system (4) is locally ISS and weakly almost ISS, then, for all $u \in \{u \in U : \gamma_2(\|u\|_\infty) < r\}$, the system is almost ISS with $\gamma = \gamma_1$.

Proof: Weakly almost ISS, expressed in (7), implies that, by the continuity of the solutions of (4), almost every solution satisfies $|x(t)| \leq \gamma_2(\|u\|_\infty) < r$ for some t, thus entering the region where the trajectories eventually satisfy the $\lim\sup$ condition expressed in (6), yielding (5). ■

Lemma 1 shows that almost ISS can be obtained by combining local ISS with weakly almost ISS, for sufficiently small inputs. The proposed ISS analysis technique is therefore based on $i)$ Lyapunov methods to attain local ISS [4], $ii)$ density function techniques to yield weakly almost ISS [1], and $iii)$ Lemma 1 to derive almost ISS.

The stability analysis technique is illustrated in Fig. 1. Lyapunov techniques yield local ISS based on ultimate boundedness and/or ISS results [4]. As shown in Fig. 1(a), Lyapunov methods find a region $\{x : \gamma_1(\|u\|_\infty) < |x(t)| < r\}$ where the Lyapunov function V decreases along the system trajectories ($\dot V < 0$), and drives the solutions to set $\{x : |x(t)| < \gamma_1(\|u\|_\infty)\}$, that is positively invariant.

However, the Lyapunov function analysis is inconclusive with respect to $\{x : |x(t)| \geq r\}$, and density functions techniques are adopted to guarantee that almost all solutions enter $\{x : |x(t)| < r\}$ for some time instant. This is obtained by finding a density function ρ such that $\mathrm{div}(\rho f) > 0$ in the region $\{x : |x(t)| > \gamma_2(\|u\|_\infty)\}$, which yields weakly almost ISS by [1, Theorem 4]. Hence, the trajectories of the system are endowed with the $\lim\inf$ characteristic depicted in Fig. 1(b), and enter the region $\{x : |x(t)| < r\}$ in finite time, as shown in Fig. 1(c), yielding almost ISS of the origin.

B. Stability of the Nonlinear Observer in the Presence of Inertial Sensor Noise

The combination of Lyapunov techniques is illustrated for the nonlinear observer described in Section II, in the presence of bounded time-varying disturbances in the angular velocity measurements. The considered set of valid disturbances $\mathbf{u}_\omega$ in (1) is described by $U = \{\mathbf{u} \in \mathbb{R}^3 : \|\mathbf{u}\|_\infty \leq u_{\max}\}$, and the observer kinematics are given by

$$\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}}\,(\hat{\boldsymbol{\omega}})_\times\,,\ \ \hat{\boldsymbol{\omega}} = \hat{\mathcal{R}}'^{I}\bar{\mathbf{X}}\mathbf{X}_r'\boldsymbol{\omega}_r - k_\omega \sum_{i=1}^{n} (\hat{\mathcal{R}}'^{I}\bar{\mathbf{x}}_i) \times \mathbf{x}_{r\,i},$$

where $k_\omega \in \mathbb{R}^+$ is the feedback gain, for more details and a motivation see [17]. Defining the attitude estimation error as $\mathcal{R} := \hat{\mathcal{R}}'\bar{\mathcal{R}}$, the closed loop error kinematics are given by

$$\dot{\mathcal{R}} = -k_\omega \mathcal{R}(\mathcal{R} - \mathcal{R}\,') - \mathcal{R}\,(\mathbf{u}_\omega)_\times\,. \tag{8}$$

Although the origin of the unforced system is almost globally asymptotically stable (GAS) [17], it can be shown that the origin of the system (8) is not ISS, namely by taking $\mathbf{u}_\omega$ as the destabilizing feedback law for a $\mathcal{R}(t_0)$ sufficiently close to the unstable equilibrium points of the unforced system. In this counterexample, $\mathbf{u}_\omega$ destabilizes only a given $\mathcal{R}(t_0)$, and hence almost ISS is not precluded.

Almost ISS of the system (8) is obtained using the stability analysis technique proposed in Section III-A and depicted in Fig. 1. The following proposition shows that the system is locally ISS, using Lyapunov stability theory.

Theorem 2: Let $k_\omega > \frac{u_{\max}}{2}$, then for any initial condition

$$\mathcal{R}(t_0) \in \{\mathcal{R} \in \mathrm{SO}(3) : \|\mathbf{I} - \mathcal{R}\|^2 < r(\|\mathbf{u}_\omega\|_\infty)\}, \tag{9a}$$

where $r(u) = 4\left(1 + \sqrt{1 - \frac{u^2}{4k_\omega^2}}\right)$, there exists T, independent of t_0, such that the trajectory of the system (8) satisfies

$$\mathcal{R}(t) \in \{\mathcal{R} \in \mathrm{SO}(3) : \|\mathbf{I} - \mathcal{R}\|^2 < \gamma_1(\|\mathbf{u}_\omega\|_\infty)\}, \tag{9b}$$

for all $t \geq t_0 + T$, where $\gamma_1(u) = 4\left(1 - \sqrt{1 - \frac{u^2}{4k_\omega^2}}\right)$.

Proof: The proof is based on the derivation of boundedness for nonlinear systems presented in [4, Theorem 4.18], using Lyapunov methods. The time derivative of the Lyapunov function $V = \frac{\|\mathbf{I} - \mathcal{R}\|^2}{2}$ along the system trajectories is given by $\dot{V} = -k_\omega \frac{\|\mathbf{I}-\mathcal{R}^2\|^2}{2} + \operatorname{tr}\left(\frac{\mathcal{R}-\mathcal{R}'}{2}(\mathbf{u}_\omega)_\times\right)$, and algebraic manipulation produces

$$\dot{V} \leq -k_\omega \|\mathbf{I}-\mathcal{R}^2\| \left(\frac{1}{2}\|\mathbf{I}-\mathcal{R}^2\| - \frac{\|\mathbf{u}_\omega\|}{k_\omega\sqrt{2}}\right),$$

where $2\operatorname{tr}(\mathbf{I}-\mathcal{R}^2) = \|\mathbf{I}-\mathcal{R}^2\|^2 = \|\mathcal{R}-\mathcal{R}'\|^2$ was used. It is immediate that $\frac{1}{2}\|\mathbf{I}-\mathcal{R}^2\| > \frac{\|\mathbf{u}_\omega\|_\infty}{k_\omega\sqrt{2}} \Rightarrow \dot{V} < 0$. Using $\|\mathbf{I}-\mathcal{R}^2\|^2 = \frac{1}{2}\left(8-\|\mathbf{I}-\mathcal{R}\|^2\right)\|\mathbf{I}-\mathcal{R}\|^2$ produces $\frac{1}{2}\|\mathbf{I}-\mathcal{R}^2\| > \frac{\|\mathbf{u}_\omega\|_\infty}{k_\omega\sqrt{2}} \Leftrightarrow \mathcal{R}\in\{\ \bar{\Omega}_t - \underline{\Omega}_t\}$, where $\underline{\Omega}_t = \{\mathcal{R} : V(\mathcal{R}) \leq \frac{1}{2}\gamma_1(\|\mathbf{u}_\omega\|_\infty)\}$, $\bar{\Omega}_t = \{\mathcal{R} : V(\mathcal{R}) \leq \frac{1}{2}r(\|\mathbf{u}_\omega\|_\infty)\}$. Consequently, $\mathcal{R} \in\{\ \bar{\Omega}_t - \underline{\Omega}_t\} \Rightarrow \dot{V} < 0$ and the set $\bar{\Omega}_t$ is a positively invariant set. The trajectories of the system starting in $\bar{\Omega}_t$ enter $\underline{\Omega}_t$ in finite time, see [4, Section 4.8] for a motivation of the level sets involved, and any solution starting in $\underline{\Omega}_t$ will remain in the set since $\dot{V} < 0$ in the corresponding boundary. The initial conditions given by (9a) satisfy $\mathcal{R}(t_0) \in \bar{\Omega}_t$; any $\mathcal{R} \in \underline{\Omega}_t$ satisfies (9b), which concludes the proof. The gain condition $k_\omega > \frac{u_{\max}}{2}$ is required so that $\{\bar{\Omega}_t - \underline{\Omega}_t\} \neq \emptyset$. ■

The results stated in Theorem 2 guarantee that any trajectory emanating from the region (9a) converges to the bounded region (9b), whose boundaries are a function of the noise to gain ratio $\frac{\|\mathbf{u}_\omega\|_\infty}{k_\omega}$. Following the proposed technique, a density function is adopted to show that almost all trajectories of the system (8) satisfy a $\liminf$ condition.

Theorem 3: The system (8) is weakly almost ISS with respect to $\mathbf{I}$. Namely, the solutions verify

$$\forall \mathbf{u}_\omega \in U\ \forall \text{a.a.} \mathcal{R}(t_0) \in \mathrm{SO}(3) \qquad \liminf_{t\to\infty} \|\mathbf{I}-\mathcal{R}(t)\|^2 < \gamma_2(\|\mathbf{u}_\omega\|_\infty),$$

where $\gamma_2(u) = \frac{8\left(\frac{u}{k_\omega}\right)^2}{1+\left(\frac{u}{k_\omega}\right)^2}$.

Proof: The result is obtained by satisfying the conditions of [1, Theorem 4], with the density function $\rho(\mathcal{R}) = \frac{1}{\operatorname{tr}^2(\mathbf{I}-\mathcal{R})}$. From the local ISS property obtained in Theorem 2, it is immediate that $\mathcal{R} = \mathbf{I}$ is a locally stable equilibrium point for $\mathbf{u}_\omega = 0$. The function $f := \operatorname{vec}\left(k\mathcal{R}(\mathcal{R}'-\mathcal{R}) - \mathcal{R}(\mathbf{u}_\omega)_\times\right)$ is locally Lipschitz over $\mathrm{SO}(3)$ and C^1 over $\mathrm{SO}(3)\setminus\{\mathbf{I}\}$. The density function $\rho(\mathcal{R})$ is of class C^1 over $\mathrm{SO}(3)\setminus\{\mathbf{I}\}$ and, given that $\mathrm{SO}(3)$ is compact, verifies $\int_{\mathrm{SO}(3)\setminus U} \rho(\mathcal{R})d\mathcal{R} < +\infty$, for all open neighborhoods U of 0_M.

The divergence is given by

$$\operatorname{div}(\rho f) = \frac{k_\omega}{\operatorname{tr}^3(\mathbf{I}-\mathcal{R})}\left(\|\mathbf{I}-\mathcal{R}\|^2 + \frac{2}{k_\omega}(\mathcal{R}-\mathcal{R}')'_\otimes \mathbf{u}_\omega\right),$$

where $\operatorname{div}(\rho f) = \rho \operatorname{div}(f) + \nabla(\rho)' f$, $\operatorname{div}(f) = -2k_\omega \operatorname{tr}(\mathcal{R})$ and $\nabla(\rho) = 2\operatorname{tr}(\mathbf{I}-\mathcal{R})^{-3}\operatorname{vec}(\mathbf{I})$, for more details on the computations of divergence and integrals in $\mathrm{SO}(3)$ see [3]. To attain the "density propagation inequality" [1, Theorem 4], given by

$$\forall u \in U\ \forall x \in M\ |x| \geq \gamma_2(|u|) \Rightarrow \operatorname{div}(\rho f) \geq Q(x), \tag{10}$$

with $Q(x) > 0$ for almost all $x \in M$, the sufficient condition

$$\|\mathbf{I}-\mathcal{R}\|^2 + \frac{2}{k_\omega}(\mathcal{R}-\mathcal{R}')'_\otimes \mathbf{u}_\omega \geq \xi, \quad \xi > 0,$$

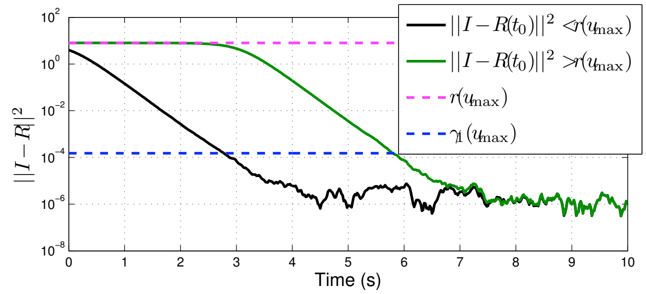


Fig. 2. Simulation results of the attitude observer, illustrating almost ISS. Convergence for $\|\mathbf{I}-\mathcal{R}(t_0)\|^2 \leq r(u_{\max})$ is guaranteed by Lyapunov methods; convergence for $\|\mathbf{I}-\mathcal{R}(t_0)\|^2 > r(u_{\max})$ is guaranteed by combining Lyapunov and density functions.

is analyzed, such that (10) is verified with $Q(\mathcal{R}) = \frac{k_\omega}{\operatorname{tr}^3(\mathbf{I}-\mathcal{R})}\xi > 0$ for almost all $\mathcal{R} \in \mathrm{SO}(3)$. The inequality is satisfied if $\|\mathbf{I}-\mathcal{R}\|^2 > 4\frac{\|\mathbf{u}_\omega\|}{k_\omega}\left\|\left(\frac{\mathcal{R}-\mathcal{R}'}{2}\right)_\otimes\right\| \Leftrightarrow$ $\|\mathbf{I}-\mathcal{R}\|^2 > \frac{8\|\frac{\mathbf{u}_\omega}{k_\omega}\|^2}{1+\|\frac{\mathbf{u}_\omega}{k_\omega}\|^2} = \gamma_2(\|\mathbf{u}_\omega\|)$. Since $\gamma_2(u)$ is a class $\mathcal{K}$ function ($\gamma_2(0) = 0$ and $\frac{\mathrm{d}\,\gamma_2(u)}{\mathrm{d}u} > 0$), the inequality (10) is verified with $\|\mathbf{I}-\mathcal{R}\|^2 > \gamma_2(\|\mathbf{u}_\omega\|) \Rightarrow \operatorname{div}(\rho f) \geq \frac{k_\omega}{\operatorname{tr}^3(\mathbf{I}-\mathcal{R})}\xi$ for some $\xi > 0$, producing the desired result. ■

As expressed in Lemma 1, the weakly almost ISS obtained in Theorem 3 is used to guarantee that the solutions enter the local ISS set derived in Theorem 2, producing almost ISS.

Theorem 4: Let $k_\omega > \frac{u_{\max}}{\sqrt{3}}$. Then, the trajectories of the system (8) satisfy

$$\forall \mathbf{u}_\omega \in U\ \forall \text{a.a.} \mathcal{R}(t_0) \in \mathrm{SO}(3) \qquad \limsup_{t\to\infty} \|\mathbf{I}-\mathcal{R}(t)\|^2 < \gamma(\|\mathbf{u}_\omega\|_\infty),$$

where $\gamma(u) = \gamma_1(u) = 4\left(1 - \sqrt{1 - \frac{u^2}{4k_\omega^2}}\right)$, i.e. the attitude observer is almost ISS with respect to $\mathbf{I}$.

Proof: The system (8) is locally ISS and weakly almost ISS, by Theorems 2 and 3, respectively. The condition $\gamma_2(\|\mathbf{u}_\omega\|_\infty) < r(\|\mathbf{u}_\omega\|_\infty)$ formulated in Lemma 1 is equivalent to $\frac{2\left(\frac{\|\mathbf{u}_\omega\|_\infty}{k_\omega}\right)^2}{1+\left(\frac{\|\mathbf{u}_\omega\|_\infty}{k_\omega}\right)^2} < 1 + \sqrt{1 - \left(\frac{\|\mathbf{u}_\omega\|_\infty}{2k_\omega}\right)^2}$, that is satisfied for $\frac{\|\mathbf{u}_\omega\|_\infty}{k_\omega} < \sqrt{3}$, which yields almost ISS. ■

Simulation results of the observer estimation error are depicted in Fig. 2. The exponential convergence for $\gamma_1(u_{\max}) < \|\mathbf{I}-\mathcal{R}(t)\|^2 < r(u_{\max})$ is justified by the fact that $\dot{V} < -\alpha V$ in that region, for some $\alpha \in \mathbb{R}^+$.

IV. LOCAL STABILITY ANALYSIS USING DENSITY FUNCTIONS

This section derives new results for local stability analysis of equilibrium points other than the origin, using density functions. By combining the proposed stability results with LaSalle's invariance principle, a new tool for global stability analysis of the origin is obtained. The proposed technique is illustrated for the case of a simple attitude observer with biased inertial measurements.

A. Stability using Density Functions and LaSalle's Invariance Principle

The proposed stability analysis results are derived for autonomous nonlinear systems of the form

$$\dot{x} = f(x), \tag{11}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, and the associated flow $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\phi_t(x_0) = x(t, x_0)$, where $x(t, x_0)$ denotes the solution of the system at time t with initial condition x_0. In the remainder of this work, it is assumed that ϕ_t is well defined [15, Chapter 7].

Assumption 1: The flow ϕ_t is unique, continuous, and exists for all t.

To formulate the stability results in the presence of multiple equilibrium points, some concepts and results are introduced, for more details the reader is referred to [15]. The values at time t and at the time interval $t \in [\tau_0, \tau]$ of the trajectories starting in the set $\mathcal{A}$, are respectively denoted by $\phi_t(\mathcal{A}) = \{x : x = \phi_t(x_0), x_0 \in \mathcal{A}\}$ and $\phi_{[\tau_0,\tau]}(\mathcal{A}) = \{\phi_t(\mathcal{A}) : t \in [\tau_0, \tau]\}$. The local inset of x_u is the set of all initial conditions inside a neighborhood U of x_u that converge to x_u without leaving U, i.e. $\mathcal{Z}_U(x_u) = \{x \in U : \forall_\epsilon \exists_T \forall_{t>T} |\phi_t(x) - x_u| < \epsilon \text{ and } \forall_{t>0} \phi_t(x) \in U\}$. The global inset of x_u, denoted as $\mathcal{W}(x_u)$, is defined by taking $\mathcal{Z}_U(x_u)$ with $U = \mathbb{R}^n$.

The following theorem is a new result in density function methodologies, and provides sufficient conditions to show that an equilibrium point is not stable, given a suitable density function. This property is of interest to exclude the stability of equilibrium points other than the origin.

Theorem 5: Let $x_u \in \mathbb{R}^n$, and suppose there exists a non-negative $\rho \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$, integrable in a neighborhood U of x_u, and with $\operatorname{div}(\rho f) > 0$ in U. Then, the global inset of x_u has zero measure.

Proof: First, it is shown that the local inset, denoted as $\mathcal{Z}_U$ with a slight abuse of notation, has zero measure. By Lemma 9 presented in Appendix A, the local inset $\mathcal{Z}_U$ is measurable. Using [12, Lemma A.1] with $D = U$ produces $0 \geq \int_{\phi_t(\mathcal{Z}_U)} \rho(x)dx - \int_{\mathcal{Z}_U} \rho(z)dz = \int_0^t \int_{\phi_\tau(\mathcal{Z}_U)} [\operatorname{div}(\rho f)](x) dx d\tau$. Since $\operatorname{div}(\rho f) > 0$ in U, then $\phi_t(\mathcal{Z}_U) \subset \mathcal{Z}_U \subset U$ has zero measure. The flow ϕ_t is a diffeomorphism and hence $\mathcal{Z}_U$ has zero measure [7]. The forward propagation of $\mathcal{Z}_U$ is $\mathcal{Z}_U$ itself, i.e. $\mathcal{Z}_U = \phi_{[0,\infty)}(\mathcal{Z}_U)$. Therefore, $\phi_{[0,\infty)}(\mathcal{Z}_U)$ has zero measure, and by Lemma 10 of Appendix A, the set $\phi_{(-\infty,\infty)}(\mathcal{Z}_U)$ has zero measure. The global inset of x_u can be expressed as $\mathcal{W}(x_u) = \phi_{(-\infty,\infty)}(\mathcal{Z}_U)$ and therefore has zero measure. ■

The combination of Theorem 5 with LaSalle's invariance principle can be used to provide almost GAS of the origin. The technique is based on using LaSalle's invariance principle to show that the trajectories approach a candidate set M [4]; and then using the $\operatorname{div}(\rho f) > 0$ property for $M \setminus \{0\}$, to show that the set of trajectories converging to $M \setminus \{0\}$ is of zero measure, and hence that the origin is almost GAS.

Lemma 6: Consider the system (11). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that the level sets $\{x : V(x) \leq c\}$ are bounded and $\dot{V}(x) \leq 0$. Let M be the largest invariant set in $\{x : \dot{V}(x) = 0\}$. Suppose that M is a countable union of isolated points, and that there is a density function that satisfies the conditions of Theorem 5 for all $x_U \in M \setminus \{0\}$. Then the origin of (11) is almost GAS.

Proof: The conditions on $V(x)$ satisfy LaSalle's invariance principle [4], [15], and hence guarantee that the trajectories approach M as $t \to \infty$, i.e. $\forall x_0 \forall_\varepsilon \exists_{T>0} \forall_{t>T} \inf_{y \in M} \|\phi_t(x_0) - y\| < \varepsilon$. By the continuity of $\phi_t(x)$, choosing $\varepsilon < \min_{x,y \in M} \|x - y\|$ shows that each solution of (11) must converge to an isolated point $x_u \in M$. By Theorem 5, the condition $\operatorname{div}(\rho f) > 0$ for a neighborhood U of every $x_u \in M \setminus \{0\}$ guarantees that the global inset of x_u has zero measure. The set of initial conditions that converge to $M \setminus \{0\}$, given by $\cup_{x_u \in M \setminus \{0\}} \mathcal{W}(x_u)$, is a countable union of zero measured sets and hence has zero measure. Consequently, almost all solutions converge to the origin, and hence the origin is almost GAS. ■

Remark 1: The results of Theorem 5 provide an alternative approach to the stability analysis based on Hartman-Grobman theorem [15]. Also, the derived stability result is also valid for non-hyperbolic equilibria. Future work will evaluate the contribution of Theorem 5 in this topic.

Remark 2: Lemma 6 is valid when the invariant set is a countable union of isolated points. The extension for more generic sets, based on generalizing the results adopted in the proof of Theorem 5, will be addressed in future work.

B. Stability of the Nonlinear Observer in the Presence of Biased Inertial Readings

In this section, the proposed stability analysis is illustrated for the attitude observer, in a case where the disturbance in (1) is a constant bias, i.e. $\dot{\mathbf{u}}_\omega = 0$. The observer kinematics are augmented to estimate the bias, and are given by $\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}} (\hat{\boldsymbol{\omega}})_\times$, $\dot{\hat{\mathbf{b}}}_\omega = k_{b_\omega} \mathbf{s}_\omega$, where $\hat{\boldsymbol{\omega}} = \hat{\mathcal{R}}'^I \bar{\mathbf{X}} \mathbf{X}'_r \left(\boldsymbol{\omega}_r - \hat{\mathbf{b}}_\omega\right) - k_\omega \mathbf{s}_\omega$, $\mathbf{s}_\omega = \sum_{i=1}^n (\hat{\mathcal{R}}'^I \bar{\mathbf{x}}_i) \times \mathbf{x}_{r\,i}$, $k_\omega, k_{b_\omega} \in \mathbb{R}^+$ are feedback gains, and $\hat{\mathbf{b}}_\omega$ is the rate gyro bias estimate, for more details on the adopted observer see [17]. The closed loop error kinematics are given by

$$\dot{\mathcal{R}} = \mathcal{R} \left[k_\omega (\mathcal{R}' - \mathcal{R}) + (\mathbf{b}_\omega)_\times\right], \dot{\mathbf{b}}_\omega = k_{b_\omega} (\mathcal{R}' - \mathcal{R})_\otimes, \tag{12}$$

where $\mathbf{b}_\omega = \hat{\mathbf{b}}_\omega - \mathbf{u}_\omega$ is the bias estimation error. The stability analysis technique is illustrated for the case where initial bias and attitude estimation errors exist along the z-axis, i.e. $\mathbf{b}(t_0) = \begin{bmatrix} 0 & 0 & b_0 \end{bmatrix}'$, $b_0 \in \mathbb{R}$, and $\mathcal{R}(t_0) = \exp(\theta_0 (\boldsymbol{\lambda}_0)_\times)$, $\theta_0 \in \mathbb{R}$, $\boldsymbol{\lambda}_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$. In this case, the trajectories of (12) satisfy $\mathcal{R}(t) = \exp(\theta(t) (\boldsymbol{\lambda}_0)_\times)$, $\mathbf{b}(t) = \begin{bmatrix} 0 & 0 & b(t) \end{bmatrix}'$, and the dynamics can be reduced to

$$\dot{\theta} = -\sin(\theta) + b, \quad \dot{b} = -\sin(\theta), \tag{13}$$

with initial conditions $\theta(t_0) = \theta_0$, $b(t_0) = b_0$.

As discussed in Section IV-A, the stability of the system (13) is first analyzed using LaSalle's invariance principle to derive an invariant set M.

Proposition 7: The trajectories of the system (13) approach $M = \{(\theta, b) : \theta = \pi k, k \in \mathbb{Z}, b = 0\}$ as $t \to \infty$.

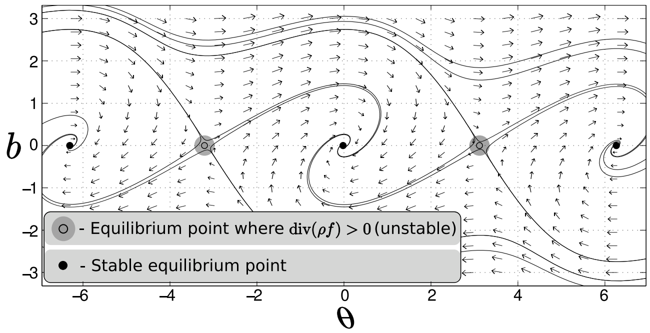


Fig. 3. Phase portrait of the reduced order attitude observer. Using the density function property $\operatorname{div}(\rho f) > 0$ in a neighborhood of the equilibrium points $(\theta, b) = (\pi + 2\pi k, 0)$, $k \in \mathbb{Z}$, shows that these are unstable.

Proof: The result is obtained by considering the Lyapunov function $V = 2(1 - \cos(\theta)) + b^2$. The time derivative is given by $\dot{V} = -2\sin^2(\theta)$ and the result is immediate from LaSalle's invariance principle. ■

The phase portrait of the system, depicted in Fig. 3, suggests that the equilibrium points in the set $E = \{(\theta, b) : \theta = 2\pi k + \pi, k \in \mathbb{Z}, b = 0\} \subset M$ are unstable. Using the technique summarized in Lemma 6, the instability of E is shown using a density function that satisfies Theorem 5, yielding that almost all trajectories approach $M \setminus E = \{(\theta, b) : \theta = 2\pi k, k \in \mathbb{Z}, b = 0\}$ as $t \to \infty$.

Proposition 8: Almost all trajectories of the system (13) approach the set $\{(\theta, b) : \theta = 2\pi k, k \in \mathbb{Z}, b = 0\}$ as $t \to \infty$.

Proof: To exclude the points in the set E, the density function $\rho = \frac{1}{2(1-\cos(\theta))+b^2}$ is adopted. The divergence is $\operatorname{div}(\rho f) = \frac{2(1-\cos(\theta))-\cos(\theta)b^2}{(2(1-\cos(\theta))+b^2)^2}$, so there is a neighborhood of every point in E where ρ is integrable and $\operatorname{div}(\rho f) > 0$. By Theorem 5, the set of initial conditions converging to each point in E has zero measure. The set E is a countable union of points, and hence the set of initial conditions that approach E has zero measure. Consequently, almost all the trajectories approach $M \setminus E$. ■

V. CONCLUSIONS

This work addressed the combination of Lyapunov and density functions, for stability analysis of nonlinear systems. Almost ISS of the origin was formulated as the combination of local ISS and weakly almost ISS, that can be derived using Lyapunov and density functions, respectively. For the case of autonomous systems, it was shown that global stability of the origin can be obtained by combining LaSalle's invariance principle, with a density function that excludes the stability of undesirable equilibrium points. The proposed stability analysis techniques were illustrated for an attitude observer with non-ideal angular velocity readings. Future work will address the topics discussed in Remarks 1 and 2.

APPENDIX

A. Set Measure Results

This section presents auxiliary results adopted in the paper.

Lemma 9: The local inset of an equilibrium point is measurable under Assumption 1.

Proof: The set $\mathcal{Z}_U$ can be written as the intersection of a "stability" and a "convergence" sets, given by $\mathcal{Z}_U = \mathcal{S} \cap \mathcal{C}$ where $\mathcal{S} = \{x \in U : \phi_t(x) \in U \text{ for all } t \geq 0\}$ and $\mathcal{C} = \{x \in U : \forall_\epsilon \exists_T \forall_{t>T} \|\phi_t(x) - x_u\| < \epsilon\}$. The set $\mathcal{S}$ can be described by $\mathcal{S} = \bigcap_{k \in \mathbb{N}_0} \mathcal{S}_k$ where $\mathcal{S}_k = \{x \in U : \phi_t(x) \in U \text{ for } t \in \begin{bmatrix} kT & kT + T \end{bmatrix}\}$. By the continuous dependence of $\phi_t(x)$ on the initial conditions [15], and on t, for each $x \in \mathcal{S}_k$ there exists δ sufficiently small, such that $\|x - y\| < \delta \Rightarrow \phi_t(y) \in U$ for the compact interval $t \in \begin{bmatrix} kT & kT + T \end{bmatrix}$. Consequently, the set $\mathcal{S}_k$ is open, thus measurable, and the set $\mathcal{S}$ is measurable. The set $\mathcal{C}$ can be described by $\mathcal{C} = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}_0} \mathcal{C}_{n,k}$, where $\mathcal{C}_{n,k} = \{x \in U : \exists_T \forall_{t \geq T} \|\phi_t(x) - x_u\| < \frac{1}{n} \text{ for all } t \geq k\}$. The set $\mathcal{C}_{n,k}$ is measurable, by the same arguments used for the measurability of $\mathcal{S}$, and $\mathcal{C}$ is a countable union and intersection of measurable sets and is measurable, which concludes the proof. ■

Lemma 10: Under Assumption 1, the set $\phi_{[\tau_0,\tau]}(\mathcal{A})$ has zero measure if and only if $\phi_{(-\infty,\infty)}(\mathcal{A})$ has zero measure.

Proof: ($\Leftarrow$) Immediate from $\phi_{[\tau_0,\tau]}(\mathcal{A}) \subset \phi_{(-\infty,\infty)}(\mathcal{A})$. ($\Rightarrow$) f is smooth, then ϕ_t is a diffeomorphism for each t [15], and $\phi_t(\phi_{[\tau_0,\tau]}(\mathcal{A}))$ has zero measure [7]. Let $t_k = (\tau - \tau_0)k$, $k \in \mathbb{Z}$, for autonomous systems $\phi_t(\phi_{[\tau_0,\tau]}(\mathcal{A})) = \phi_{[\tau_0+t,\tau+t]}(\mathcal{A})$ and hence $\phi_{(-\infty,\infty)}(\mathcal{A}) = \bigcup_{k \in \mathbb{Z}} \phi_{[\tau_0+t_k,\tau+t_k]}(\mathcal{A})$ is a countable union of zero measure sets, and thus has zero measure [14]. ■

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