A note on Isomorphism and Identity

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Abstract

This note argues that, insofar as contemporary mathematics is concerned, there is overwhelming evidence that if mathematical objects are structures, then isomorphism should not be taken as their identity condition. This goes against a common version of structuralism in the philosophical literature. Four areas are presented in which identifying isomorphic structures or objects leads to contradiction or inadequacy. This is followed by a philosophical discussion on possible ways to approach the distinction, and a section on the possibility of proceeding intensionally, as is done in e.g. the Univalent Foundations program.

1 Preliminaries

1.1 Isomorphism in category theory

In many versions of structuralism it is assumed, often without argument, that the proper identity condition for structures is isomorphism. The purpose of this note is to show that, unless identity is given a rather different treatment than classically, this makes structuralism inapplicable to mathematics. These lessons seem to be rather well-known in the mathematics community; we find Barr and Barr [BB11], in the preface to their translation of Grothendieck's famous *Tôhoku* paper [Gro57], remarking as follows on Grothendieck's use of '=' for isomorphism in the 50's:

> The structuralists who founded Bourbaki wanted to make the point that isomorphic structures should not be distinguished, but category theorists now understand the distinction between isomorphism and equality. For example, all of Galois theory is dependent on the automorphism group which is an incoherent notion in the structuralist paradigm.

The mentioned understanding does not seem to have spread quite as well to philosophy yet. It is my hope that this note can help a bit here. [1]

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[1] A reader sufficiently damaged by linguistic philosophy may have noted that the quote mentions 'equality' rather than identity. In mainstream mathematics, however, there seems

As Barr & Barr note, the appropriate framework for discussing isomorphisms mathematically is category theory. Recall, a *category* $\mathcal{C}$ is a pair of collections $\mathsf{obj}_\mathcal{C}, \mathsf{hom}_\mathcal{C}$ together with mappings $\mathsf{cod}, \mathsf{dom} : \mathsf{hom}_\mathcal{C} \to \mathsf{obj}_\mathcal{C}$, a mapping $\mathsf{id} : \mathsf{obj}_\mathcal{C} \to \mathsf{hom}_\mathcal{C}$ and a partial binary operation $g \circ f$ on $\mathsf{hom}_\mathcal{C}$, defined wherever $\mathsf{cod}\, f = \mathsf{dom}\, g$. Where $\mathsf{dom}\, f = A$ and $\mathsf{cod}\, f = B$, we write $f : A \to B$, and we let $\mathsf{hom}_\mathcal{C}(A, B)$, for $A, B \in \mathsf{obj}_\mathcal{C}$, be the subcollection of $\mathsf{hom}_\mathcal{C}$ containing all $f : A \to B$. The axioms $\mathsf{id}\, B \circ f = f \circ \mathsf{id}\, A = f$, for all $f : A \to B$, and $(h \circ g) \circ f = h \circ (g \circ f)$, for all f, g, h such that the compositions are defined, are imposed.

A commonly occurring example of category in mathematics is the following:

(*i*) $\mathsf{obj}_\mathcal{C}$ is a class of mathematical objects or structures, such as the class of all sets, the class of all groups, all vector spaces, or all topological spaces.

(*ii*) $\mathsf{hom}_\mathcal{C}(A, B)$ is the set of all structure-preserving mappings from A to B. In the examples of (*i*), we may take functions, group homomorphisms, linear transformations, and continuous functions, which gives us the categories **Set**, **Grp**, **Vec** and **Top**, although these are not the only choices for any of these classes of objects. When the elements of $\mathsf{hom}_\mathcal{C}$ are structure-preserving mappings, they are often referred to as the *morphisms* of $\mathcal{C}$.

(*iii*) dom and cod give the domain and codomain (or source and target) of each mapping in $\mathsf{hom}_\mathcal{C}$.

(*iv*) $\mathsf{id}\, A$ is the trivial identity mapping on A.

(*v*) $g \circ f$ is the composition of the mappings f and g.

Specification of a category entails a definition of what it means for two objects to be isomorphic: $A \cong B$ holds, by definition, iff there are morphisms $f : A \to B$ and $g : B \to A$ such that $g \circ f = \mathsf{id}\, A$ and $f \circ g = \mathsf{id}\, B$. This coincides, in all known categories, with the more traditional, "internal" notion of isomorphism between objects as a mapping that preserves and reflects all structure. In fact, from the viewpoint of contemporary mathematics, it is the category theoretical concept of isomorphism that defines what structure is, much as it is the concept of equinumerousity that defines what numbers are.

This, however, brings with it a certain *relativity* of structure. The real numbers $\mathbb{R}$ can be defined as, say, a certain set of Dedekind cuts on the rationals. Nevertheless, their structure differs a lot depending on whether you treat them as just a set, an ordered set, a monoid under multiplication, a group under addition, a ring, a field, or a one-dimensional vector space. Likewise, topology using open maps is different from the usual topology built on continuous maps, and first-order model theory with only elementary equivalences describes a different kind of structure than first-order model theory with the usual model homomorphisms. The upshot is that, category-theoretically, an object's structure is not something intrinsic to that object, but depends on relations to other objects.

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to be no systematic difference in how these words are used, and I will follow this practice and use the words 'identity' and 'equality' interchangeably.

1.2 Structures and objects

Before proceeding, it is important to clarify the possible distinction between *identifying objects* and *identifying structures*. Mathematics treats of many objects that are intrinsically the same: Dedekind's natural numbers make up a set in which we "entriely neglect the special character of the elements" [Ded01, §73]. Cantor's cardinalities are sets of "pure ones", all the same, arrived at by intellectual projection from sets of distinct elements [Can62, p. 283]. And the points of a space are, traditionally, considered to be completely similar, and only distinguishable by their relations to one another. Kant subscribes to a version of this thesis in the *Critique*:

> The conception of a cubic foot of space, however I may think it, is in itself completely identical. But two cubic feet in space are nevertheless distinct from each other from the sole fact of their being in different places [...] [Kan10, p. 203]

In mathematics, identifying all intrinsically similar objects is *clearly* not admissible. But the traditional "objects" of mathematics, e.g. numbers, elements and points, are generally not taken to be the objects in a category. In **Set**, the objects are sets rather than elements of sets, in **Vec** and **Top**, the objects are spaces rather than points, and in **Grp** they are groups rather than elements of groups. Natural numbers are commonly described all at one using a *natural numbers object N*, which corresponds to Dedekind's notion of a simply infinite system, rather than a single number.[2]

So we have to be careful not to automatically interpret the *philosophical* notion of mathematical object as the *category-theoretical* one, which is, so to say, of a higher order. On the other hand, anything that can stand in structural relationships, such as being isomorphic with X, being embeddable in X, and being a quotient structure of X, is an object of a category. Furthermore, as we noted, it is this fact that gives it the structure it has.

Structuralism, as a philosophy of mathematics, will be taken to be a thesis that all, or at least many, of the objects of mathematics *are* structures. In particular, we will take this to hold for groups, sets, topological spaces, vector spaces, or categories. The informal gloss of isomorphism as *sameness of structure* makes it very tempting to interpret the identity conditions of such objects to be given by isomorphism; in fact, it seems to be a typical case of Fregean abstraction, much as he takes *direction* to be abstracted from the relation of parallelism, or *number* from equinumerosity [Fre53]. Nevertheless, as the next section will show, there are strong reasons why this temptation should be resisted.

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[2]In some cases, elements of an object X are *representable* by morphisms $1 \to X$, where 1 is the terminal object of the category. But for this representation to be accurate quite a lot of extra structure on the category is required.

2 Counterexamples to *isomorphism → identity*

We will now give four examples of areas where assuming that isomorphism implies identity leads to contradiction or inadequacy.

2.1 Sets

A *full* subcategory of a category $\mathcal{C}$ is a category $\mathcal{C}'$ whose objects are some of those of $\mathcal{C}$, such that $\mathsf{hom}_{\mathcal{C}'}(A, B) = \mathsf{hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathsf{obj}\,\mathcal{C}'$. A *skeleton* of a category $\mathcal{C}$ is a full subcategory $\mathcal{C}_0$ of $\mathcal{C}$ such that for all $A \in \mathsf{obj}\,\mathcal{C}$, there is an element B in $\mathsf{obj}\,\mathcal{C}'$ such that $A \cong B$ (i.e. the inclusion is *essentially surjective*), and

$$A \cong B \Rightarrow A = B$$

holds for all $A, B \in \mathsf{obj}\,\mathcal{C}_0$.

A skeleton is always equivalent to the category it is a skeleton of, and skeletons of equivalent categories are themselves equivalent. The isomorphism $\Rightarrow$ identity thesis can be interpreted as the requirement that all categories should be skeletal.

Now consider the category **Set** of sets, and one of its skeletons **Set**$_0$. Since isomorphism in **Set** is just one-to-one correspondence, it follows that **Set**$_0$ contains just one set of each cardinality. But this structure does not satisfy the ZFC axioms; we would have, for instance, that

$$\{\varnothing\} = \{\{\varnothing\}\}$$

which gives a contradiction with the singleton (or pair) axiom.

One reply to this argument would be that plain functions are *not* the appropriate structure-preserving transformations among sets, so that isomorphism is something stronger than mere one-to-one correspondence. A different choice would be to take the membership relation into account, and let $\mathsf{hom}(A, B)$ be those functions f from A to B such that

$$X \in Y \Leftrightarrow f(X) \in f(Y)$$

But this is still not strong enough. Consider the function $f : \{0, 1\} \to \{1, 2\}$ defined as $f(X) = X + 1$, with the natural numbers interpreted as von Neumann numerals. Since $X \in X + 1$ for all ordinals, we have that $X \in Y \Leftrightarrow f(X) \in f(Y)$, and f is clearly one-to-one. But $\{0, 1\} = 2$, so if we identify $\{0, 1\}$ and $\{1, 2\}$, we get that

$$\{0, 1\} = \{1, 2\} = \{1, \{0, 1\}\} \Leftrightarrow \{0, 1\} = 0 \Leftrightarrow 0 = 2$$

which still contradicts ZFC, to say nothing of basic arithmetic.

There are, of course, even stronger conditions one can impose on the morphisms in **Set** in order to guarantee that all invertible morphisms are identities. One example is

$$\forall Y \ (Y \in X \leftrightarrow Y \in f(X))$$

which, by extensionality, entails that $f(X) = X$ is the only invertible morphism on its domain. But mathematicians generally do *not* require this to talk about sets as isomorphic, and, if anything, such a strong requirement means that the usefulness of the isomorphism concept is completely lost.

A strong requirement on set morphisms would also "infect" the rest of mathematics. Most mathematical structures can be exhibited as *concrete categories*, i.e. members of a category $\mathcal{C}$ together with a faithful functor $U : \mathcal{C} \to \mathbf{Set}$, which is called the *forgetful* functor. But all functors preserve isomorphisms, so if $U(X) \ncong U(Y)$, we must have $X \ncong Y$, for any $X, Y \in \mathsf{obj}\,\mathcal{C}$. Thus no two groups, vector spaces, first-order models, or topologies, with differnt carrier sets, can be isomorphic either.

In fact, the choice of arbitrary functions as the morphisms of $\mathbf{Set}$ is just right for everyday mathematics. What is not possible is to use these morphisms to identify isomorphic sets.

2.2 Categories

This is a good place to make a distinction betwen two different versions of the *isomorphism implies identity* claim:

(i) Isomorphic objects are identical, i.e. if there is an isomorphism $f : A \to B$, then $A = B$. This is what we referred to in the last subsection as *all categories being skeletal*.

(ii) Isomorphisms are identities, i.e. if f is an isomorphism, then $f = \mathsf{id}\,A$ for some A. This can be referred to as a requirement that *objects have no nontrivial automorphisms*.

The second of these is clearly stronger than the first, and it is not hard to see that it is far too strong. It makes the notion of a nontrivial automorphism group impossible, as the Barrs note in the quote in the beginning. One slightly more formal proof of its inadequacy for category theory is the following.

Let $\mathcal{C}$ be any category with binary products, i.e. such that for each $A, B \in \mathsf{obj}\,\mathcal{C}$, there is an object $A \times B$ and morphisms $\pi_A^{A \times B} : A \times B \to A$ and $\pi_B^{A \times B} : A \times B \to B$ such that, for any $X \in \mathsf{obj}\,\mathcal{C}$ with morphisms $f : X \to A$ and $g : X \to B$, there is a unique morphism $f \times g : X \to A \times B$ that makes the following diagram commute:

Whenever $(A \times B, \pi_A^{A \times B}, \pi_B^{A \times B})$ and $(B \times a, \pi_A^{B \times A}, \pi_B^{B \times A})$ are products of A and B, there is a unique isomorphism $\tau : A \times B \to B \times A$, called the *twisting*

isomorphism [Bor94, p. 40], such that $\pi_A^{B \times A} = \pi_A^{A \times B} \circ \tau$ and $\pi_B^{B \times A} = \pi_B^{A \times B} \circ \tau$. This entails that any two products of A and B are isomorphic.

Let P be a product $(B \times B, \pi_1, \pi_2)$ and let Q be the product $(B \times B, \pi_2, \pi_1)$, and let $f, g : C \to B$. We then have a commutative diagram

Now assume that the only isomorphisms are identities, so that $\tau = \mathsf{id}(B \times B)$. Then, by the diagram,

$$f = \pi_1 \circ (f \times g) = \pi_2 \circ \tau \circ (f \times g) = \pi_2 \circ (f \times g) = g$$

But f and g were arbitrary, so we have just shown that any category with products and no nontrivial automorphisms must be a preorder, i.e. have at most one morphism between every pair of objects. Partial orders are, however, definitely not enough for category theory. [3]

Thus the second, stronger, condition, i.e. that all isomorphisms are identities, makes category theory impossible. How about the weaker requirement (i)? Let **Cat** be the category of all small categories, with functors as morphisms. For each pair $\mathcal{C}, \mathcal{D}$, there is a *functor category* $\mathcal{D}^{\mathcal{C}}$ with the functors $F : \mathcal{C} \to \mathcal{D}$ as objects, and the natural transformations $\alpha : F \to G$ as morphisms. The *endomorphism category* $\mathcal{C}^{\mathcal{C}}$ is a special case of functor category.

Let $\mathcal{C}$ and $\mathcal{D}$ be any two categories with isomorphic endomorphism categories. If isomorphism implies equality, we have that $\mathcal{C}^{\mathcal{C}} = \mathcal{D}^{\mathcal{D}}$. But in **Cat**, distinct categories have distinct endomorphism categories, so, contrapositively, it follows that $\mathcal{C} = \mathcal{D}$. Is this a reasonable identification to make?

That the answer is no can be seen by considering the full subcategory **Grp** of **Cat**, which has one-object categories in which all morphisms are invertible as objects. Such a category is easily seen to be the same thing as a group, with the elements of the group as the morphisms [Awo10, ch. 4]. For a *simple* group $\mathcal{G}$, the endomorphism category $\mathcal{G}^{\mathcal{G}}$ has no non-invertible morphisms, so it is itself a group, and, in fact, $\mathcal{G}$'s (outer) *automorphism group*.

But there are several non-isomorphic simple groups that have the same automorphism group. An example is given by the alternating groups A_n for $n > 4$, which, except when $n = 6$, have a (the) two-element group as automorphism group. Thus, identifying these groups would identify almost all alternating groups, but none of these are even isomorphic.

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[3]This proof is loosely inspired by one attributed to Isbell by MacLane [Mac98, p. 164]. One difference is that MacLane's proof deals with monoidal categories, while I have attempted to stay within non-enriched category theory. Although I have not been able to find Isbell's original argument, my understanding of the problem has benefited greatly from discussions with members of the *Categories* mailing list.

2.3 Braids

A *braid* is an arrangement of strings, such as the ones below. Braids are counted as the same if they can be continuously deformed into one another without cutting strings or permuting the endpoints; thus braids A and B below are the same, but not A and C, nor C and D.


$$A \qquad C$$

$$B \qquad D$$
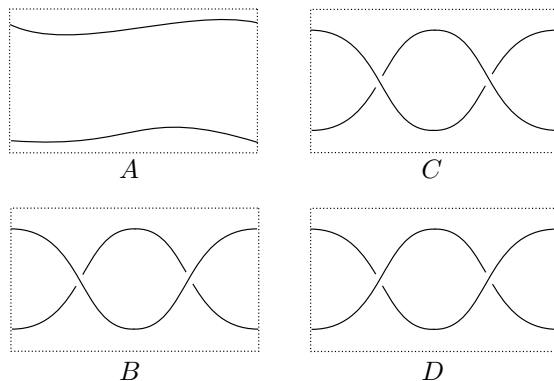
Braids can be composed in two ways: horizontally, by attaching the right end of one braid to the left end of another, and vertically, by placing one braid directly above the other. We write horizontal composition by attaching Y to the right of X as $X \circ Y$, and vertical composition by placing Y underneath X by $X \otimes Y$. Both of these operations are clearly associative. $\circ$ has a unit for each number n of strings, consisting of just n straight strings (in the case $n = 2$, it is braid A above). The unit of $\otimes$ is the empty braid with no strings at all.

For each number n of strings the $\circ$ operator induces a group structure known as the *braid group* B_n. D, above, is the inverse of C, since composing them horizontally gives the unit braid A of 2 strings.

$\circ$ is obviously not commutative, and neither is $\otimes$. However, there is a close connection between $X \otimes Y$ and $Y \otimes X$: for each vertical composition $X \otimes Y$ there is a transformation $\gamma_{XY} : X \otimes Y \to Y \otimes X$ called a *braiding*, obtained by passing the strings of X over those of Y. Since it is also possible to go the other way by passing Y's strings *under* X's, γ is an isomorphism, and so $X \otimes Y \cong Y \otimes X$. Note, however, that we do *not* have that

$$\gamma_{XY} = \gamma_{YX}^{-1}$$

Now consider the following braids, consisting of the braids E, F, their vertical products $E \otimes F$ and $F \otimes E$, and the γ_{EF} and γ_{FE} braidings.

$$E \qquad\qquad\qquad F$$

$$F \qquad \gamma_{EF} \qquad\qquad E \qquad \gamma_{FE}$$
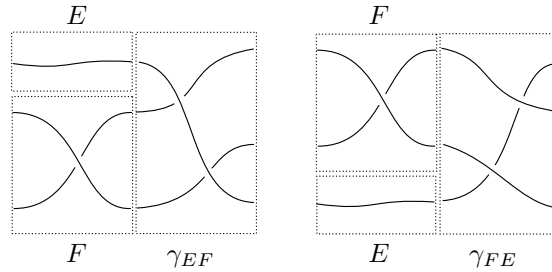
As we have remarked, $E \otimes F$ and $F \otimes E$ are isomorphic, but we can *not* identify them: that would give $\gamma_{EF} = \gamma_{FE}$, but these are clearly distinct. γ_{EF} is the braid that passes the uppermost string above the two others, and γ_{FE} is the braid that passes the *two* uppermost strings above the third. Identifying $E \otimes F$ and $F \otimes E$ conflicts with the braid structure.

Braid categories (i.e. categories in which the morphisms are braids) give a rather visual example of the inadmissibility of identifying isomorphic objects. But they are also an important subject in their own right, with applications to group theory, string theory, and topological quantum field theory. The feature of them that hinders identification of the isomorphic is also closely related to our last example, that of homotopy theory.[4]

2.4   Homotopies

Since the mathematics of this section are fairly involved, we will just give an overview here. Nevertheless, the subject gives an example of an area where, I believe, the inadequacy of identifying isomorphic objects is connected to actual problems in contemporary mathematical research.

Homotopy theory is a branch of topology that, roughly, studies spaces and continuous transformations using higher-order continuous transformations among the transformations themselves. A *homotopy* between continuous functions $f, g : X \to Y$ is a continuous function $h : [0,1] \times X \to Y$, where $[0,1]$ is the unit interval, such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$. When there is a homotopy between continuous functions $f, g : X \to Y$, they are said to be *homotopic*.

Let $X, *$ be a *pointed space*, i.e. a topological space with a selected base point $*$. Of special interest are the homotopies of *loops*, which are continuous functions $f : [0,1] \to X$ such that $f(0) = f(1) = *$. How these behave often tells us a lot about X. Two loops f, g compose as

$$(g \circ f)(t) = \begin{cases} f(2t) & \text{if } t < {}^1\!/_2 \\ g(2t - 1) & \text{if } t \geqslant {}^1\!/_2 \end{cases}$$

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[4]The argument of this section can be given much more generally by using braided monoidal categories, cf. [Mac98, pp. 260–265]. We have avoided doing so in order to keep the discussion as concrete as possible, although this brings the disadvantage that the similarity with the next section is not as apparent.

The composition of such loops is not associative, and neither are there strict identity loops, so we do not have a category. However, the quotient space of such loops under the relation of homotopy *is* a category, and, in fact, for a path-connected space, even a group. It is called the *fundamental group* of X.

From the homotopy definition we have used, we can see that it is iterable: the paths on X are themselves members of a topological space $X^{[0,1]}$ consisting of the continuous functions from $[0,1]$ to X.[5] What we get when we consider all iterations is a sequence of groups π_n called the *homotopy groups*. The elements of these are homotopy classes of transformations

$$h : [0,1] \to (\cdots((X^{[0,1]})^{[0,1]})^{\cdots})^{[0,1]}$$

where the right-hand side contains $n-1$ occurences of $[0,1]$. But, by the exponential law $(Z^Y)^X \cong Z^{Y \times X}$, these are reformulable as classes of transformations $h : [0,1]^n \to X$ subject to the boundary conditions

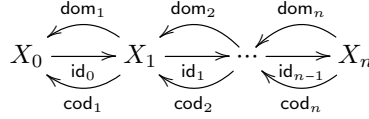$$h(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) = h(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n)$$

for $1 \leqslant k \leqslant n$. Intuitively, these transformtions can be seen functions from an n-dimensional cube where all of the faces have been identified down to a single point. These cubes, in turn, are homeomorphic to n-spheres, so the homotopy groups π_n are more often defined directly as the groups arising from composing homotopy classes of mappings of the n-sphere S^n to X (see, for example, [May99, p. 63]). The *homotopy n-type* of a space X is the class of spaces whose homotopy groups π_k, for $k \leqslant n$, are isomorphic with X's. Pointed spaces are *weakly equivalent* if all their homotopy groups are isomorphic.

The homotopy groups and homotopy n-types of many spaces are notoriously hard to compute, and attempts to do so have been a driving force in the development of algebraic topology during the 20th century. Since the 1980's, partly due to Grothendieck's preliminary investigations [Gro83], higher category theory has often been expected to turn out to be useful here. A category is, after all, a kind of generalisation of a group. But an interesting deveopment has taken place: it has turned out, due to arguments structurally rather similar to those for braids, that *strict* higher categories are insufficient to model the higher homotopies of spaces, and, in particular, the n-sphere.

One version of higher categories can be modeled as a *reflexive globular set*: a tuple X_k of sets for $0 \leqslant k \leqslant n$, with functions $\mathsf{dom}_{k+1}, \mathsf{cod}_{k+1} : X_{k+1} \to X_k$ and $\mathsf{id}_k : X_n \to X_{k+1}$ for $0 \leqslant k < n$.[6]

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[5] Actually, this is not always true, but requires X to be compactly generated and Hausdorff. That it is, is a common assumption in algebraic topology. The most widely used topology for function spaces is the *compact-open topology*, which for metric spaces coincides with the common uniform or "sup" topolgy. Cf. the careful discussion in [Bro06, pp. 181–197].

[6] This approach is used in e.g. [KV91].

$$X_0 \underset{\mathsf{cod}_1}{\overset{\mathsf{dom}_1}{\underset{\mathsf{id}_0}{\longrightarrow}}} X_1 \underset{\mathsf{cod}_2}{\overset{\mathsf{dom}_2}{\underset{\mathsf{id}_1}{\longrightarrow}}} \cdots \underset{\mathsf{cod}_n}{\overset{\mathsf{dom}_n}{\underset{\mathsf{id}_{n-1}}{\longrightarrow}}} X_n$$

An n-category can be interpreted as such a tuple, with X_0 as the collection of objects. For every pair $c, d \in X_0$, $\mathsf{dom}_1^{-1}(c) \cap \mathsf{cod}_1^{-1}(d)$ gives the object-set of an $n-1$-category, with dom, cod and id restricted to this set in the natural way.

What makes a reflexive globular set an *n-category* is the presence of *compositions*. For each set X_k, there are k composition operators $\underset{i}{\circ}$, $1 \leqslant i \leqslant k$. In a *strict n*-category, all of these operators are assumed to be associative, to commute with one another, and to have strict units. In a *weak n*-category, at least one of these requirements holds only up to an appropriate equivalence.

A strict 1-category is just a category. An example of a strict 2-category is **Cat**, which has all small categories as X_0, functors between small categories as X_1, and natural transformations between these functors as X_2.

A weak 2-category is also called a *bicategory*. Explicitly, such a structure can be represented as a reflexive globular set X_0, X_1, X_2 together with, for each composable triple of 1-morphisms $f, g, h \in X_1$, a natural isomorphism $\alpha : h \underset{1}{\circ}(g \underset{1}{\circ} f) \to (h \underset{1}{\circ} g)\underset{1}{\circ} f$, and for each $f \in X_1$, two natural isomorphisms $\rho : f \underset{1}{\circ}(\mathsf{id}_0 \mathsf{dom}_1 f) \to f$ and $\lambda : (\mathsf{id}_0 \mathsf{cod}_1 f)\underset{1}{\circ} f \to f$. These are furthermore required to satisfy *coherence axioms*, which guarantee that the diagrams using them commute.

Every bicategory is equivalent to a strict 2-category, so in 2 dimensions, there is no obstacle to replacing the isomorphisms α, ρ, λ with identities. But in three dimensions, this does not hold. The weak 3-categorical analogues of bicategories, known as *tricategories* [Gur07], are *not* all equivalent to strict 3-categories: there are tricategories in which, although $h \underset{1}{\circ}(g \underset{1}{\circ} f) \cong (h \underset{1}{\circ} g)\underset{1}{\circ} f$, the resulting morphisms cannot be identified without losing desciptive power.

Weak 3-categories model homotopy 3-types, and the fact that they are not all equivalent to strict 3-categories makes it possible that strict 3-categories are not enough for this task. As we mentioned, this is in fact the case: the homotopy 3-type of the 3-dimensional sphere is not representable in terms of a strict 3–category [Sim11, ch. 4]. This means that at least some kind of weakening of n-categories will be necessary for homotopy theory. This, in turn, requires keeping isomorphic objects or morphisms apart.

3   Discussion

The last section gave four examples of why isomorphism should not be taken to imply identity. The two concepts are related, but clearly distinct. This section speculates on some possible philosophical reasons for the distinction.

3.1 Proofs and truthmakers

A good place to start with the analysis of any mathematical relation is to investigate how it is proved to hold; proof is, after all, the primary way to obtain mathematical knowledge. Beginning with isomorphism, this is fairly straightforward: to show that $A \cong B$, one typically constructs an isomorphism between them, i.e. in most cases, one defines a mapping, and then one constructs an inverse to that mapping.

With equality, things are more difficult. In many cases, it depends on conventions already in place:

(i) Among numbers, showing that $a = b$ typically involves transforming a and b, using the equalities assumed in Peano arithmetic, into the same numeral n so that $a = b$ is shown to be equivalent to $n = n$.

(ii) Among sets, showing that $a = b$ typically involves showing that $x \in a \leftrightarrow x \in b$ for all x.

(iii) Among functions, showing that $f = g$ typically involves showing that they have the same domain and codomain and that $f(x) = g(x)$ for all x in the domain.

In all of these cases, however, we already *have* equalities assumed in the framework that we can use: PA *is*, essentially, a system of equalities, and extensionality and function extensionality are necessary to prove equalities among sets or functions, respectively.

The identity conditions of an axiomatisation are thus part of the axiomatisation, and insofar as axiomatisations are conventional, these conditions are themselves conventions. Nothing stops a mathematician from considering analogues of sets or functions that are not extensional, although she should of course not call them 'sets' or 'functions' then, since those words have been reserved for the extensional notions.

This means that equality among a class of terms is generally to be seen as part of the meaning explanation of these terms.[7] The imposition of identity conditions, given the other axioms, is however not entirely free, but is subject to the indiscernibility of identicals

$$a = b \Rightarrow \forall X \ (X(a) \leftrightarrow X(b))$$

The indiscernibility of identicals, in turn, fixes what it takes to prove *distinctness* of a and b: the existence of a relevant property X such that $X(a)$ but $\neg X(b)$, which we will refer to as a *separating property* for a, b. This is *not* conventional in the same way as rules for establishing identity. In particular, it is not dependent on any convention explicitly relying on identity itself. Already

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[7]This is even an explicit requirement in some frameworks, such as Martin-Löf type theory, which we will discuss briefly below.

when those axioms of a theory that do not mention identity are specified, we have what we need to be able to prove distinctness.[8]

This indicates that one difference between isomorphism and identity is that, for isomorphism, the epistemologically fundamental notion is indeed isomorphism, since we can prove that by constructing an appropriate mapping. By contrast, for equality, it is *distinctness* that is epistemologically fundamental, since that is what we typically can give independent proof of.

This argument can be transposed to the ontological sphere by talking of *truthmakers* rather than proofs.[9] A truthmaker for $a \cong b$ would then be an isomorphism between a and b, while $a \ncong b$ holds simply if there are no such truthmakers. A truthmaker for $a \neq b$ would be a property X such that $X(a)$ but $\neg X(b)$, while $a = b$ merely indicates the absence of such a truthmaker.

With one possible exception, the truthmakers of $a \cong b$ and $a \neq b$ should be seen as logically independent of one another: the existence of a separating property for a, b does not imply the existence of an isomorphism between them, and neither does the existence or the non-existence of an isomorphism imply the existence of a separating property.

However, we *do* usually have that if $a \ncong b$, then $a \neq b$. This is a result of isomorphism being defined as an equivalence relation, and thus reflexive. The "deeper" category-theoretic reason is that, in defining a type of mathematical structure, we are obliged to also define the structure-preserving mappings (i.e. morphisms) between objects of this type, and to do so in a way that provides an *identity morphism* for each object a. It is a simple exercise to show that identity morphisms are always isomorphisms, so the condition that well-behaved structures make up categories is sufficient to guarantee that $a = b \Rightarrow a \cong b$. Thus, on this conception, while $a = b$ needs no truthmaker, the identity morphism of a (or b) is a truthmaker for $a \cong b$.

This also indicates that when we do *not* have strict identities, there is no reason to expect identity to imply isomorphism either. This could turn out to be the case in certain forms of weak categories. In some cases, we may have canonical isomorphisms that, while not identities, still associate an automorphism with each object. Consider, for example, a space X with morphisms being paths $[0, 1] \rightarrow X$, as in section 2.4, but without identifying homotopic paths. The constant path $f(t) = a$ connects a to itself, but it is not an isomorphism: composing it with its converse (or pseudo-inverse) $g(t) = f(1 - t)$ does not get us to an identity, and as we mentioned, there are no strict identity morphisms in X at all.

This does not mean that $a = a$ does not hold for all $a \in X$, of course. But since there are no identity morphisms, and, *a fortiori*, no isomorphisms, identity does not imply isomorphism in this structure. We thus have a case in which the

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[8]Is there a corresponding method of showing that $a = b$ by using the identity of indiscernibles? Not that I am aware of; the possible relevant properties are usually infinitely many, so showing that they are all instantiated in a iff they are in b seems hopeless without having some kinds of general principles to start with, such as extensionality axioms.

[9]For intuitionistic mathematics this would not make a difference at all since truthmakers, under the BHK interpretation, *are* proofs. See [Sun94].

two concepts are entirely separate.

3.2   UF and the intensionality of structure

The discussion so far has been relying on classical extensional logic: it is *classical-logical* identity which cannot be the same as isomorphism. When we leave the safe confines of classical logic, however, the landscape of possibilities expands. The current programme of *Univalent Foundations* [Voe10] builds on a principle called the *Univalence Axiom*, which entails, for many algebraic structures, that isomorphism implies identity [CD12]. But the logic of UF is not classical logic, but intensional Martin-Löf type theory. In this logic, we have, as Martin-Löf [Mar84, pp. 59] explains, *four* different forms of identity:

(i) The *definitional equality* $a =_{df.} b$. This is "the equivalence relation generated by abbreviatory definitions, changes of bound variables and the principle of substituting equals for equals" [Mar84, p. 31]. It is a purely syntactic notion.

(ii) The *equality of elements* $a = b \in X$, where X is a type of which a and b are elements. This is perhaps the version most like classical identity in that it allows replacing equals for equals arbitrarily, subject only to type restrictions.

(iii) The *equality of types* $X = Y$, which consists in them having equal elements.

(iv) The *identity type* $Id_X(a,b)$: every pair of elements of X has specific type $Id_X(a,b)$ of *proofs that* $a = b$, and if $a \in X$, there is a canonical element $r(a)$ of this type called the *reflexivity proof*, interpretable as the canonical proof that $a = a$.

In extensional M-L type theory, the existence of an element of $Id_X(a,b)$ implies $a = b \in X$. In the intensional version, however, the non-emptiness of $Id_X(a,b)$ is strictly weaker than $a = b \in X$, and instead motivates a type of induction principle [NPS90, pp. 57–60].

In M-L type theory as used in Homotopy Type Theory, which is the interpretation of type theory in which UF is implemented, the identity type takes center stage. The univalence axiom entails that is that if A and B are isomorphic, then the identity type $Id_U(A,B)$, where U is a universe, is non-empty. *Properties* $P(x)$ where $x \in A$ are, roughly, identified with functions from A to U, with the informal explanation that $P(x)$, for any x, is the type of proofs that x satisfies P.[10] An induction theorem guarantees that if $f \in Id_A(a,b)$ and $P(a)$ is non-empty, then so is $P(b)$. This gives a kind of internal version of the indiscernibility of identicals for the elements of A [Pro13, p. 48].

The *general* indiscernibility of identicals does not follow from the existence of an element of $Id_X(a,b)$, however. Suppose that $a, b \in X$, and that $p \in Id_X(a,b)$.

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[10]More exactly, properties are types *dependent on A*. But for the purposes of this discussion it is handy to just think of them as functions into U.

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We have that $r(a) \in Id_X(a,a)$, but unless $a = b \in X$, we cannot have $r(a) \in Id_X(b,b)$. The induction principle for Id_X gives us only that $f(r(a)) \in Id_X(b,b)$ for a function $f : Id_X(a,a) \to Id_X(b,b)$.

What inhabitance of $Id_X(a,b)$ does is to allow us to *transport* structures on a to structures on b while retaining some of of their properties, the most important one being inhabitedness of types. It does not mean that structures defined on a are *themselves* defined on b as well, unlike classical equality. This can be framed as an issue of *covariance* vs. *invariance*: if $p \in Id_X(a,b)$, then structures on a *covary* with structures on b, with p giving the method of covariation. On the other hand, if $a = b \in X$, then structures on a are *invariant* with respect to replacing a with b.[11]

Awodey [Awo13] has recently argued for the use of UF and HTT as a foundation for structuralism which does satisfy the principle that isomorphic objects are equal. As he points out, the principle is not attained by collapsing classes of isomorphic objects, but by the weakening of the relation of identity. M-L type theory with intensional identity but without the Univalence axiom is indeed so weak that type checking becomes decidable. Since each proposition P is a type (i.e. the type of its proofs), being able to check whether P is the empty type 0 furthermore allows one to decide any statement. It remains to be seen how much of classical mathematics can still be represented in this foundation, and to what extent the addition of the Univalence axiom increases its power.

We have, in this note, been concerned with classical extensional identity, so our critique of identifying isomorphism and equality on itself poses no obstacles to the viability of the Univalent Foundations program, or programs like it. In intensional logics, equality often comes out as weaker. This may even be taken as a definition of what intensionality consists in, as Quine does when discussing opaque contexts [Qui80, pp. 139–159]. Since isomorphism is, as we have argued, also strictly weaker than classical identity, it is not implausible that it should be possible to model it using weaker forms of identity.

When Univalence is added, however, UF does not give a weaker *system* than classical type theory, but one incompatible with it. UF is *proof relevant*: unlike in extensional intuitionistic mathematics, not only is the existence of objects dependent on the existence of proofs, but the identity conditions of those objects depend non-trivially on the identity conditions of the proofs as well. This gives a radical form of non-extensionality, which is evinced in the fact that adding the Univalence Axiom to *classical* type theory, which validates $\neg\neg A \to A$, results in all types being inhabited. Given the internal interpretation of properties, this would mean that all elements of all types have all properties, which would trivialize the system [Pro13, pp. 106–107].

Thus, if we follow this path, and base mathematics thoroughly on intensional logic, we will end up with a very different view of mathemamatics than the usual structuralist one, as it has been described in e.g. [Hel94, Res97, Sha00]. As far as I know, even structuralists advocating constructive methods, such as [Chi04],

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[11]For a long, careful discussion on covariance and invariance, and the relationship to structuralism, see [Rod14], esp. chapter 8.

have generally assumed identity to be extensional. In UF, however, rather than being inherent to an object, what properties an object have will have to be treated as dependent on how the object in question is referred to. Although, in the internal language of HTT, we can prove that $P(a)$ iff $P(b)$ as soon as $Id_X(a,b)$ is non-empty, we cannot prove the more general principle of arbitrary substitution of equals for equals.

According to Awodey, equality in HTT means having the same structural properties, and 'property definable in HTT' gives an explication of what 'structural property' might mean, so that any $p \in Id_X(a,b)$ can be interpreted as a proof that a and b have the same structural properties. Like any explication, however, I believe that we should not expect it to be universal. In contrast, I would like to point out that it is very useful for mathematics to be able to *not* have a single notion of what counts as structural properties, and *a fortiori*, what counts as isomorphism.

Quite apart from our reliance on extensional identity, this indicates a significant difference between the way we have been working with structure here, and the way Awodey does in his work on UF. We have, freely, allowed ourselves to talk about the same object being in two different categories. This is what gives rise to the relativity of structure I alluded to in the first section: the same object can be structured in various ways by being considered as a member of different categories. In any type theory, including HTT, if we take categories to be types, such talk makes no sense.

3.3 Conclusion

We can state the message of this paper as a trilemma. One of the following has to be false:

(i) Mathematical objects such as groups, categories and spaces are structures.

(ii) Isomorphic structures are identical.

(iii) Identity in mathematics is extensional.

The conjunction of (i) and (ii) imply what Awodey [Awo13] calls the *Principle of Structuralism*, i.e. that isomorphic objects are identical. Non-structuralists such as set-theoretic foundationalists discard (i) (and, in most cases, (ii) as well). One can also consider dropping (ii) alone. Doing so means that one can be a structuralist and still use extensional logic. The downside, however, is that it invites the question: given that there is not just one 2-element group, *how many are there*? The set-theoretic foundationalist has an easy answer: one for each two-element set. But if we are to give a non-substantialist answer, what could we say?

One possibility is to stay close to the substiantialist, and posit one "copy" of a structure for everything that instantiates it. This will mean that structures, rather than being *universals*, will be more like *tropes* in that every instance of a structure will be its own entity. Perhaps this would be the minimal way to

change classic structuralism in order to accommodate extensional identity. But it means that the mathematical objects in question will not be independent, but necessarily tied to the things that have them.

A different way out, which comes closer to the intensionalist paradigm, would be to deny structures objecthood, and thus identity conditions, altogether. However, doing so will make it hard to take account of actual mathematics, which does treat groups, categories, and other mathematical entities as objects.

For myself, I suspect that all three propositions might be wrong, or at least not good assumptions to make. While one could possibly create a theory of structure which is wide enough to cover all mathematical objects, it may still be useful to be able to differentiate between an object and its structure in some cases, so we should not assume (i). And while extensional identity has been the norm in 20th century mathematics, much good mathematics is done by logicians and computer scientists without that assumption, so we should not make that one either.

If we deny (i) or (iii), it is possible to hold (ii) to be true. But this should be interpreted as saying that we can give an explication of the notion of structure on which (ii) holds, rather than that any explication *has* to satisfy it. A category-theoretic way to approach the problem would be to define a category **Str** of all (small) structurestogether with functors from each other (small) category to **Str** which are interpreted as assigning each object a structure. (ii) then reduces to the question of whether **Str** should be assumed to be skeletal or not. As such a theory does not yet exist, I do not wish to presuppose either answer.

Much of what I have argued here is based on a rather straightforward interpretation of categories and the isomorphisms in them. As I have noted frequently, the structure of an object is dependent on what category it is considered as an element of. This, in turn, means that assigning an object to a category gives us a structural interpretation of what it is. What is special about category theory is that it does so by comparison with other objects rather than by reference to any internal constitution.

Taking structure to be category-relative lets us identify objects of different structures, although we do not, of course, *have* to make such identifications. Rather, it allows us to work with structure as something *we* impose on the mathematical world, rather than something that objects come with already assigned. A simple example of a view of this type a principle would be a combination of Frege's insight that number is concept-relative with a non-Platonist understanding of concepts. In fact, cardinality is itself a type of structure, namely the type of structure expressed by isomorphisms in **Set**. So just as whether we treat something as one deck of cards or 52 cards is up to us, whether we treat an object as a group, a set, or a topological space, can be taken to be so as well.

There is something rather Kantian, or perhaps neo-Kantian, about this approach. Kant was, after all, one of the first modern structuralists (see, for example, [Kan10, p. 204]), although Poicaré's advocacy of the thesis now seems to be more well-known. Contemporary structuralists tend to follow much of the rest of contemporary philosophy in treating metaphysical and epistemological questions separately. In contrast, on the neo-Kantian view I have indicated here,

a category in the mathematical sense can be seen as a kind of schema though which we can understand and interpret the mathematical world, or in other words, a category in the Kantian sense. This would make MacLane's purloining of that word for his and Eilenberg's theory of natural equivalences a particularly happy one.

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