



# LUND UNIVERSITY

## A novel approach to balanced truncation of nonlinear systems

Nilsson, Oskar; Rantzer, Anders

2009

[Link to publication](#)

*Citation for published version (APA):*

Nilsson, O., & Rantzer, A. (2009). *A novel approach to balanced truncation of nonlinear systems*. Paper presented at European Control Conference, 2009, Budapest, Hungary.

*Total number of authors:*

2

### General rights

Unless other specific re-use rights are stated the following general rights apply:

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Read more about Creative commons licenses: <https://creativecommons.org/licenses/>

### Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

LUND UNIVERSITY

PO Box 117  
221 00 Lund  
+46 46-222 00 00



# A novel approach to balanced truncation of nonlinear systems

Oskar Nilsson\*, Anders Rantzer  
Department of Automatic Control  
Lund University  
Box 118, SE-221 00 Lund, Sweden  
{oskar,rantzer}@control.lth.se

**Abstract**—This paper addresses the problem of state reduction of nonlinear continuous-time systems. A novel method that relates to balanced truncation is presented and applied to examples. The method is computationally efficient and is applicable to relatively large systems.

## I. INTRODUCTION

Model reduction is an attractive tool in many contexts. For example, modeling is a mayor undertaking in a model based development process. Models for various purposes yield different requirements on e.g. precision and simulation speed. A systematic method to reduce model complexity would be a very useful tool to aid this process. Further, the reduction method can also be applied to the controller in order to meet controller hardware constraints or to simplify closed-loop analysis.

Model reduction of linear systems is a mature research topic and well-known methods featuring error bounds and preserved stability are available. However, in practise, one is often confronted with nonlinear systems and model reduction for this model class is so far an open research problem.

In this paper, a new method for simplification of nonlinear input-output models is outlined. The method relates to balanced truncation and uses a state transformation followed by truncation of some states. It's applicability is shown through examples.

## II. MODEL REDUCTION METHODS FOR NONLINEAR SYSTEMS

Model reduction of nonlinear systems is a research area under heavy development. The currently available methods can be divided into the following categories.

### A. Heuristic methods

Probably the most common way to simplify nonlinear models is through heuristic methods. For example, indirect model reduction is performed in all modeling-work when complexity is chosen to match the intended model purpose. There are three common ways to reduce complexity:

- To discard effects that by intuition or experience have a relatively weak impact on the dynamics of interest.
- Separation of time-scales and replacing relatively fast dynamics with static gains.

- Averaging several effects into one pseudo-effect.

All three approaches require great knowledge and intuition of the modeled object. However, attempts to perform these simplification steps in a systematic automatized manner has been investigated, see for example [1]. The second mentioned method can be applied in a more formal manner, it is commonly called the *singular perturbation method*, see [2].

### B. Linear methods

Here, a linearization, around an equilibrium or trajectory, is made followed by the application of some linear model reduction method. The obvious downside of this procedure is that the end result will be a linear model that can only be expected to perform well in a region close to the mentioned equilibrium or trajectory. Further, the size of this region depends on how nonlinear the original system is.

### C. Balancing of nonlinear systems

Balanced truncation is a popular method for model reduction of linear systems introduced in [3]. Recent research has extended this method to also cover the nonlinear case, see [4], its discrete-time counterpart in [5] and for differential-algebraic systems in [6]. Here, a balancing nonlinear coordinate-change is applied followed by truncation of states. The method has strong mathematical support but due to the required numerical effort only models with very moderate size have so far been considered.

### D. Pseudo-linear methods

These methods try to extend ideas of reduction of linear systems to the nonlinear case. In similarity to the method mentioned in Section II-C they apply a coordinate-change followed by truncation, however here the coordinate change is linear. This restriction to linear subspaces makes applicability to large systems possible. The main difference between the following methods is how the coordinate-change is found.

A very commonly used method for nonlinear model reduction is the so called *Proper Orthogonal Decomposition* method, introduced in [7], [8]. Here principal component analysis is performed on state-space data and the subspace that captures the majority of the variance is chosen. A common application is discretized partial differential equations, see [9]. The standard version of this method does not take any output-signal into consideration and can therefore be disadvantageous for control purposes.

\*O. Nilsson is financially supported by Toyota Motor Corporation and Sweden Japan Foundation

A recent contribution is found in [10], the so called *empirical gramian* approach extends ideas from *balanced truncation* of linear systems to the nonlinear case. Here state-space data are collected while impulse input-signals in different directions are applied. The data is then used to estimate a constant controllability gramian matrix. Similarly, a constant observability gramian matrix is constructed from simulation data generated by different initial values distributed on the unit sphere.

In [11] the so called *Trajectory Piecewise-Linear Approach* is presented. The method applies linear methods on linearizations distributed over one or several trajectories. Here the main focus is not only on reducing the number of states but also improving simulation speed.

The method introduced in this paper also belongs to this class of reduction methods where a linear coordinate change is used. Further, as in [10] it applies the notion of gramians and in similarity with [11], linearizations distributed over trajectories are used.

### III. PRELIMINARIES

The method presented in this paper is based on theory concerning linear time-varying systems. Consider the linear continuous-time time-varying system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \quad t \in [0, t_f],$$

where  $x$  is the state-vector,  $u$  the input-signal and  $y$  the output-signal. Further,  $A$ ,  $B$ ,  $C$  and  $D$  are time-varying matrices of appropriate dimensions. As in [4] the notion of so called energy functions is used. The *controllability energy function* is the amount of energy required in the input-signal to reach a specific state. In the linear-time varying case this can be stated as the optimal control problem

$$L_c(x_0, t) = \min_{\substack{u \in L_2(0, t) \\ x(0)=0 \\ x(t)=x_0}} \frac{1}{2} \int_0^t \|u(\tau)\|^2 d\tau. \quad (1)$$

That is,  $L_c(x_0, t)$  is the minimal amount of energy in  $u$  required to reach a certain state  $x_0$  at time  $t$ , starting from the zero initial state.

Further, the *observability energy function* determines the energy induced in the output, given a certain initial state and a zero input-signal. In this case it can be stated as

$$L_o(x_0, t) = \frac{1}{2} \int_t^{t_f} \|y(\tau)\|^2 d\tau, \quad x(t) = x_0, \quad u \equiv 0 \quad (2)$$

That is, the amount of energy an initial state  $x_0$  at time  $t$  induces in the output-signal over the time-interval  $[t, t_f]$ . The concept of these energy function is illustrated in Fig. 1. The usefulness of these functions for model reduction is clear. If a large amount of energy is required to reach a certain state and if the same state yields a small output energy, this state is unimportant for the input-output behaviour of the system.

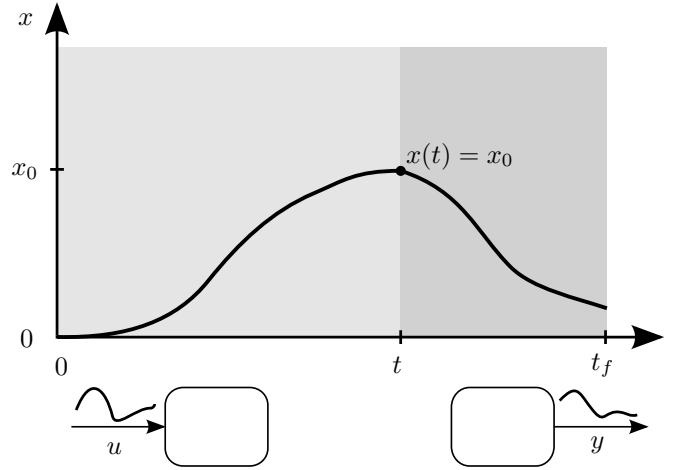


Fig. 1. Visualization of the energy functions. The left part illustrates the minimal input energy required to reach  $x_0$  at time  $t$ . In the right part the initial state  $x_0$  yields the mentioned output energy while the control signal is zero.

The energy functions can be determined through the following Lyapunov equations

$$\begin{aligned} \dot{P}(t) &= A(t)P(t) + P(t)A^T(t) + B(t)B^T(t) \\ \dot{Q}(t) &= -Q(t)A(t) - A^T(t)Q(t) - C^T(t)C(t) \end{aligned}$$

with  $t \in [0, t_f]$  and the boundary conditions  $P(0) = 0$  and  $Q(t_f) = 0$ . The matrices  $P$  and  $Q$  are commonly called the controllability gramian and observability gramian, respectively. Further, the solutions to (1) and (2) can be written as the quadratic forms

$$L_c(x_0, t) = \frac{1}{2} x_0^T P^{-1}(t) x_0, \quad L_o(x_0, t) = \frac{1}{2} x_0^T Q(t) x_0$$

The gramians  $P$  and  $Q$ , and their analogues for other system classes, are central to many model reduction methods. They show how strongly states are connected to the input and output and thereby supplies essential information of which state-subspace is of most significance.

### IV. METHOD DESCRIPTION

Let the system to be reduced have the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= g(x, u) \end{aligned} \quad (3)$$

where  $u \in \mathbf{R}^l$ ,  $x \in \mathbf{R}^n$  and  $y \in \mathbf{R}^m$ . To find states that are redundant or that have small importance for the input-output relationship, linearizations of the system dynamics will be used. Local importance of states would be revealed if one linearizes the system around a stationary point. A combination of several linearization points could then indicate which the important states are in the nonlinear system. However, some states may only have an active role during transient behaviour, which will later be commented in Example 3. Instead, linearization around a trajectory will be used as a tool to find an approximate low-order model.

Recall the theory concerning linear time-varying systems presented in the prior section. The time-varying gramians give information about state importance even in transient regions of state-space. These gramians can be computed in the neighborhood of simulated trajectories using linearization of the system dynamics. When there exists a linear coordinate transformation that disconnects some states from the input-output relationship, this will be revealed in those localized gramians.

The choice of training trajectory, around which linearization is made, is an important aspect of the reduction procedure. The corresponding training input should be chosen as a typical input-signal, which is rich enough to excite all dynamics important to the intended model use.

One possible scenario is that a state is nearly constant but non-zero, it might then be replaced by a constant value without losing much accuracy. However, the method yields a linear coordinate change (not an affine one), which is followed by truncation of states. Therefore, a preconditioning coordinate change that shifts the states so that their mean value over the training trajectory is zero should be applied to (3). Also, if the model is equipped with multiple input- and output-signals they should be scaled so that they possess the same amplitude. This scaling also leaves room for the user to specify how important he/she finds the accuracy of the different input- and output-signals.

The following sections explain the main steps involved in the method.

#### A. Linearization along trajectory

The first step is to choose a so called training input-signal. This is an important step of the method of which the performance is highly dependent. As a general rule the input should be chosen to obey physical restrictions on the signal and to excite all relevant dynamics. To find such a signal might be a challenging task. However, one could also see this as an advantage of the method. If the reduced model is only going to be used for some restricted purposes, the model could probably be reduced to a greater extent. Through the choice of training input the user can show which behaviour is relevant for the reduced system to reproduce. Hence, the signal should be chosen corresponding to realistic usage of the model.

The first step of the procedure is to simulate the system over the time-interval  $t = [0, t_f]$  with the training input-signal. The system is then linearized along the state-trajectory the training input gave rise to. The result is a time-varying linear system

$$\begin{aligned}\Delta\dot{x}(t) &= A(t)\Delta x(t) + B(t)\Delta u(t) \\ \Delta y(t) &= C(t)\Delta x(t) + D(t)\Delta u(t)\end{aligned}\quad t \in [0, t_f]$$

where  $\Delta u$ ,  $\Delta x$  and  $\Delta y$  denote deviations from the nominal trajectories. Further,  $A$ ,  $B$ ,  $C$  and  $D$  are time-varying

matrices defined by

$$\begin{aligned}A(t) &= \frac{\partial f}{\partial x}(x(t), u(t)) & B(t) &= \frac{\partial f}{\partial u}(x(t), u(t)) \\ C(t) &= \frac{\partial g}{\partial x}(x(t), u(t)) & D(t) &= \frac{\partial g}{\partial u}(x(t), u(t))\end{aligned}$$

#### B. Compute the time-varying gramians

In similarity with balanced truncation the method uses the notion of gramians. As mentioned, for time-varying systems the controllability gramian can be computed through simulation of the differential equation

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) + B(t)B^T(t) \quad (4)$$

with  $P(0) = 0$ . Similarly, the observability gramian is determined by

$$\dot{Q}(t) = -Q(t)A(t) - A^T(t)Q(t) - C^T(t)C(t) \quad (5)$$

with the boundary condition  $Q(t_f) = 0$ . The controllability gramian  $P(t)$  reveals how large deviation in input-signal is needed to perturb  $x(t)$ . If a certain state component is hard to perturb for all times, one can suspect that this state is in general hard to affect in the nonlinear system. Similarly,  $Q(t)$  shows how much the output-signal is affected if  $x(t)$  is perturbed. If the output-signal is weakly influenced by a certain state-perturbation, independently of when the perturbation is made, it can be suspected that this state-output connection is weak also for the nonlinear system.

#### C. Determine the average gramians

As mentioned, the gramians  $P(t)$  and  $Q(t)$  contain local information along the trajectory of how strongly states are connected to the input and output. In order to remove the time dependency and isolate the overall important states with a constant state-transformation, one could use the *average gramians*

$$\bar{P} = \frac{1}{t_f} \int_0^{t_f} P(\tau) d\tau \quad \bar{Q} = \frac{1}{t_f} \int_0^{t_f} Q(\tau) d\tau \quad (6)$$

These time-invariant matrices contain information of how strongly the states are connected to the input and output on average over the training trajectory. For example, if a certain linear state combination is unobservable from the output in all points of the trajectory, it will be revealed in  $\bar{Q}$ . Further, a rank deficiency of the matrix  $\bar{P}\bar{Q}$  indicates that some states are obsolete and can be truncated from the model without changing the input-output relationship.

#### D. Find balancing coordinate-change

This step is performed to extract the relevant state subspace using the information gathered in the average gramians. The chosen approach treats  $\bar{P}$  and  $\bar{Q}$  as if they belonged to a linear time-invariant system. By following the standard balanced truncation procedure for linear systems, a coordinate change  $z = Tx$  can be found, see [12], such that the

average gramians become equal and diagonal with decreasing diagonal elements.

$$T\bar{P}T^T = T^{-T}\bar{Q}T^{-1} = \bar{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

The diagonal elements  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$  corresponds to the Hankel singular values in balanced truncation of linear systems, where they show how important states are for the input-output relationship. Although no error-bound is available, in contrast to the linear case, these values will be used to determine which model order to choose for the reduced system.

#### E. Truncate states

Truncating states corresponding to relatively small singular values and keeping  $\hat{n}$  states is equivalent to removing rows and columns in  $T$  and  $T^{-1}$ , respectively.

$$\begin{aligned} T \in \mathbf{R}^{n \times n} &\Rightarrow T_l \in \mathbf{R}^{\hat{n} \times n} \\ T^{-1} \in \mathbf{R}^{n \times n} &\Rightarrow T_r \in \mathbf{R}^{n \times \hat{n}} \end{aligned} \quad (7)$$

Applying the truncated coordinate change to the original system formulation in (3) gives rise to the reduced order system

$$\begin{aligned} \dot{\hat{z}} &= T_l f(T_r \hat{z}, u) \\ \hat{y} &= g(T_r \hat{z}, u) \end{aligned} \quad (8)$$

where  $\hat{z} \in \mathbf{R}^{\hat{n}}$ . Deriving the reduced system through symbolic substitution in (8) is generally not an attractive option. Commonly, the original set of equations is sparse, i.e. all state equations do not involve all states. The sparsity is lost with a dense coordinate change and truncation of states. Therefore, the total computation time is not necessarily reduced for the right-hand-side functions, which can e.g. be seen in [13]. One possibility to redeem this is to extend the presented method with a piece-wise approximation of  $f$  and  $g$ , as mentioned in [11]. Additionally, for continuous-time systems not only evaluation time of the right-hand-side functions are of importance. Integration time does also depend on the choice of solver, numerical stiffness etc. These properties have not been considered in this work.

For further illustration, the method is demonstrated in the following examples.

### V. EXAMPLES

*Example 1 (Exact reduction):* Just as a demonstration, the method will here be applied to the following toy example. The nonlinear system

$$\begin{aligned} \dot{x}_1 &= -3x_1^3 + x_1^2x_2 + 2x_1x_2^2 - x_2^3 \\ \dot{x}_2 &= 2x_1^3 - 10x_1^2x_2 + 10x_1x_2^2 - 3x_2^3 - u \\ y &= 2x_1 - x_2 \end{aligned}$$

has exactly the same input-output relationship as the system

$$\dot{y} = -y^3 + u. \quad (9)$$

It is a challenge for any model reduction procedure to detect that a reduction like this is possible. A general methodology for such problems has been presented, but first a simple proof of the equivalence in this particular case is given.

Note that the system can be rewritten as

$$\begin{aligned} \dot{x}_1 &= -(2x_1 - x_2)^2x_1 + (x_1 - x_2)^3 \\ \dot{x}_2 &= -(2x_1 - x_2)^2x_2 + 2(x_1 - x_2)^3 - u \\ y &= 2x_1 - x_2 \end{aligned}$$

With the new variables  $z_1 = 2x_1 - x_2$ ,  $z_2 = x_2 - x_1$ , this means that

$$\begin{aligned} \dot{z}_1 &= -z_1^3 + u \\ \dot{z}_2 &= -z_1^2z_2 - z_2^3 - u \\ y &= z_1 \end{aligned}$$

In particular, the state  $z_2$  does not appear in the output and does not affect  $z_1$ . Hence, it can be truncated and (9) holds.

In the example, a linear coordinate transformation followed by state truncation gave a simplified model without approximation error. The goal of the described method is to provide a systematic way to find such transformations whenever they exist and otherwise to find good approximations.

Now the method will be applied to the original system description. Simulating the system along various trajectories and computing the observability gramian according to the differential equation

$$\dot{Q}(t) = -Q(t)A(t) - A^T(t)Q(t) - C^T(t)C(t)$$

with  $Q(t_f) = 0$  one observes that the rank of  $Q(t)$  never exceeds one for any trajectory. In particular,  $\bar{Q}$  and  $\bar{P}\bar{Q}$  are singular and one state can be truncated without affecting the input-output relationship. For demonstration, the nonlinear system was simulated with a certain input-signal. Following the method the average gramians were determined to

$$\bar{P} = \begin{bmatrix} 0.0018 & -0.0145 \\ -0.0145 & 0.2936 \end{bmatrix} \quad \bar{Q} = \begin{bmatrix} 1.3058 & -0.6529 \\ -0.6529 & 0.3264 \end{bmatrix}$$

and the matrix  $\bar{Q}$  is, as expected, singular<sup>1</sup>. The corresponding coordinate change  $z = Tx$  is then determined according to the standard balanced truncation method. Further, the Hankel singular values become in this case  $\sigma_1 = 0.3422$  and  $\sigma_2$  is very close to zero (not exactly due to numerical reasons). In accordance with the size of  $\sigma_2$ ,  $z_2$  is truncated and substitution according to (8) yields the nonlinear system

$$\begin{aligned} \dot{z}_1 &= -1.23z_1^3 - 0.901u \\ y &= -1.11z_1 \end{aligned}$$

which is equivalent to  $\dot{y} = -y^3 + u$ .

<sup>1</sup>using a larger numerical precision of  $\bar{P}$  and  $\bar{Q}$  than printed here

*Example 2 (A seven-state system):* The procedure can be applied to larger examples and also when loss-less truncation is not possible. Consider the seven-state system

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + u \\ \dot{x}_2 &= -x_2^3 - x_1^2 x_2 + 3x_1 x_2^2 - u \\ \dot{x}_3 &= -x_3^3 + x_5 + u \\ \dot{x}_4 &= -x_4^3 + x_1 - x_2 + x_3 + 2u \\ \dot{x}_5 &= x_1 x_2 x_3 - x_5^3 + u \\ \dot{x}_6 &= x_5 - x_6^3 - x_5^3 + 2u \\ \dot{x}_7 &= -2x_6^3 + 2x_5 - x_7 - x_5^3 + 4u \\ y &= x_1 - x_2^2 + x_3 + x_4 x_3 + x_5 - 2x_6 + 2x_7\end{aligned}$$

Following the described procedure, the system is linearized along a simulated training trajectory. As mentioned, the training input-signal should be chosen to reflect intended model use. However, this system lacks physical interpretation and just as an example the input-signal was chosen as a 10Hz square-wave signal with amplitude one. Again, following the procedure, the gramians are calculated according to (4) and (5). Further, the balancing coordinate change  $T$  is computed and the Hankel singular values are shown in Fig. 2. The relative size of these values indicate the importance of the new states for the input-output relationship.

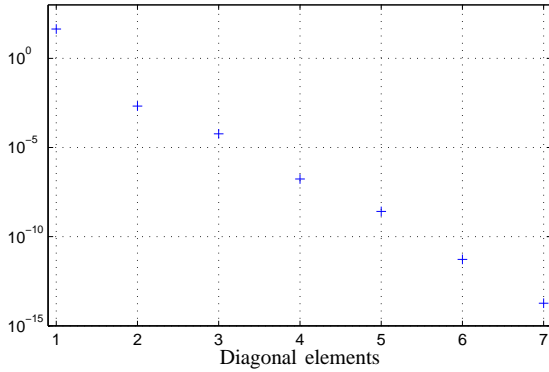


Fig. 2. Hankel singular values for the system in Example 2

If, for example, the nonlinear system is truncated to one state the reduced system becomes

$$\begin{aligned}\dot{z}_1 &= -0.492z_1 - 0.0879z_1^3 + 5.08u \\ \tilde{y} &= 1.34z_1 + 0.0792z_1^2\end{aligned}$$

A comparison achieved by simulating the original and reduced system with the same input signal  $u(t)$  can be seen in Fig. 3. The input-signal is different from the one used for model reduction and has been chosen to be the sum of a sinusoidal and a square-wave signal.

*Example 3 (A mass-spring-damper system):* In this example, the method is applied to a two-dimensional multiple-input multiple-output mass-spring-damper system. Fig. 4 shows six masses connected with springs and dampers. The input-signal is an external force, in horizontal and vertical

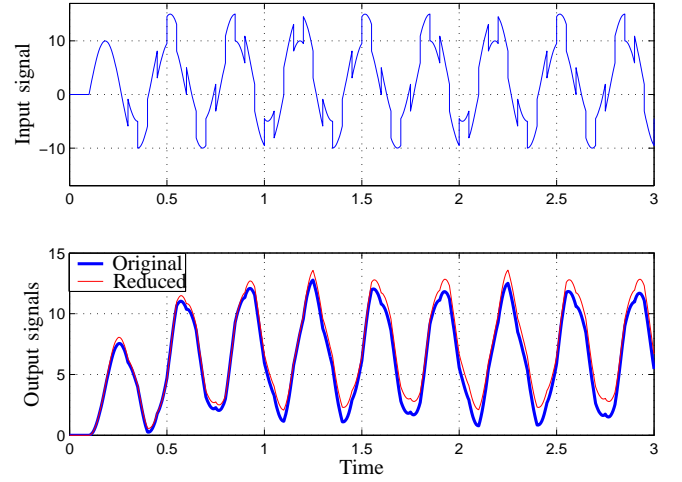


Fig. 3. Validation results for the system in Example 2

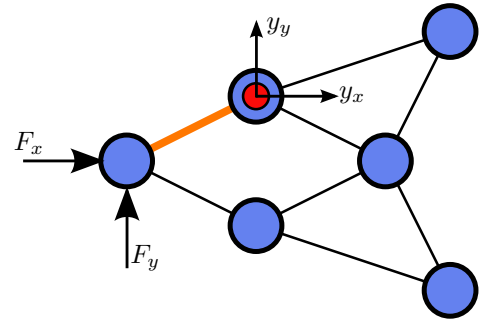


Fig. 4. Mass-spring-damper system in Example 3. Forces on the left mass as input-signals and position of the marked mass as output.

direction, on the leftmost mass. The output-signal is the position coordinates of the top middle mass.

A thin line in the figure represents a linear spring-damper with an unforced length  $l_0$  according to the figure. The masses, except the two rightmost ones, are also connected to the ground with linear spring-dampers.

The motion equations for each mass consist of four differential equations

$$\begin{aligned}\dot{p}_x &= v_x & \dot{p}_y &= v_y \\ \dot{v}_x &= \frac{1}{M} \sum_i F_{x,i} & \dot{v}_y &= \frac{1}{M} \sum_i F_{y,i}\end{aligned}$$

where  $p_x$  and  $p_y$  are the position coordinates with the corresponding velocities  $v_x$  and  $v_y$ . The mass is denoted  $M$  and the forces  $F_{x,i}$  and  $F_{y,i}$  are the forces in horizontal and vertical direction inflicted by spring-damper  $i$

$$\begin{aligned}F_{x,i} &= \left( K(l_i - l_{0i}) + D \frac{d}{dt}(l_i) \right) \cos \theta_i \\ F_{y,i} &= \left( K(l_i - l_{0i}) + D \frac{d}{dt}(l_i) \right) \sin \theta_i\end{aligned}$$

Here  $l_i$  is the length of spring-damper  $i$ ,  $D$  the damping coefficient and  $K$  the spring coefficient. In this example all coefficients have been set to one,  $M = K = D = 1$ . Further, the angle  $\theta_i$  is the angle of the spring-damper. Here, only

small angle perturbations are considered and  $\theta_i$  is therefore assumed to be constant.

The thick line is a nonlinear damper that gives a force proportional to the deformation rate to the power of three,

$$F_{x,i} = D \left( \frac{d}{dt}(l_i) \right)^3 \cos \theta_i \quad F_{y,i} = D \left( \frac{d}{dt}(l_i) \right)^3 \sin \theta_i$$

Linearization of the model around any stationary point would neglect this nonlinear damper, it only affects the linearization during transient behaviour. In the case of the leftmost mass, the external forces also contribute to the equations.

The model has four states per mass, yielding a total of 24 states, and can be written on the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}$$

In this example the described method is compared to the Proper Orthogonal Decomposition method mentioned in Section II-D. The method can briefly be described in these three steps.

- 1) Simulate the nonlinear system

$$\dot{x} = f(x, u)$$

and collect snapshots of the state vector in a matrix

$$X = [x(t_0) \quad x(t_1) \quad \dots \quad x(t_N)] \in \mathbf{R}^{n \times N}$$

- 2) Factorize  $X$  with the singular value decomposition

$$U \Sigma V^T = X$$

- 3) Choose truncation level after size of singular values in  $\Sigma$ . Truncate  $U \in \mathbf{R}^{n \times n}$  to  $\hat{U} \in \mathbf{R}^{n \times \hat{n}}$  so that  $x \approx \hat{U} \hat{x}$  where  $\hat{x} \in \mathbf{R}^{\hat{n}}$ . Then the reduced model becomes

$$\begin{aligned} \dot{\hat{x}} &= \hat{U}^T f(\hat{U} \hat{x}, u) \\ \hat{y} &= g(\hat{U} \hat{x}, u) \end{aligned}$$

Reduction to 8 states is performed with both methods using the same training trajectory. A simulation result can be seen in Fig. 5. The input is different from the reduction training input. A better result is obtained with the described method, which partly is due to the fact that the Proper Orthogonal Decomposition method does not take the output function  $g(x, u)$  into consideration.

## VI. SUMMARY

A method for simplification of nonlinear input-output models has been outlined. The given procedure is focused on reducing the number of states using information obtained by linearization around trajectories.

The number of states is one factor contributing to simulation time and even though it does not necessarily diminish in the general case simulation time has been reduced in the presented examples.

No proofs concerning preserved stability or error bounds are presented. However, the methodology is closely tied to existing theory on error bounds and promising results are shown in form of examples and simulation data.

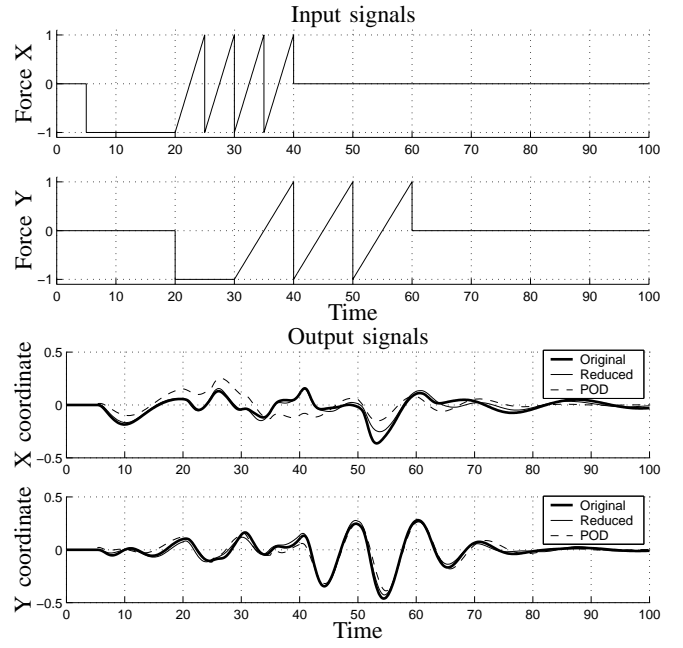


Fig. 5. Simulation results for example 3. Method comparison for model reduction from 24 to 8 states.

## REFERENCES

- [1] J. Broz, C. Clauss, T. Halfmann, P. Lang, R. Martin, and P. Schwarz, "Automated symbolic model reduction for mechatronical systems," in *Proc. of 2006 IEEE International Symposium on Computer-Aided Control Systems Design*, (Munich, Germany), pp. 408–415, IEEE, Oct. 2006.
- [2] H. K. Khalil, *Nonlinear systems*. Upper Saddle River, New Jersey: Prentice Hall, third ed., 2002.
- [3] B. Moore, "Principal component analysis in linear systems: controllability, observability, and model reduction," *IEEE Transactions on Automatic Control*, vol. 26, pp. 17–32, February 1981.
- [4] J. Scherpen and K. Fujimoto, "Nonlinear balanced realization based on singular value analysis of Hankel operators," in *Proceedings of the 42nd IEEE Conference on Decision & Control*, (Maui, USA), pp. 6072–6077, IEEE, December 2003.
- [5] J. Scherpen and K. Fujimoto, "Balancing and model reduction for discrete-time nonlinear systems based on Hankel singular value analysis," in *Proceedings of the 16th International Symposium on Mathematical Theory of Networks and Systems*, (Leuven, Belgium), July 2004.
- [6] J. Sjöberg, T. Glad, and K. Fujimoto, "Model reduction of nonlinear differential-algebraic equations," Apr. 2007.
- [7] M. Loève, "Fonctions aléatoires de second ordre," *Comptes Rendus Acad. Sci. Paris*, pp. 220–469, 1945.
- [8] K. Karhunen, "Zur spektraltheorie stochastischer prozesse," *Ann. Acad. Sci. Fennicae*, vol. Ser.A1, no. 34, 1946.
- [9] P. Astrid, *Reduction of process simulation models: a proper orthogonal decomposition approach*. PhD thesis, Technische Universiteit Eindhoven, Netherlands, 2004.
- [10] S. Lall, J. Marsden, and S. Glavaski, "A subspace approach to balanced truncation for model reduction of nonlinear control systems," *International Journal of Robust and Nonlinear Control*, vol. 12, pp. 519–535, 2002.
- [11] D. Vasilyev, M. J. Rewieński, and J. White, "Macromodel generation for biomems components using a stabilized balanced truncation plus trajectory piecewise-linear approach," *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, vol. 25, Feb. 2006.
- [12] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Upper Saddle River, New Jersey: Prentice Hall, 1998.
- [13] Z. Liu and J. Wagner, "Nonlinear model reduction for dynamic and automotive system descriptions," *Journal of Dynamic Systems, Measurement, and Control*, vol. 124, pp. 637–647, December 2002.