Coherent effects in single scattering and random errors in antenna technology

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Abstract

Lack of knowledge prevents us from exactly calculating the behavior of electromagnetic fields. We study two extremes in this respect: scattering against randomly distributed particles (no idea of the position or orientation of the scatterers), and random errors in antenna technology (small deviations from what we think are the proper parameters). Random variables are used to model our lack of knowledge, and far field expressions are studied. Using the concept of characteristic functions from probability theory, results for arbitrary probability distributions are obtained. We explain an anomaly in the forward scattering direction in single scattering theory, present simple formulas for the directivity, side lobe level, and beam efficiency for a general array antenna with random errors, and a simple formula for the scattering coefficient from a general frequency selective structure with random errors.

1 Introduction

This paper has two themes, single scattering of electromagnetic waves against a cluster of randomly distributed particles, and random errors in antenna technology. These seemingly distant subjects are joined by essentially the same analysis, and represent two extremes of our degree of knowledge. In the scattering case, it is assumed we know very little about the true positions of the scatterers, since they may be part of an aerosol where the thermal movement is constantly changing the particle distribution. In the antenna case, the uncertainty is small and due to errors in the manufacturing. In this case, it is desirable to estimate the resulting error in the antenna parameters.

The canonical problem is to study expressions of the form $|\sum_{n=1}^{N}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_n}\boldsymbol{F}_n|^2$, which represents the far field power pattern from a collection of sources with far field amplitude $\boldsymbol{F}_n$ and placed in $\boldsymbol{r}_n$, $n = 1, \ldots, N$. If these quantities were exactly given, we could also calculate the power pattern exactly. This is clearly not possible, and we must instead investigate what can be said in spite of our incomplete knowledge.

The problem posed is not a new one. Previous results in scattering theory are presented in many textbooks, among which we mention two primarily studying random scattering [4, 16]. In antenna theory, the widely referenced papers [11, 12] discuss error estimates similar to this paper, and related material is also found in [2, 14] and [6, Sec. 2–9]. Related ideas are also found in the vast literature treating scattering from rough surfaces, see [8] for a review.

Most of the above references make the common assumption that the randomness has a Gaussian probability distribution. Due to the central limit theorem, see for instance [5, pp. 177–181], this is an excellent approximation in cases where the randomness can be considered as a sum of many independent contributions. However, random errors appearing in manufacturing processes are often not Gaussian, which demonstrates the need of treating a more general case. In this paper, we show that it is possible to derive quite general results from an arbitrary probability distribution, by consistently using the concept of a characteristic function from probability theory.

There is an anomaly in the forward direction in scattering theory. An incident plane wave is always scattered in phase in the forward direction, no matter how randomly placed the scatterers are, implying the forward scattered field is proportional to the number of scatterers, N. This means the scattered intensity in the forward direction is proportional to N^2. However, in all the other directions, the scattered intensity is proportional only to N, since the contributions from each particle are mutually uncorrelated. We show that it is possible to find a continuous transition between these seemingly contradictory behaviors in a very narrow angle in the forward direction.

As for the error analysis in antenna technology, the main contribution of this paper is a general and explicit expression for the radiation pattern from an array antenna and a frequency selective structure, subject to errors in its constituents. It is seen that the main effect of errors in position and phase of the antenna elements is a decrease of the deterministic radiation pattern and uncorrelated contributions from each element. The errors due to uncertainty in the amplitude of the elements give an isotropic contribution to the radiation pattern. We also give simplified expressions for the antenna parameters directivity, side lobe level, and beam efficiency.

This paper is organized as follows. In Section 2 we give a brief review of the probability theory needed for this paper, and in Section 3 we define the far field approximation of an electromagnetic field. Sections 4 and 5 deal with the scattering and the antenna application, respectively. In Section 6 we give some numerical examples of the theory presented in the preceding sections, and the conclusions are given in Section 7.

2 Probabilistic background

A *random variable* is a common model of a quantity which we do not know the true value of. A real valued random variable X is a mapping from a sample space Γ to the real numbers, *i.e.*, $X : \Gamma \to \mathbb{R}$. Each element $\gamma \in \Gamma$ corresponds to a given realization of the physical problem, *i.e.*, when fixing γ the number $X(\gamma)$ gives the exact value of the quantity modeled by X in this particular realization.

The *ensemble average* of a function g of the random variable X, is defined as

$$\langle g(X)\rangle \equiv \int_\Gamma g(X(\gamma))\,\mathrm{d}P(\gamma) = \int_{\mathbb{R}} g(x) f_X(x)\,\mathrm{d}x. \tag{2.1}$$

The first equality is the definition of the ensemble average as the mean value of the random variable over the sample space Γ, where the probability measure $\mathrm{d}P$ satisfies $\int_\Gamma \mathrm{d}P = 1$. In the second equality we introduced the probability density $f_X \geq 0$ for the random variable X. Not all random variables have a density, but the conclusions in this paper are still valid even if the density can only be defined as a measure. The important difference between the two expressions in (2.1) is that the first is an integral defined on the sample space (which often has a very high dimension making the integral difficult to calculate), and the second is an integral over the range of the random variable, which is usually much easier to calculate.

When the function g above is the exponential function, we have a particularly interesting interpretation of the ensemble average. This is

$$\left\langle e^{ikX}\right\rangle = \int_{\mathbb{R}} e^{ikx} f_X(x)\,dx = \hat{f}_X(k), \tag{2.2}$$

i.e., , the Fourier transform of the probability density of the random variable. The function $\hat{f}_X$ is known in probability theory as the *characteristic function* of the random variable X [5, p. 100]. It is straight-forward to show the important properties $|\hat{f}_X(k)| \leq 1$, and $\hat{f}_X(0) = 1$.

We close this section by mentioning that two random variables X and Y are *independent* if and only if the relation

$$\langle g(X)h(Y)\rangle = \langle g(X)\rangle \langle h(Y)\rangle \tag{2.3}$$

holds for all measurable functions g and h [5, p. 62]. If it holds for a certain choice g_0 and h_0, the random variables $g_0(X)$ and $h_0(Y)$ are said to be *uncorrelated*.

3 Generation of electromagnetic waves

Electromagnetic waves are generated by time-varying currents. In this paper, we restrict ourselves to time harmonic electric currents; the results are easily extended to include magnetic currents. If the currents are contained within a bounded volume V, the electric field at large distances is given by the far field approximation,

$$\boldsymbol{E}(\boldsymbol{r})_{\text{far field}} = \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{kr}\boldsymbol{F}(\boldsymbol{k}), \tag{3.1}$$

where we assumed the time convention $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r})e^{-i\omega t}$. The far field amplitude $\boldsymbol{F}(\boldsymbol{k})$ is a function of the wave vector (propagation direction) $\boldsymbol{k}$ only, and is given by

$$\boldsymbol{F}(\boldsymbol{k}) = \frac{-i\eta}{4\pi}\boldsymbol{k}\times\left(\boldsymbol{k}\times\int_V \boldsymbol{J}(\boldsymbol{r}')e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}\,dV(\boldsymbol{r}')\right), \tag{3.2}$$

where η is the wave impedance of an isotropic medium surrounding the sources. From this expression we see that the far field is essentially the spatial Fourier transform of the current density $\boldsymbol{J}$. This current can be generated in many ways. In the scattering context, the current is induced by an incident field, and in the antenna context, it is given from a feeding network.

Often, the most easily measured quantity of an electromagnetic wave is the average power or intensity, especially in optics. The average intensity is

$$\langle I\rangle = \frac{\langle|\boldsymbol{E}|^2\rangle}{2\eta} = \frac{\eta k^2}{2(4\pi r)^2}\iint \langle \boldsymbol{J}_\perp(\boldsymbol{r}_1)\cdot\boldsymbol{J}_\perp^*(\boldsymbol{r}_2)\rangle\, e^{-i\boldsymbol{k}(\boldsymbol{r}_1-\boldsymbol{r}_2)}\,dV(\boldsymbol{r}_1)\,dV(\boldsymbol{r}_2), \tag{3.3}$$

where $\boldsymbol{J}_\perp = -k^{-2}\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{J})$ is the part of $\boldsymbol{J}$ orthogonal to $\boldsymbol{k}$. From this relation it is readily seen that (after a suitable change of variables) the average intensity is proportional to the Fourier transform of the function $\int \langle \boldsymbol{J}_\perp(\boldsymbol{x})\cdot\boldsymbol{J}_\perp^*(\boldsymbol{x}+\boldsymbol{r})\rangle\,dV(\boldsymbol{x})$,

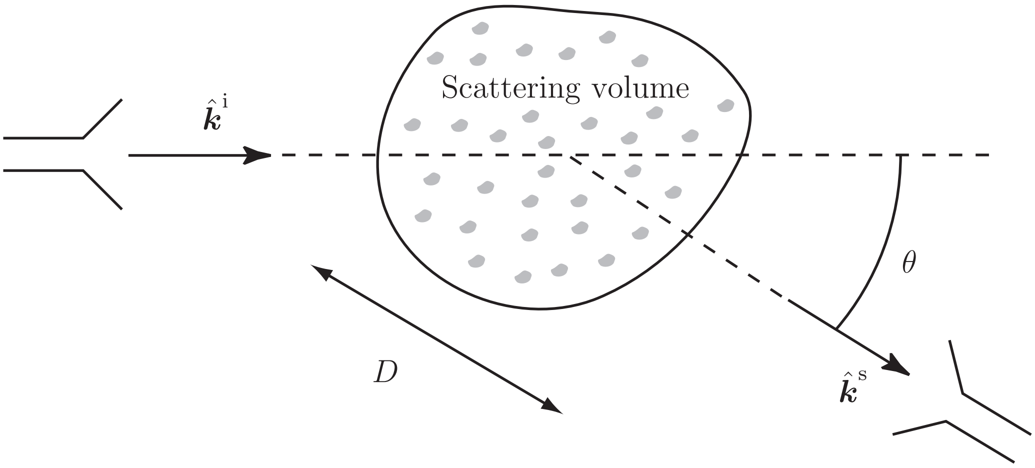

Figure 1: The bistatic scattering arrangement. The incident wave propagates in the direction $\hat{\boldsymbol{k}}^{\mathrm{i}}$, and the scattering is studied in direction $\hat{\boldsymbol{k}}^{\mathrm{s}}$. The angle between $\hat{\boldsymbol{k}}^{\mathrm{i}}$ and $\hat{\boldsymbol{k}}^{\mathrm{s}}$ is θ, given by $\hat{\boldsymbol{k}}^{\mathrm{i}} \cdot \hat{\boldsymbol{k}}^{\mathrm{s}} = \cos\theta$. The N individual scatterers are contained in a scattering volume with approximate linear size D.

which is the integral of the autocorrelation function of the current density. The importance of the correlation function in connection to dissipation in a medium has been thoroughly discussed in statistical physics (the fluctuation-dissipation theorem), see for instance [7, pp. 384–389] or [10, pp. 570–573]. In those cases, the interest is on small fluctuations from equilibrium due to thermal agitation. This paper is concerned with more large scale phenomena, in particular the case where the current is generated in N mutually disjoint volumes.

4 Single scattering approximation

We study the common bistatic arrangement of a scattering experiment as in Figure 1. In the single scattering approximation, we assume each scattering particle is subject to the incident field only, neglecting the fields scattered from the other particles. In this paper, we assume the scattering is weak enough not to cause a substantial decrease in the amplitude of the incident wave. For a given particle placed in the origin ($\boldsymbol{r} = \boldsymbol{0}$), it is possible to calculate a scattering matrix $\mathbf{S}(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}})$, which relates the scattered far field to an incident plane wave $\boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r}) = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}^{\mathrm{i}}\cdot\boldsymbol{r}}$ through the relation

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) = \frac{\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}^{\mathrm{s}}\cdot\boldsymbol{r}}}{kr}\boldsymbol{F}(\hat{\boldsymbol{k}}^{\mathrm{s}}) = \frac{\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}^{\mathrm{s}}\cdot\boldsymbol{r}}}{kr}\mathbf{S}(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \cdot \boldsymbol{E}_0, \tag{4.1}$$

where $\boldsymbol{F}$ is the far field amplitude of the scattered field, which depends on the scattering direction $\hat{\boldsymbol{k}}^{\mathrm{s}}$, the propagation direction $\hat{\boldsymbol{k}}^{\mathrm{i}}$, and the polarisation $\boldsymbol{E}_0$ of the incident wave. If the scatterer is not placed in the origin but rather in $\boldsymbol{r}'$, where $r' \ll r$, this corresponds to an additional phase $k(\hat{\boldsymbol{k}}^{\mathrm{i}} - \hat{\boldsymbol{k}}^{\mathrm{s}})\cdot\boldsymbol{r}' = k\boldsymbol{q}\cdot\boldsymbol{r}'$ in the scattered far field, where $\boldsymbol{q} = \mathbf{0}$ corresponds to forward scattering,

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) = \frac{\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}^{\mathrm{s}}\cdot\boldsymbol{r}}}{kr}\mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}'}\mathrm{S}(\hat{\boldsymbol{k}}^{\mathrm{s}},\hat{\boldsymbol{k}}^{\mathrm{i}})\cdot\boldsymbol{E}_0. \tag{4.2}$$

We now study the intensity $I = |\boldsymbol{E}|^2/2\eta$ of the scattered field. The square of the electric field scattered from N individual scatterers in the single scattering approximation is

$$|\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r})|^2 = \left|\sum_{n=1}^{N}\frac{\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}^{\mathrm{s}}\cdot\boldsymbol{r}}}{kr}\mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}_n}\mathrm{S}_n(\hat{\boldsymbol{k}}^{\mathrm{s}},\hat{\boldsymbol{k}}^{\mathrm{i}})\cdot\boldsymbol{E}_0\right|^2 = \frac{1}{(kr)^2}\left|\sum_{n=1}^{N}\mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}_n}\mathrm{S}_n(\hat{\boldsymbol{k}}^{\mathrm{s}},\hat{\boldsymbol{k}}^{\mathrm{i}})\cdot\boldsymbol{E}_0\right|^2. \tag{4.3}$$

In practice, we cannot exactly know all the parameters involved in this calculation. For instance, when studying scattering of laser light from a turbulent gas, it is impossible to know the positions $\boldsymbol{r}_n$ of all the individual scatterers. It is also possible that we do not know the geometry or material parameters of the particles exactly, and thus cannot determine the scattering matrices $\mathbf{S}_n$. Apart from the statistical nature of our knowledge, the very procedure of measuring often introduces averaging in time and/or space.

Multiplying (4.3) with $(kr)^2$ and taking the ensemble average implies

$$\begin{aligned}
\left\langle (kr)^2|\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r})|^2\right\rangle &= \left\langle\left|\sum_{n=1}^{N}\mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}_n}\mathrm{S}_n(\hat{\boldsymbol{k}}^{\mathrm{s}},\hat{\boldsymbol{k}}^{\mathrm{i}})\cdot\boldsymbol{E}_0\right|^2\right\rangle \\
&= \left\langle\sum_{n=1}^{N}\sum_{n'=1}^{N}\mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot(\boldsymbol{r}_n-\boldsymbol{r}_{n'})}\boldsymbol{E}_0^*\cdot\mathrm{S}_{n'}^{\dagger}(\hat{\boldsymbol{k}}^{\mathrm{s}},\hat{\boldsymbol{k}}^{\mathrm{i}})\cdot\mathrm{S}_n(\hat{\boldsymbol{k}}^{\mathrm{s}},\hat{\boldsymbol{k}}^{\mathrm{i}})\cdot\boldsymbol{E}_0\right\rangle \\
&= \boldsymbol{E}_0^*\cdot\left[\sum_{n=1}^{N}\left\langle\mathrm{S}_n^{\dagger}(\hat{\boldsymbol{k}}^{\mathrm{s}},\hat{\boldsymbol{k}}^{\mathrm{i}})\cdot\mathrm{S}_n(\hat{\boldsymbol{k}}^{\mathrm{s}},\hat{\boldsymbol{k}}^{\mathrm{i}})\right\rangle\right. \\
&\quad\left.+\sum_{n=1}^{N}\sum_{\substack{n'=1\\n'\neq n}}^{N}\left\langle\mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}_n}\right\rangle\left\langle\mathrm{e}^{-\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}_{n'}}\right\rangle\left\langle\mathrm{S}_{n'}^{\dagger}(\hat{\boldsymbol{k}}^{\mathrm{s}},\hat{\boldsymbol{k}}^{\mathrm{i}})\right\rangle\cdot\left\langle\mathrm{S}_n(\hat{\boldsymbol{k}}^{\mathrm{s}},\hat{\boldsymbol{k}}^{\mathrm{i}})\right\rangle\right]\cdot\boldsymbol{E}_0,
\end{aligned} \tag{4.4}$$

where we assumed all the random variables $\boldsymbol{r}_n$ and $\mathbf{S}_n$, $n = 1,\ldots,N$, are mutually independent and separated the double sum in diagonal terms and cross terms. The notation $\mathbf{S}_n^{\dagger}$ stands for the conjugated transposed matrix of $\mathbf{S}_n$ (the Hermitian conjugate).

4.1 Identical scatterers

Assuming all the scatterers are assigned the same probability densities in position and scattering matrix, we can further simplify (4.4), since the various expectation values are independent of n and n'. In the following we adopt the convention to use the index 0 for a typical representative X_0 of a sequence of random variables

X_n, $n = 1, \dots, N$, where all the X_n have the same probability distribution. We introduce the notation

$$
\begin{aligned}
\left\langle \mathbf{S}_0^2(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \right\rangle &= \left\langle \mathbf{S}_n^{\dagger}(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \cdot \mathbf{S}_n(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \right\rangle, \\
\left\langle \mathbf{S}_0(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \right\rangle^2 &= \left\langle \mathbf{S}_{n'}^{\dagger}(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \right\rangle \cdot \left\langle \mathbf{S}_n(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \right\rangle, \\
\hat{f}_{\boldsymbol{r}_0}(k\boldsymbol{q}) &= \left\langle \mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}_n} \right\rangle,
\end{aligned} \tag{4.5}
$$

which is intuitive but slightly violates the use of an exponent for matrices. This allows us to write (4.4) as

$$
\left\langle (kr)^2 |\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r})|^2 \right\rangle = \boldsymbol{E}_0^* \cdot \left[N \left\langle \mathbf{S}_0^2(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \right\rangle + N(N-1) |\hat{f}_{\boldsymbol{r}_0}(k\boldsymbol{q})|^2 \left\langle \mathbf{S}_0(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \right\rangle^2 \right] \cdot \boldsymbol{E}_0. \tag{4.6}
$$

Observe that even if the particles are identical, their scattering matrices may not be identical, due to different orientations of non-spherical particles. In most cases, it is therefore expected that $\langle \mathbf{S}_0^2 \rangle \neq \langle \mathbf{S}_0 \rangle^2$. Both matrices are hermitian and positive semi-definite by construction.

The second term, proportional to $N(N-1)$, is often neglected. It is now clear that this is justified only when $|\hat{f}_{\boldsymbol{r}_0}(k\boldsymbol{q})|^2 \ll N$. But since $\hat{f}_{\boldsymbol{r}_0}(k\boldsymbol{q})$ is a characteristic function of a random variable, this can only be true away from the forward scattering direction, $\boldsymbol{q} = \hat{\boldsymbol{k}}^{\mathrm{i}} - \hat{\boldsymbol{k}}^{\mathrm{s}} = \boldsymbol{0}$, since $\hat{f}_{\boldsymbol{r}_0}(\boldsymbol{0}) = 1$. How fast the factor $|\hat{f}_{\boldsymbol{r}_0}(k\boldsymbol{q})|^2$ tends to zero depends on the statistics for the positions of the scatterers, but a rough estimate is given by the "uncertainty relation" as follows.

For a given Fourier transform pair $(f, \hat{f})$ we define the half-width (or standard deviation) in space and reciprocal space $W_{\boldsymbol{r}}$ and $W_{\boldsymbol{k}}$, respectively, as

$$
W_{\boldsymbol{r}}^2 = \frac{\int |\boldsymbol{r}|^2 |f(\boldsymbol{r})|^2 \, \mathrm{d}V(\boldsymbol{r})}{\int |f(\boldsymbol{r})|^2 \, \mathrm{d}V(\boldsymbol{r})}, \quad W_{\boldsymbol{k}}^2 = \frac{\int |\boldsymbol{k}|^2 |\hat{f}(\boldsymbol{k})|^2 \, \mathrm{d}V(\boldsymbol{k})}{\int |\hat{f}(\boldsymbol{k})|^2 \, \mathrm{d}V(\boldsymbol{k})}, \tag{4.7}
$$

with the result that $W_{\boldsymbol{r}} W_{\boldsymbol{k}} \geq 1/2$, see for instance [15, p. 314]. Equality is obtained for Gaussian distributions. It is clear that $k|\boldsymbol{q}|$ must be larger than at least $W_{\boldsymbol{k}} > 1/(2W_{\boldsymbol{r}}) \approx 1/D$, where D is the diameter of the scattering volume, before the term proportional to $N(N-1)$ can be neglected. With $|\boldsymbol{q}| = |\hat{\boldsymbol{k}}^{\mathrm{i}} - \hat{\boldsymbol{k}}^{\mathrm{s}}| = \sqrt{2(1-\cos\theta)}$, where θ is the angle between $\hat{\boldsymbol{k}}^{\mathrm{i}}$ and $\hat{\boldsymbol{k}}^{\mathrm{s}}$, a small angle approximation implies $|\boldsymbol{q}| \approx \theta$. The term proportional to $N(N-1)$ gives a significant contribution when

$$
\theta < \frac{1}{kD} = \frac{\lambda}{2\pi D}, \tag{4.8}
$$

which is a narrow angle if the scattering volume is several wavelengths. A more careful estimate for a particular case is given in Section 6.1.

4.2 The optical theorem

We close this section on single scattering by discussing a possible misinterpretation of the optical theorem. The optical theorem states that the total (or extinction) cross section is given by the scattering amplitude in the forward direction,

$$\sigma_{\mathrm{t}} = \frac{P_{\mathrm{a}} + P_{\mathrm{s}}}{|\boldsymbol{E}_{\mathrm{i}}|^2/2\eta} = \frac{4\pi}{k^2} \operatorname{Im} \left\{ \frac{\boldsymbol{E}_0^* \cdot \mathbf{S}(\hat{\boldsymbol{k}}^{\mathrm{i}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \cdot \boldsymbol{E}_0}{|\boldsymbol{E}_0|^2} \right\}, \tag{4.9}$$

where P_{a} is the total absorbed power, and $P_{\mathrm{s}} = r^2 \int |\boldsymbol{E}_{\mathrm{s}}|^2/2\eta \, \mathrm{d}\Omega(\hat{\boldsymbol{k}}^{\mathrm{s}})$ is the total scattered power. This is an exact relation, which holds for every realization. This means it also holds when taking the mean value on both sides. Using (4.6) to calculate $\langle P_{\mathrm{s}} \rangle$ and the fact that the absorbed power $\langle P_{\mathrm{a}} \rangle \geq 0$, the optical theorem leads to an interesting relationship for the scattering matrix,

$$\begin{aligned} \frac{1}{4\pi} \int \left[N \left\langle \mathbf{S}_0^2(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \right\rangle + N(N-1)|\hat{f}_{\boldsymbol{r}_0}(k\boldsymbol{q})|^2 \left\langle \mathbf{S}_0(\hat{\boldsymbol{k}}^{\mathrm{s}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \right\rangle^2 \right] \mathrm{d}\Omega(\hat{\boldsymbol{k}}^{\mathrm{s}}) \\ \leq N \operatorname{Im} \left\langle \mathbf{S}_0(\hat{\boldsymbol{k}}^{\mathrm{i}}, \hat{\boldsymbol{k}}^{\mathrm{i}}) \right\rangle \end{aligned} \tag{4.10}$$

with equality for lossless scatterers. The inequality is taken in the sense that it applies for all (hermitian) quadratic forms over the matrices. It seems this inequality can be broken simply by letting $N \to \infty$, which would make the second term in the integral arbitrarily large. However, a dense packing of scatterers also implies multiple scattering, which is neglected in the present formulation. In order to maintain the conditions necessary for single scattering, the scattering volume must be made large as N increases. This means the support of the function $\hat{f}_{\boldsymbol{r}_0}(k\boldsymbol{q})$, which is essentially the Fourier transform of the scattering volume, shrinks to a small neighborhood of the forward direction ($\boldsymbol{q} = \boldsymbol{0}$), and the integral remains bounded. This demonstrates the need for caution when applying the result (4.6).

5 Random errors in antenna technology

In this section we study random errors in a deterministic structure. We assume all quantities associated with the deterministic case can be computed, although this task may indeed be a challenge of its own.

5.1 Array antennas

An array antenna is composed of N more or less identical antenna elements, distributed in a given volume. Each element is driven by a current which may have a different phase for different elements. An illustration is given in Figure 2. The mean value of the far field intensity in direction $\boldsymbol{k}$ is then

$$\langle I_{\text{far field}} \rangle = \frac{\langle |\boldsymbol{E}_{\text{far field}}|^2 \rangle}{2\eta} = \frac{1}{2\eta(kr)^2} \left\langle \left| \sum_{n=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_n - \mathrm{i}\phi_n} \boldsymbol{F}_n(\boldsymbol{k}) \right|^2 \right\rangle, \tag{5.1}$$

where $\boldsymbol{F}_n(\boldsymbol{k})$ is the element far field amplitude for element n. The difference between this expression and the scattered intensities studied in Section 4, is that the

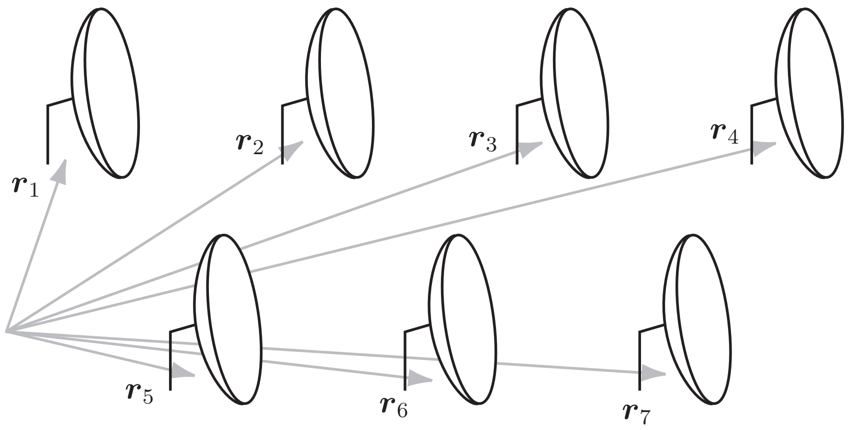

Figure 2: An example of an array antenna, consisting of 7 identical elements placed in $\boldsymbol{r}_n$, $n = 1, \ldots, 7$.

amplitudes $\boldsymbol{F}_n$ are not generated by an incident field, but are calculated from a set of given current densities $\boldsymbol{J}_n$ as

$$\boldsymbol{F}_n(\boldsymbol{k}) = \frac{-\mathrm{i}\eta}{4\pi}\boldsymbol{k} \times \left(\boldsymbol{k} \times \int \boldsymbol{J}_n(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}V(\boldsymbol{r})\right), \tag{5.2}$$

where the domains of $\boldsymbol{J}_n$, $n = 1, \ldots, N$, are mutually disjoint. In this section we treat the random variables as partly known, that is

$$\begin{aligned} \boldsymbol{r}_n &= \langle \boldsymbol{r}_n \rangle + \Delta\boldsymbol{r}_n \\ \phi_n &= \langle \phi_n \rangle + \Delta\phi_n \\ \boldsymbol{F}_n &= \langle \boldsymbol{F}_n \rangle + \Delta\boldsymbol{F}_n, \end{aligned} \tag{5.3}$$

where $\Delta\boldsymbol{r}_n$, $\Delta\phi_n$ and $\Delta\boldsymbol{F}_n$ have zero mean and small variances, and are assumed to have probability densities independent of n. Typical representatives of the random variables $(\Delta\boldsymbol{r}_n, \Delta\phi_n, \Delta\boldsymbol{F}_n)$, $n = 1, \ldots, N$, are denoted $(\Delta\boldsymbol{r}_0, \Delta\phi_0, \Delta\boldsymbol{F}_0)$. This arrangement corresponds to us having some knowledge of the design of the antenna (the mean values), and assumes the errors are equally probable in all elements. Expanding the mean value in diagonal and cross terms as in Section 4, we find

$$\begin{aligned} \left\langle \left| \sum_{n=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_n - \mathrm{i}\phi_n} \boldsymbol{F}_n(\boldsymbol{k}) \right|^2 \right\rangle &= \sum_{n=1}^{N} \left\langle |\boldsymbol{F}_n(\boldsymbol{k})|^2 \right\rangle \\ &+ \sum_{n=1}^{N} \sum_{\substack{n'=1 \\ n' \neq n}}^{N} \left\langle \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_n - \mathrm{i}\phi_n} \right\rangle \left\langle \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_{n'} + \mathrm{i}\phi_{n'}} \right\rangle \langle \boldsymbol{F}_n(\boldsymbol{k}) \rangle \cdot \langle \boldsymbol{F}^*_{n'}(\boldsymbol{k}) \rangle . \end{aligned} \tag{5.4}$$

Making use of the decomposition (5.3) and $\left\langle \mathrm{e}^{\mathrm{i}kX} \right\rangle = \hat{f}_X(k)$ for a random variable X, this becomes

$$
\begin{aligned}
\left\langle \left| \sum_{n=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_n - \mathrm{i}\phi_n} \boldsymbol{F}_n(\boldsymbol{k}) \right|^2 \right\rangle &= N \left\langle |\varDelta\boldsymbol{F}_0(\boldsymbol{k})|^2 \right\rangle + \sum_{n=1}^{N} |\left\langle \boldsymbol{F}_n(\boldsymbol{k}) \right\rangle|^2 \\
&+ |\hat{f}_{\varDelta\boldsymbol{r}_0}(\boldsymbol{k})|^2 |\hat{f}_{\varDelta\phi_0}(1)|^2 \sum_{n=1}^{N} \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\langle\boldsymbol{r}_n\rangle - \langle\boldsymbol{r}_{n'}\rangle) - \mathrm{i}(\langle\phi_n\rangle - \langle\phi_{n'}\rangle)} \left\langle \boldsymbol{F}_n(\boldsymbol{k}) \right\rangle \cdot \left\langle \boldsymbol{F}^*_{n'}(\boldsymbol{k}) \right\rangle \\
&= N \left\langle |\varDelta\boldsymbol{F}_0(\boldsymbol{k})|^2 \right\rangle + (1 - |\hat{f}(\boldsymbol{k})|^2) \sum_{n=1}^{N} |\left\langle \boldsymbol{F}_n(\boldsymbol{k}) \right\rangle|^2 \\
&+ |\hat{f}(\boldsymbol{k})|^2 \left| \sum_{n=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\langle\boldsymbol{r}_n\rangle - \mathrm{i}\langle\phi_n\rangle} \left\langle \boldsymbol{F}_n(\boldsymbol{k}) \right\rangle \right|^2, \qquad (5.5)
\end{aligned}
$$

where $\hat{f}(\boldsymbol{k}) = \hat{f}_{\varDelta\boldsymbol{r}_0}(\boldsymbol{k})\hat{f}_{\varDelta\phi_0}(1)$. This expression has the interesting feature that the last two terms only contain quantities which are known from the antenna designer's point of view, *i.e.*, the deterministic far field patterns, and locations and phases of the antenna elements. The last term is exactly the deterministic far field pattern, multiplied by the factor $|\hat{f}(\boldsymbol{k})|^2$. Since $\hat{f}$ is a characteristic function it satisfies $|\hat{f}| \leq 1$, and we see that an uncertainty in position $\boldsymbol{r}_n$, phase ϕ_n, and amplitude $\boldsymbol{F}_n$ of the different elements results in a decreased deterministic contribution (last term), and the first two terms correspond to incoherent, or diffuse, contributions.

The random variable $\varDelta\boldsymbol{F}_n$ is the Fourier transform of the correlation function of the current fluctuations in element n, as discussed at the end of Section 3. If the fluctuations are due to thermal agitation only, the term $N\left\langle |\varDelta\boldsymbol{F}_0(\boldsymbol{k})|^2 \right\rangle$ represents the black-body radiation and is proportional to the physical temperature of the antenna, see for instance [16, p. 147].

In Appendix A, the antenna parameters directivity D, side lobe level SLL, and beam efficiency BE, are derived as functions of the probability variables. When the errors are small and normally distributed, the following simplified formulas are obtained:

$$
D \leq D_{\mathrm{d}} - \left[\frac{N\delta_F^2}{k^2 U_{\mathrm{d0}}}(D_{\mathrm{d}} - 1) + (k^2\delta_{\boldsymbol{r}}^2 + \delta_\phi^2) \sum_{n=1}^{N} (D_{\mathrm{d}} - D_{\mathrm{d}}^{(n)}) \frac{U_{\mathrm{d0}}^{(n)}}{U_{\mathrm{d0}}} \right] \qquad (5.6)
$$

$$
\mathrm{SLL} \leq \mathrm{SLL}_{\mathrm{d}} + (1 - \mathrm{SLL}_{\mathrm{d}}) \left[\frac{N\delta_F^2}{k^2 U_{\mathrm{dmax}}} + (k^2\delta_{\boldsymbol{r}}^2 + \delta_\phi^2) \sum_{n=1}^{N} \frac{U_{\mathrm{dmax}}^{(n)}}{U_{\mathrm{dmax}}} \right] \qquad (5.7)
$$

$$
\mathrm{BE} \leq \mathrm{BE}_{\mathrm{d}} - \left[\frac{N\delta_F^2}{k^2 U_{\mathrm{d0}}} \left(\mathrm{BE}_{\mathrm{d}} - \frac{|\Omega_0|}{4\pi} \right) + (k^2\delta_{\boldsymbol{r}}^2 + \delta_\phi^2) \sum_{n=1}^{N} \left(\mathrm{BE}_{\mathrm{d}} - D_{\mathrm{d}}^{(n)} \frac{|\Omega_0|}{4\pi} \right) \frac{U_{\mathrm{d0}}^{(n)}}{U_{\mathrm{d0}}} \right], \qquad (5.8)
$$

where the index (d) indicates the deterministic values (or design values), and δ_F, $\delta_{\boldsymbol{r}}$ and δ_ϕ are the standard deviations of $\varDelta\boldsymbol{F}_0$, $\boldsymbol{r}_0$, and ϕ_0, respectively. The radiation

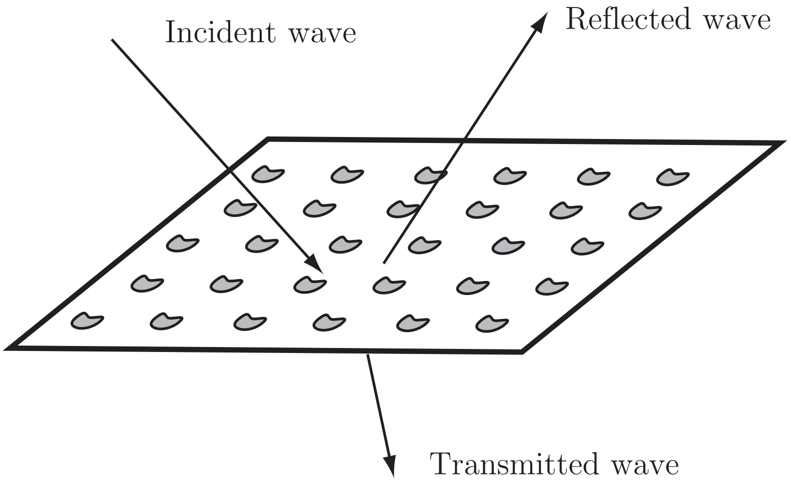

Figure 3: Example of a frequency selective structure (FSS). The periodicity of the metal surface creates a spatial filter which only transmits waves with certain wave numbers.

intensity U is defined as $\langle r^2|\boldsymbol{E}_{\text{far field}}|^2\rangle / 2\eta$, and the index 0 indicates the mean value over the unit sphere, *i.e.*, $U_0 = (4\pi)^{-1}\int U\,\mathrm{d}\Omega$. The index (n) denotes a quantity associated with element n, and $|\Omega_0|$ is the solid angle within which the main lobe is contained. Note the great resemblance between the different formulas, which is due to the fact that they are all derived from the same radiation pattern (5.5).

5.2 Frequency selective structures

The results obtained for array antennas can also be applied to frequency selective structures (FSS), used in radome applications. A frequency selective structure is a periodic pattern of scatterers, often consisting of metal patches or apertures in a metallic sheet as in Figure 3. Compared to the situation in Section 4, the scatterers are close and we must take multiple scattering in consideration. In this section, we study the effects of random displacements of the scatterers.

When the structure is illuminated by a plane incident wave $\boldsymbol{E}_{\mathrm{i}} = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}^{\mathrm{i}}\cdot\boldsymbol{r}}$, currents $\boldsymbol{J}_n$ are induced in each unit cell. With cell n positioned at $\boldsymbol{r}_n$, each current will inherit the phase $\phi_n = k\hat{\boldsymbol{k}}^{\mathrm{i}}\cdot\boldsymbol{r}_n$ as in the single scattering case in Section 4. The scattered far field is then written

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) = \frac{\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}^{\mathrm{s}}\cdot\boldsymbol{r}}}{kr}\sum_{n=1}^{N}\mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}_n}\boldsymbol{F}_n(k\hat{\boldsymbol{k}}^{\mathrm{s}}), \tag{5.9}$$

where $\boldsymbol{q} = \hat{\boldsymbol{k}}^{\mathrm{i}} - \hat{\boldsymbol{k}}^{\mathrm{s}}$, and the far field amplitude $\boldsymbol{F}_n$ is the far field amplitude of cell n calculated as if the cell were in the origin. We further make the approximation that a translation of the patch or aperture in cell n by $\Delta\boldsymbol{r}_n$ only changes its origin $\boldsymbol{r}_n$, and not the far field amplitude $\boldsymbol{F}_n$. The multiple scattering is included in the deterministic far field amplitudes $\boldsymbol{F}_n = \langle \boldsymbol{F}_n \rangle$, which are calculated from the unperturbed problem. The only changes necessary in (5.5) to accommodate the FSS situation are then $\hat{f}(\boldsymbol{k}) \to \hat{f}(k\boldsymbol{q})$ and $|\varDelta \boldsymbol{F}_n|^2 \to 0$:

$$\left\langle \left| \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}_n} \boldsymbol{F}_n(k\hat{\boldsymbol{k}}^{\mathrm{s}}) \right|^2 \right\rangle = \left(1 - |\hat{f}(k\boldsymbol{q})|^2\right) \sum_{n=1}^{N} \left| \left\langle \boldsymbol{F}_n(k\hat{\boldsymbol{k}}^{\mathrm{s}}) \right\rangle \right|^2 + |\hat{f}(k\boldsymbol{q})|^2 \left| \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\langle \boldsymbol{r}_n \rangle} \left\langle \boldsymbol{F}_n(k\hat{\boldsymbol{k}}^{\mathrm{s}}) \right\rangle \right|^2. \tag{5.10}$$

The last term is the deterministic scattering from the FSS, multiplied by a factor $|\hat{f}(k\boldsymbol{q})|^2 \leq 1$. The first term is the diffuse contribution, originating from the non-periodicity of the FSS. This term is responsible for the radiation in directions other than the grating lobes, and consists of the non-interacting radiation from the different cells.

A real FSS is often curved to conform with a given radome surface, but it is common to study the model problem of an infinite, plane, periodic, structure. All the far field amplitudes $\langle \boldsymbol{F}_n \rangle$ are then equal, denoted $\langle \boldsymbol{F}_0 \rangle$, and we sum over infinitely many amplitudes. This calls for a normalization, and we normalize (5.10) with the power incident on the structure. If the FSS consists of N unit cells with unit normal $\hat{\boldsymbol{n}}$ and cell area A, the incident power is $P_{\mathrm{i}} = |\boldsymbol{E}_0|^2/2\eta \cdot NA|\hat{\boldsymbol{k}}^{\mathrm{i}} \cdot \hat{\boldsymbol{n}}|$. The scattered power per unit solid angle is $U_{\mathrm{s}} = \langle r^2 |\boldsymbol{E}_{\mathrm{s}}|^2 \rangle /2\eta$, and the power scattering coefficient per unit solid angle is computed from (5.10) as

$$\frac{U_{\mathrm{s}}}{P_{\mathrm{i}}} = \lim_{N\to\infty} \frac{\langle r^2 |\boldsymbol{E}_{\mathrm{s}}|^2 \rangle}{|\boldsymbol{E}_0|^2 NA|\hat{\boldsymbol{k}}^{\mathrm{i}} \cdot \hat{\boldsymbol{n}}|} = \left[1 - |\hat{f}(k\boldsymbol{q}_\perp)|^2 + |\hat{f}(k\boldsymbol{q}_\perp)|^2 \lim_{N\to\infty} \frac{1}{N} \left| \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}k\boldsymbol{q}_\perp\cdot\langle \boldsymbol{r}_n \rangle} \right|^2 \right] \frac{\left| \left\langle \boldsymbol{F}_0(k\hat{\boldsymbol{k}}^{\mathrm{s}}) \right\rangle \right|^2}{|\boldsymbol{E}_0|^2 k^2 A |\hat{\boldsymbol{k}}^{\mathrm{i}} \cdot \hat{\boldsymbol{n}}|}, \tag{5.11}$$

Observe that we have taken explicit consideration to the fact that the error in position $\varDelta \boldsymbol{r}$ only occurs in the plane of the FSS, using the index $\perp$ to indicate vectors in that plane (orthogonal to the surface normal $\hat{\boldsymbol{n}}$).

Using the property $\lim_{N\to\infty} N^{-1} \sin^2(Nt/2)/\sin^2(t/2) = 2\pi\delta(t)$ of Fejér kernels, see for instance [1, p. 88], it is straightforward to show

$$\lim_{N\to\infty} \frac{1}{N} \left| \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}k\boldsymbol{q}_\perp\cdot\langle \boldsymbol{r}_n \rangle} \right|^2 = \frac{(2\pi)^2}{A} \sum_{m,n} \delta^{(2)}(k\boldsymbol{q}_\perp + m\boldsymbol{A} + n\boldsymbol{B}), \tag{5.12}$$

where $\delta^{(2)}$ is the two-dimensional Dirac delta distribution, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are basis vectors in the reciprocal lattice. With $\boldsymbol{a}$ and $\boldsymbol{b}$ being basis vectors in physical space, the reciprocal basis is defined as $\boldsymbol{A} = 2\pi\boldsymbol{a}/|\boldsymbol{a}|^2$ and $\boldsymbol{B} = 2\pi\boldsymbol{b}/|\boldsymbol{b}|^2$. The expression in (5.12) is an angular distribution with support in the grating directions only, *i.e.*, the deterministic array factor for an infinite periodic structure. The frequency selective surface is often designed so that only the delta distribution associated with $m = n = 0$ comes into play, corresponding to the specular directions given by $\boldsymbol{q}_\perp = \boldsymbol{0}$, that is, $\hat{\boldsymbol{k}}^{\rm s}_\perp = \hat{\boldsymbol{k}}^{\rm i}_\perp$.

We note that in the specular directions, the factor $|\hat{f}(k\boldsymbol{q}_\perp)|^2$ is exactly 1, corresponding to no attenuation of the deterministic contribution to the intensity. But since there is a positive contribution from the diffuse intensity, energy conservation is violated. This is due to the fact that the current is calculated through a perturbation analysis which does not take sufficient care of energy conservation. The expressions given in this section should be used to estimate the intensity in regions where the deterministic contribution is very small. For instance, when the deterministic calculations imply total transmission and zero reflection, the diffuse contribution shows there is still a small reflected field.

We conclude this section by giving the small error limit of the power scattering coefficient per unit solid angle. Assuming the errors $\varDelta\boldsymbol{r}_n$ are symmetrically distributed in the FSS plane with zero mean and variance $\delta_{\boldsymbol{r}}$, we have $\hat{f}(k\boldsymbol{q}_\perp) = 1-|k\delta_{\boldsymbol{r}}\boldsymbol{q}_\perp|^2/2+\mathrm{O}((k\delta_{\boldsymbol{r}})^4)$, independent of the distribution [13, p. 278]. This implies $|\hat{f}(k\boldsymbol{q}_\perp)|^2 = 1-|k\delta_{\boldsymbol{r}}\boldsymbol{q}_\perp|^2+\mathrm{O}((k\delta_{\boldsymbol{r}})^4)$, and we have

$$\frac{U_{\rm s}}{P_{\rm i}} = \left[|k\delta_{\boldsymbol{r}}\boldsymbol{q}_\perp|^2 + (1-|k\delta_{\boldsymbol{r}}\boldsymbol{q}_\perp|^2)\frac{(2\pi)^2}{A}\sum_{m,n}\delta^{(2)}(k\boldsymbol{q}_\perp + m\boldsymbol{A} + n\boldsymbol{B})\right]\frac{\left|\left\langle \boldsymbol{F}_0(k\hat{\boldsymbol{k}}^{\rm s})\right\rangle\right|^2}{|\boldsymbol{E}_0|^2k^2A|\hat{\boldsymbol{k}}^{\rm i}\cdot\hat{\boldsymbol{n}}|}. \tag{5.13}$$

Since $\delta^{(2)}(k\boldsymbol{q}_\perp) = k^{-2}\delta^{(2)}(\boldsymbol{q}_\perp)$, it is seen that upon integration over all scattering directions, the quotient between diffuse power and deterministic power grows as k^4, unless the angular dependence of the far field amplitude $\boldsymbol{F}_0(k\hat{\boldsymbol{k}}^{\rm s})$ varies too much.

6 Numerical examples

In this section we give a few numerical examples of the calculations presented in the previous sections.

6.1 Uniform distribution of identical scatterers in a cube

Take N identical, isotropic scatterers which are uniformly distributed within a cube of side $2R$, *i.e.*, $f(\boldsymbol{r}) = 1/(2R)^3$ for $\max(|x|,|y|,|z|) < R$ and zero elsewhere. The characteristic function is then

$$\begin{aligned}\hat{f}_{\boldsymbol{r}_0}(\boldsymbol{k}) &= \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f_{\boldsymbol{r}_0}(\boldsymbol{r})\,\mathrm{d}V(\boldsymbol{r})\\ &= \frac{1}{(2R)^3}\left(\int_{-R}^{R}\mathrm{e}^{\mathrm{i}k_x x}\,\mathrm{d}x\right)\left(\int_{-R}^{R}\mathrm{e}^{\mathrm{i}k_y y}\,\mathrm{d}x\right)\left(\int_{-R}^{R}\mathrm{e}^{\mathrm{i}k_z z}\,\mathrm{d}x\right)\\ &= \frac{\sin k_xR}{k_xR}\frac{\sin k_yR}{k_yR}\frac{\sin k_zR}{k_zR},\end{aligned} \tag{6.1}$$

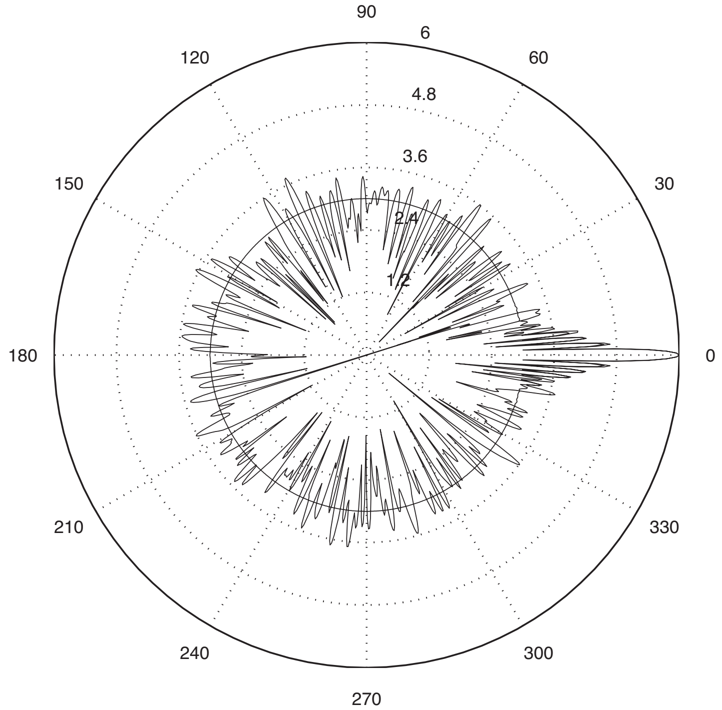

Figure 4: Plot of $\log_{10}\left|\sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}_n}\right|^2$ in the x-y plane, where $\boldsymbol{q} = \hat{\boldsymbol{k}}^{\mathrm{i}} - \hat{\boldsymbol{k}}^{\mathrm{s}}$ and $\hat{\boldsymbol{k}}^{\mathrm{i}} = \hat{\boldsymbol{x}}$. The scatterers are randomly distributed with uniform probability within a cube of side $2R$, and the parameters are $N = 1000$ and $kR = 100$, implying $\theta_{\mathrm{crit}} = N^{1/2}/kR = 0.056 = 18°$. The smooth line is the ensemble average of all realizations, *i.e.*, $N + N(N-1)|\hat{f}_{\boldsymbol{r}_0}(k\boldsymbol{q})|^2$.

and we have

$$\left\langle \left| \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}k\boldsymbol{q}\cdot\boldsymbol{r}_n} \right|^2 \right\rangle = N + N(N-1)\left(\frac{\sin kq_xR}{kq_xR}\right)^2 \left(\frac{\sin kq_yR}{kq_yR}\right)^2 \left(\frac{\sin kq_zR}{kq_zR}\right)^2. \quad (6.2)$$

From (4.8) we expect the second term to give a substantial contribution when $\theta < 1/(k2R)$. However, using the above expression we can make a better estimate. Near the forward direction $\hat{\boldsymbol{k}}^{\mathrm{i}} = \hat{\boldsymbol{x}}$ we have $\boldsymbol{q} \approx \theta\hat{\boldsymbol{y}}$, where $|\theta| \ll 1$, which means only one of the $\sin kqR/kqR$ factors above contribute to the damping of the second term, implying $N/(k\theta R)^2 \ll 1$. In this case, we would expect a substantial contribution from the second term when $|\theta| < N^{1/2}/kR = \theta_{\mathrm{crit}}$.

In Figure 4 is found a simulation of a given realization of this problem, along with a curve corresponding to the ensemble average. With the parameters $N = 1000$ and $kR = 100$, the critical angle is $\theta_{\mathrm{crit}} = 18°$, and it is seen that there are indeed some lobes close to the forward direction, approximately within this angle.

6.2 Random errors in a linear antenna array

A simple example of an array antenna is a linear array of identical dipole elements uniformly distributed along the z-axis, *i.e.*,

$$\begin{aligned} \langle \boldsymbol{F}_n(\boldsymbol{k}) \rangle &= F_0 \sin\theta \hat{\boldsymbol{\theta}} \\ \langle \boldsymbol{r}_n \rangle &= nd\hat{\boldsymbol{z}} \\ \langle \phi_n \rangle &= n\beta, \end{aligned} \tag{6.3}$$

where θ is the angle between the z-axis and $\boldsymbol{k}$. Assuming there is no error in amplitude, *i.e.*, $\varDelta \boldsymbol{F}_n = \boldsymbol{0}$, the expression (5.5) can then be explicitly calculated:

$$\begin{aligned} \left\langle \left| \sum_{n=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_n - \mathrm{i}\phi_n} \boldsymbol{F}_n(\boldsymbol{k}) \right|^2 \right\rangle &= (1 - |\hat{f}(\boldsymbol{k})|^2) N F_0^2 \sin^2\theta \\ &+ |\hat{f}(\boldsymbol{k})|^2 F_0^2 \sin^2\theta \left[\frac{\sin(N(kd\cos\theta + \beta)/2)}{\sin((kd\cos\theta + \beta)/2)} \right]^2, \end{aligned} \tag{6.4}$$

see for instance [3, p. 259]. We assume a spherically symmetric Gaussian probability distribution of the positions and neglect the variations in phase, to obtain $|\hat{f}(\boldsymbol{k})|^2 = \mathrm{e}^{-k^2\delta^2}$, where δ is the standard deviation of position. A plot of the deterministic and averaged radiation pattern for $N = 10$, $d = \lambda/2$, $\beta = 0$, and $k\delta = 2\pi \cdot 0.05$ is found in Figure 5. It is seen that the deep nulls in the deterministic radiation pattern are lifted mainly due to shifts of the nulls in different realizations.

We calculate the antenna parameters directivity, side lobe level, and beam efficiency, given in Appendix A, for position error $\delta_{\boldsymbol{r}}$ only, and plot the results in Figure 6. In this figure is also found a comparison on how good the simplified expressions (5.6), (5.7), and (5.8) are. It is seen that up to $k\delta_{\boldsymbol{r}} = 2\pi \cdot 0.1$ the formulas are accurate within 1 dB.

6.3 Frequency selective surface

A frequency selective surface can be made using a pattern of hexagonal rings as indicated in Figure 6.3. Poulsen presents the full geometry in [9], and has kindly supplied the data necessary to compute the deterministic far field amplitudes for this example. In the absence of grating lobes, equation (5.13) for the power scattering coefficient per unit solid angle becomes

$$\frac{U_{\mathrm{s}}}{P_{\mathrm{i}}} = \left[|k\delta_{\boldsymbol{r}}\boldsymbol{q}_\perp|^2 + \frac{(2\pi)^2}{A}\delta^{(2)}(k\boldsymbol{q}_\perp) \right] \frac{\left| \left\langle \boldsymbol{F}_0(k\hat{\boldsymbol{k}}^{\mathrm{s}}) \right\rangle \right|^2}{|\boldsymbol{E}_0|^2 k^2 A |\hat{\boldsymbol{k}}^{\mathrm{i}} \cdot \hat{\boldsymbol{n}}|}. \tag{6.5}$$

Integrating this expression over the top half sphere Ω_+, we obtain the fraction of power which is reflected, $P_{\mathrm{r}}/P_{\mathrm{i}} = \int_{\Omega_+} U_{\mathrm{s}}(\hat{\boldsymbol{k}}^{\mathrm{s}})\,\mathrm{d}\Omega(\hat{\boldsymbol{k}}^{\mathrm{s}})/P_{\mathrm{i}}$. This is plotted in Figure 6.3, where we also show the part of the reflected power corresponding to the diffuse intensity.

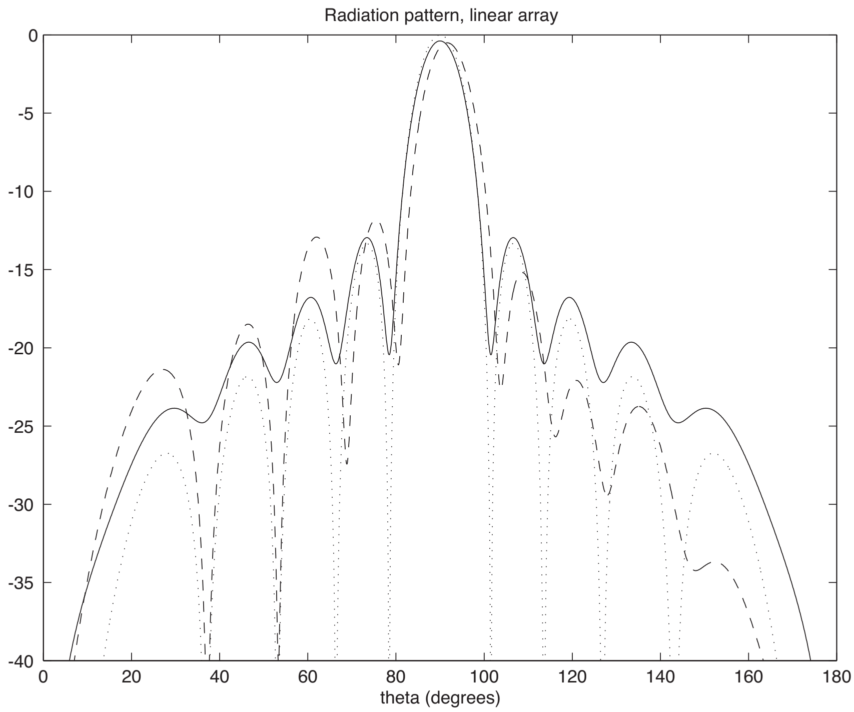

Figure 5: The radiation pattern from a linear array of 10 identical dipoles, uniformly spaced by $\lambda/2$, and position error $k\delta = 2\pi \cdot 0.05$. The dotted line is the deterministic radiation pattern, the dashed line is the pattern of a given realization, and the solid line is the ensemble averaged radiation pattern.

From Figure 6.3 it is seen that the diffuse part of the reflected power is very small up to the resonance frequency at 10 GHz, even though the average error in position is about 10 % of the cell size. At higher frequencies the diffuse power is more or less constant at −20 dB. The anomalies seen for frequencies higher than 17 GHz are due to the need of a more accurate calculation of the far field amplitudes $\boldsymbol{F}_0$ at these frequencies, only a few basis functions are used here.

7 Discussion and conclusions

This paper treats essentially two applications: single scattering against randomly distributed particles, and random errors in antenna technology. The common element is the calculation of expressions of the kind $\left\langle |\sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_n}\boldsymbol{F}_n|^2 \right\rangle$, where $\boldsymbol{r}_n$ and $\boldsymbol{F}_n$ are random variables. We have shown that it is possible to explicitly calculate these expressions, in terms of the characteristic function of the probability density and the deterministic part of the random variables.

The closed form of our results allows an explicit estimate of when the "$N(N-1)$"-term in single scattering theory cannot be neglected. The extra information which can be extracted if this contribution can be measured, is mainly concerned with

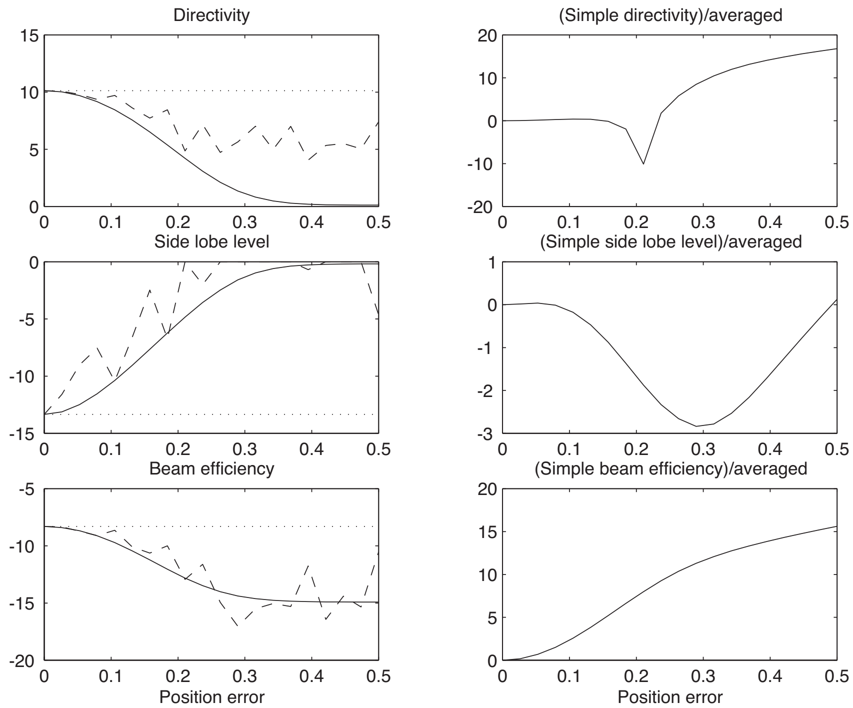

Figure 6: Left column: directivity, side lobe level, and beam efficiency as functions of position error only (dB units). The solid lines are for the exact calculation of the averaged parameters, the dotted are the design values, and the dashed are computed for a single realization for each position error. Right column: the ratio between simplified formulas (5.6), (5.7), and (5.8) and the exact calculation of the parameters (dB units). The position error is in fractions of a wavelength. Observe that the averaged parameters of the array antenna are essentially those of an isotropic antenna at position error $\delta_{\boldsymbol{r}} = \lambda/2$.

the shape of the scattering volume, since the lobes in the scattered power pattern centered round the forward direction is essentially the Fourier transform of the scattering volume. However, since the interesting contribution is near the forward direction, it is technically difficult to distinguish the scattered field from the incident field.

Random errors in antenna technology are treated in great generality in this paper, giving explicit estimates on the expected radiation pattern as well as several important antenna parameters when the antenna is subject to perturbations. The estimates are given in terms of the deterministic design values, and the errors in phase, amplitude and position of the antenna elements. We also give explicit error estimates on the behavior of a quite general frequency selective structure.

From the results in Section 6, we see that the simplified, and computationally effective, formulas (5.6), (5.7), and (5.8) give good result for errors up to roughly a tenth of a wavelength. It should be noted that even though the expressions were derived using a Gaussian probability distribution, they are actually still valid for a

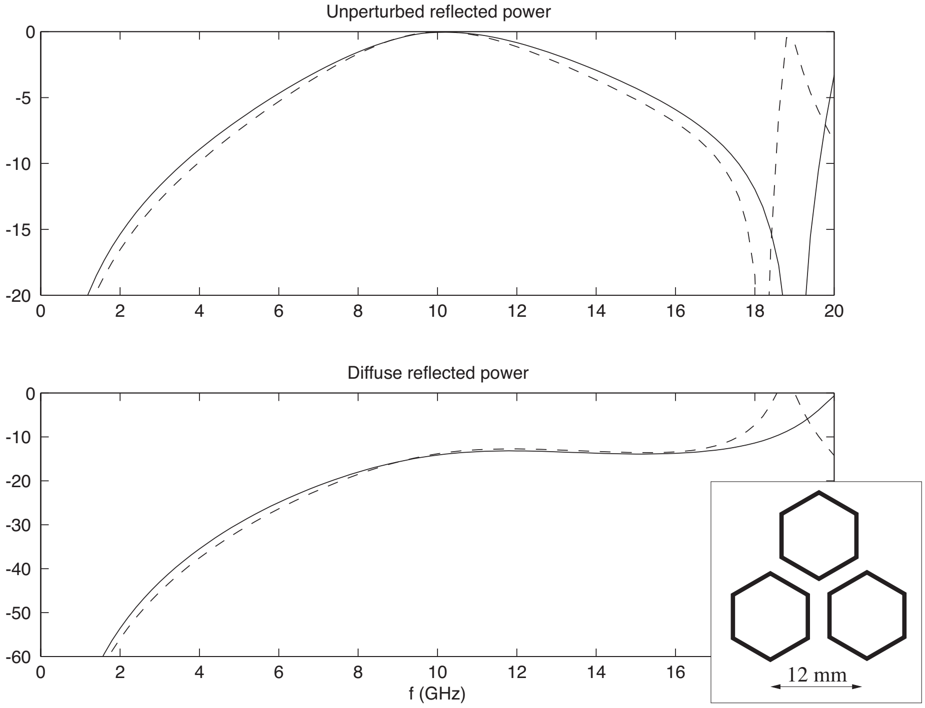

Figure 7: Simulated reflection from a frequency selective surface with random errors. The angle of incidence is given by $\theta = 30°$ and $\phi = 0°$, and the geometry of the unit cell is indicated. The solid and dashed lines correspond to TE and TM polarizations, respectively. The lattice has side 12 mm, and the standard deviation of position is $\delta_{\boldsymbol{r}} = 1\,\mathrm{mm}$. The upper diagram corresponds to the reflected power from an unperturbed FSS, *i.e.*, the second term inside the brackets in (6.5). The lower diagram corresponds to the diffuse part of the reflected power from a perturbed FSS, *i.e.*, the first term inside the brackets in (6.5).

general probability distribution which allows a truncation of the Taylor expansion of its characteristic function. We have chosen to exclude this technical derivation from this paper.

The results given in this paper should be of interest to a wide variety of scientists and engineers. In particular the error estimates for antenna technology are important when considering the amount of overdesign necessary to obtain specific design goals.

8 Acknowledgements

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Appendix A Calculation of antenna parameters

This appendix presents the somewhat technical derivations of three antenna parameters, when the array antenna is subject to errors in position, phase and amplitude of its elements. The aim is to find expressions for the perturbed antenna parameters as functions of the unperturbed parameters, *i.e.*, the deterministically calculated parameters. We give expressions which have a minimum of approximations, as well as simplified expressions which are easier to handle.

A.1 Directivity

The (maximum) directivity of an antenna is defined as [3, p. 39]

$$D = D_{\max} = \frac{U_{\max}}{U_0} = \frac{\max_{\boldsymbol{k}} U}{(4\pi)^{-1} \int U \, \mathrm{d}\Omega(\boldsymbol{k})}, \tag{A.1}$$

where $U = \langle r^2 |\boldsymbol{E}|^2 \rangle / 2\eta$ is the radiation intensity (radiated power per unit solid angle), and U_0 is the radiation intensity which would have been produced if the source had been isotropic. Recall that $\langle |\boldsymbol{E}|^2 \rangle$ and thereby U are proportional to the previously derived factor

$$\begin{aligned} \left\langle \left| \sum_{n=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_n - \mathrm{i}\phi_n} \boldsymbol{F}_n(\boldsymbol{k}) \right|^2 \right\rangle &= N \left\langle |\Delta \boldsymbol{F}_0(\boldsymbol{k})|^2 \right\rangle + (1 - |\hat{f}(\boldsymbol{k})|^2) \sum_{n=1}^{N} | \langle \boldsymbol{F}_n(\boldsymbol{k}) \rangle |^2 \\ &+ |\hat{f}(\boldsymbol{k})|^2 \left| \sum_{n=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\langle \boldsymbol{r}_n \rangle - \mathrm{i}\langle \phi_n \rangle} \langle \boldsymbol{F}_n(\boldsymbol{k}) \rangle \right|^2 . \end{aligned} \tag{A.2}$$

Denote the directivity and radiation intensity from element n by $D_\mathrm{d}^{(n)}$ and $U_\mathrm{d}^{(n)}$, respectively, where the index (d) stands for deterministic or design value. The directivity and radiation intensity of the entire array antenna when there are no perturbations (deterministic case) is denoted by D_d and U_d, respectively. Assume the errors are spherically symmetric, *i.e.*, $\Delta\boldsymbol{F}_0$ and $\hat{f}$ depend only on $|\boldsymbol{k}| = k$. The maximum radiation intensity $U_\mathrm{max} = DU_0$ is then

$$U_\mathrm{max} \leq k^{-2}N\left\langle|\Delta\boldsymbol{F}_0(k)|^2\right\rangle + (1-|\hat{f}(k)|^2)\sum_{n=1}^{N} D_\mathrm{d}^{(n)}U_\mathrm{d0}^{(n)} + |\hat{f}(k)|^2 D_\mathrm{d}U_\mathrm{d0} \tag{A.3}$$

with equality for identical elements. The integration over solid angle gives

$$U_0 = \frac{1}{4\pi}\int U\,\mathrm{d}\Omega = k^{-2}N\left\langle|\Delta\boldsymbol{F}_0(k)|^2\right\rangle + (1-|\hat{f}(k)|^2)\sum_{n=1}^{N} U_\mathrm{d0}^{(n)} + |\hat{f}(k)|^2 U_\mathrm{d0}, \tag{A.4}$$

and the directivity is

$$\begin{aligned} D &\leq \frac{k^{-2}N\left\langle|\Delta\boldsymbol{F}_0(k)|^2\right\rangle + (1-|\hat{f}(k)|^2)\sum_{n=1}^{N} D_\mathrm{d}^{(n)}U_\mathrm{d0}^{(n)} + |\hat{f}(k)|^2 D_\mathrm{d}U_\mathrm{d0}}{k^{-2}N\left\langle|\Delta\boldsymbol{F}_0(k)|^2\right\rangle + (1-|\hat{f}(k)|^2)\sum_{n=1}^{N} U_\mathrm{d0}^{(n)} + |\hat{f}(k)|^2 U_\mathrm{d0}} \\ &= D_\mathrm{d} - \frac{k^{-2}N(D_\mathrm{d}-1)\left\langle|\Delta\boldsymbol{F}_0(k)|^2\right\rangle + (1-|\hat{f}(k)|^2)\sum_{n=1}^{N}(D_\mathrm{d}-D_\mathrm{d}^{(n)})U_\mathrm{d0}^{(n)}}{k^{-2}N\left\langle|\Delta\boldsymbol{F}_0(k)|^2\right\rangle + (1-|\hat{f}(k)|^2)\sum_{n=1}^{N} U_\mathrm{d0}^{(n)} + |\hat{f}(k)|^2 U_\mathrm{d0}}. \end{aligned} \tag{A.5}$$

Since the directivity $D_\mathrm{d} \geq 1$ and we can assume $D_\mathrm{d} \geq D_\mathrm{d}^{(n)}$ (usually an array antenna is constructed with the specific purpose to increase the directivity), the fraction is positive and it is seen that the directivity in general decreases when the antenna has random errors.

A simplified expression for the directivity can be found when the errors are small. This corresponds to $|\hat{f}(k)|^2 \to 1$, and the denominator in (A.5) is approximated by U_d0. We assume the location and phase errors are normally distributed, implying $|\hat{f}(k)|^2 = \mathrm{e}^{-k^2\delta_{\boldsymbol{r}}^2-\delta_\phi^2}$, where $\delta_{\boldsymbol{r}}$ and δ_ϕ are the standard deviations of location and phase, respectively. For small deviations, this means $1-|\hat{f}(k)|^2 \approx k^2\delta_{\boldsymbol{r}}^2 + \delta_\phi^2$. The variance of the amplitude error is denoted $\langle|\Delta\boldsymbol{F}_0|^2\rangle = \delta_F^2$, and we have

$$D \leq D_\mathrm{d} - \left[\frac{N\delta_F^2}{k^2U_\mathrm{d0}}(D_\mathrm{d}-1) + (k^2\delta_{\boldsymbol{r}}^2+\delta_\phi^2)\sum_{n=1}^{N}(D_\mathrm{d}-D_\mathrm{d}^{(n)})\frac{U_\mathrm{d0}^{(n)}}{U_\mathrm{d0}}\right]. \tag{A.6}$$

A.2 Side lobe level

The side lobe level is calculated as the ratio of the maximum radiation intensity outside the main lobe to the maximum radiation intensity. In this and the following section, we assume the main lobe is contained in the solid angle Ω_0, and that this angle does not change appreciably with errors present. The side lobe level is then given by

$$\begin{aligned}\mathrm{SLL} &= \frac{\max_{\boldsymbol{k}\notin\Omega_0} U}{U_{\max}} \\ &= \frac{k^{-2}N\,\langle|\boldsymbol{\mathit{\Delta F}}_0(k)|^2\rangle + (1-|\hat{f}(k)|^2)\max_{\boldsymbol{k}\notin\Omega_0}\sum_{n=1}^{N} U_{\mathrm{d}}^{(n)} + |\hat{f}(k)|^2\mathrm{SLL}_{\mathrm{d}}U_{\mathrm{dmax}}}{k^{-2}N\,\langle|\boldsymbol{\mathit{\Delta F}}_0(k)|^2\rangle + (1-|\hat{f}(k)|^2)\max_{\boldsymbol{k}}\sum_{n=1}^{N} U_{\mathrm{d}}^{(n)} + |\hat{f}(k)|^2U_{\mathrm{dmax}}}.\end{aligned} \tag{A.7}$$

The same approximations leading to (A.6), *i.e.*, approximating the denominator with U_{dmax} and $1-|\hat{f}(k)|^2 \approx k^2\delta_{\boldsymbol{r}}^2 + \delta_\phi^2$ and $\max_{\boldsymbol{k}\notin\Omega_0}\sum_{n=1}^{N} U_{\mathrm{d}}^{(n)} \le \sum_{n=1}^{N} U_{\mathrm{dmax}}^{(n)}$, now give

$$\mathrm{SLL} \le \mathrm{SLL}_{\mathrm{d}} + (1-\mathrm{SLL}_{\mathrm{d}})\left[\frac{N\delta_F^2}{k^2U_{\mathrm{dmax}}} + (k^2\delta_{\boldsymbol{r}}^2+\delta_\phi^2)\sum_{n=1}^{N}\frac{U_{\mathrm{dmax}}^{(n)}}{U_{\mathrm{dmax}}}\right], \tag{A.8}$$

where $U_{\mathrm{dmax}} = D_{\mathrm{d}}U_{\mathrm{d0}}$ as in the previous section.

A.3 Beam efficiency

The beam efficiency of an antenna with a main lobe is the ratio of the power within the lobe to the total power emitted [3, p. 63], *i.e.*, if the main lobe is contained in the solid angle Ω_0 we have

$$\mathrm{BE} = \frac{\int_{\Omega_0} U\,\mathrm{d}\Omega}{4\pi U_0}, \tag{A.9}$$

where $U_0 = (4\pi)^{-1}\int_{4\pi} U\,\mathrm{d}\Omega$ is defined in Section A.1. Using the approximation $\int_{\Omega_0} U_{\mathrm{d}}^{(n)}\,\mathrm{d}\Omega \le |\Omega_0|U_{\mathrm{dmax}}^{(n)} = |\Omega_0|D_{\mathrm{d}}^{(n)}U_{\mathrm{d0}}^{(n)}$, the same procedure as in Section A.1 implies

$$\begin{aligned}\mathrm{BE} &\le \frac{1}{4\pi}\frac{k^{-2}N\,\langle|\boldsymbol{\mathit{\Delta F}}_0(k)|^2\rangle\,|\Omega_0| + (1-|\hat{f}(k)|^2)\sum_{n=1}^{N}|\Omega_0|D_{\mathrm{d}}^{(n)}U_{\mathrm{d0}}^{(n)} + |\hat{f}(k)|^2\mathrm{BE}_{\mathrm{d}}4\pi U_{\mathrm{d0}}}{k^{-2}N\,\langle|\boldsymbol{\mathit{\Delta F}}_0(k)|^2\rangle + (1-|\hat{f}(k)|^2)\sum_{n=1}^{N}U_{\mathrm{d0}}^{(n)} + |\hat{f}(k)|^2U_{\mathrm{d0}}} \\ &= \mathrm{BE}_{\mathrm{d}} - \frac{k^{-2}N(\mathrm{BE}_{\mathrm{d}}-|\Omega_0|/4\pi)\,\langle|\boldsymbol{\mathit{\Delta F}}_0(k)|^2\rangle + (1-|\hat{f}(k)|^2)\sum_{n=1}^{N}(\mathrm{BE}_{\mathrm{d}}-D_{\mathrm{d}}^{(n)}|\Omega_0|/4\pi)U_{\mathrm{d0}}^{(n)}}{k^{-2}N\,\langle|\boldsymbol{\mathit{\Delta F}}_0(k)|^2\rangle + (1-|\hat{f}(k)|^2)\sum_{n=1}^{N}U_{\mathrm{d0}}^{(n)} + |\hat{f}(k)|^2U_{\mathrm{d0}}}.\end{aligned} \tag{A.10}$$

Once again applying the approximation of small, normally distributed errors as in the derivation of (A.6), we find

$$\mathrm{BE} \le \mathrm{BE}_{\mathrm{d}} - \left[\frac{N\delta_F^2}{k^2U_{\mathrm{d0}}}\left(\mathrm{BE}_{\mathrm{d}} - \frac{|\Omega_0|}{4\pi}\right) + (k^2\delta_{\boldsymbol{r}}^2+\delta_\phi^2)\sum_{n=1}^{N}\left(\mathrm{BE}_{\mathrm{d}} - D_{\mathrm{d}}^{(n)}\frac{|\Omega_0|}{4\pi}\right)\frac{U_{\mathrm{d0}}^{(n)}}{U_{\mathrm{d0}}}\right]. \tag{A.11}$$

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