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Some Distance Properties of Tailbiting Codes[1]

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Abstract — **The active tailbiting segment distance for convolutional codes is introduced. Together with the earlier defined active burst distance, it describes the error correcting capability of a tailbiting code encoded by a convolutional encoder. Lower bounds on the new active distance as well as an upper bound on the ratio between tailbiting length and memory of encoder such that its minimum distance $d_{\min}$ equals the free distance d_{free} of the corresponding convolutional code are presented.**

I. Introduction

Tailbiting codes can be obtained by terminating rate $R = b/c$ convolutional codes into block codes of length l c-tuples [1]. For simplicity, we consider only binary codes. The error correcting capability of a block code is estimated via $d_{\min}$. There is no description which error patterns with more than $\lfloor \frac{d_{\min}-1}{2} \rfloor$ errors can be corrected. In order to describe the error correcting capability of a tailbiting code beyond the minimum distance argument, we define the active tailbiting segment distance a_j^{tbs}. We also give an upper bound on the length of a tailbiting code so that $d_{\min}$ equals d_{free} of the corresponding convolutional code. This is useful when analyzing concatenated coding schemes containing tailbiting encoders.

II. The Active Tailbiting Segment Distance

Consider the convolutional code $\mathcal{C}$ encoded by a rate $R = b/c$ encoder. The binary matrix $\boldsymbol{\sigma}_t$ denotes the encoder state at time t. Let $\mathcal{S}_{[t_1,t_2]}^{\boldsymbol{\sigma}_s,\boldsymbol{\sigma}_e}$, $0 \le t_1 < t_2$, be the set of state sequences $\boldsymbol{\sigma}_{[t_1,t_2]} = \boldsymbol{\sigma}_{t_1} \dots \boldsymbol{\sigma}_{t_2}$ that start in state $\boldsymbol{\sigma}_s$ and terminate in state $\boldsymbol{\sigma}_e$, and do not have two consecutive zero states with zero input in between. Let j_b be the smallest positive integer such that $\mathcal{S}_{[0,j_b+1]}^{\mathbf{0},\mathbf{0}} \neq \emptyset$. The *jth order active burst distance* [2] is $a_j^b \triangleq \min_{\mathcal{S}_{[0,j+1]}^{\mathbf{0},\mathbf{0}}} \{ w_{\text{H}}(\boldsymbol{v}_{[0,j]}) \}$, $j \ge j_b$. For any code $\mathcal{C}$, a_j^b is invariant over the set of its canonical encoders [2]. It is lower-bounded by a linearly increasing function $a_j^b \ge \alpha j + \beta^b$, $j \ge j_b$, where α is the *asymptotic slope*, and β^b is chosen as large as possible. Let j_r be the smallest number of steps that, starting in the allzero state, take us to any reachable state.

Definition 1 *The* jth order active tailbiting segment distance *is* $a_j^{tbs} \triangleq \min_{\mathcal{S}_{[j_r,j_r+j+1]}^{\boldsymbol{\sigma},\boldsymbol{\sigma}}} \{ w_{\text{H}}(\boldsymbol{v}_{[j_r,j_r+j]}) \}$, *where $\boldsymbol{\sigma}$ denotes any possible encoder state.*

Theorem 1 *The active tailbiting segment distance is lower-bounded by $a_j^{tbs} \ge \alpha(j+1)$, for all $j \ge 0$.*

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III. Properties of Tailbiting Codes Via the Active Distances

We define an *incorrect path* at the receiver to be any trellis path differing from the *transmitted path*. For any $k_1, k_2 < l$, let $e_{[k_1,k_2]}$ denote the Hamming weight of the error pattern $\boldsymbol{e}_{[k_1,k_2]} = \boldsymbol{e}_{k_1}\boldsymbol{e}_{k_1+1}\dots\boldsymbol{e}_{k_2}$, where $\boldsymbol{e}_i$, $0 \le i < l$, are c-tuples and all indices are evaluated modulo l. Then, we have

Theorem 2 *Consider a tailbiting code $\mathcal{C}^{tb}$ of length l c-tuples encoded by a convolutional encoder. Then, for $j_b < l$, a maximum likelihood (ML) decoder corrects all error patterns $\boldsymbol{e}_{[0,l-1]}$ that satisfy $e_{[t,\ t+j \bmod l]} < \min\{a_j^b/2,\ a_{l-1}^{tbs}/2\}$ for $0 \le t < l$, $j_b \le j < l$. For $j_b \ge l$, all error patterns that satisfy $e_{[0,l-1]} < a_{l-1}^{tbs}/2$ are corrected.*

Example 1 *A tailbiting code of length $l = 18$ 2-tuples encoded by $G(D) = (1 + D + D^2 \quad 1 + D^2)$ with $d_{\min} = 5$ is used on a binary symmetric channel. From Theorem 2 follows that*

$$\boldsymbol{e}_{[0,17]} = 10\,00\,00\,01\,00\,00\,00\,00\,01\,00\,00\,00\,10\,00\,00\,00\,00\,00$$

is corrected by an ML decoder although the error pattern contains four channel errors which exceeds $\lfloor \frac{d_{\min}-1}{2} \rfloor$.

Consider a tailbiting code of length l c-tuples with minimum distance $d_{\min}$. The *free tailbiting length* l_{free} is the shortest length l for which $d_{\min}$ will remain equal to d_{free} of the corresponding convolutional code for all tailbiting lengths greater than or equal to l_{free}. The free tailbiting length is upper-bounded by $l_{\text{free}} \le \lfloor d_{\text{free}}/\alpha \rfloor$.

IV. Ensemble Properties of the Active Tailbiting Segment Distance

The concept of the active distances can be generalized to time-varying convolutional encoders.

Theorem 3 *There exists a rate $R = b/c$ convolutional code $\mathcal{C}$ encoded by a time-varying encoder of memory m such that*
$$a_j^{tbs} > \rho(j+1)c + O(\log m), \quad \text{for } j = O(m) \ge j_o \quad m \to \infty$$
where $\rho = h^{-1}(1-R)$ is the Gilbert-Varshamov parameter, $h(\,)$ is the binary entropy function, and j_o is the smallest integer satisfying $(1-R)(j+1)c \ge 4\log m$.

Using the Heller asymptotic bound we obtain

Theorem 4 *There exists a tailbiting code $\mathcal{C}^{tb}$ encoded by a time-varying encoder of memory m, such that $\lim_{m\to\infty} l_{\text{free}}/m \le \frac{1}{2\rho}$.*

References

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