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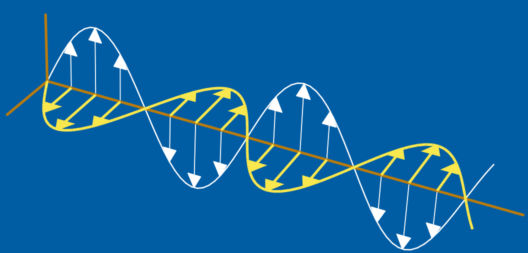

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Abstract

A passive system is one that cannot produce energy, a property that naturally poses constraints on the system. A system on convolution form is fully described by its transfer function, and the class of Herglotz functions, holomorphic functions mapping the open upper half plane to the closed upper half plane, is closely related to the transfer functions of passive systems. Following a well-known representation theorem, Herglotz functions can be represented by means of positive measures on the real line. This fact is exploited in this paper in order to rigorously prove a set of integral identities for Herglotz functions that relate weighted integrals of the function to its asymptotic expansions at the origin and infinity.

The integral identities are the core of a general approach introduced here to derive sum rules and physical limitations on various passive physical systems. Although similar approaches have previously been applied to a wide range of specific applications, this paper is the first to deliver a general procedure together with the necessary proofs. This procedure is described thoroughly, and exemplified with examples from electromagnetic theory.

1 Introduction

The concept of passivity is fundamental in many applications. Intuitively, a passive system is one that does not in itself produce energy (if the system does not consume energy either, it is called lossless); hence the energy-content of the output signal is limited to that of the input. Passivity poses severe constraints, or physical limitations, on a system. The aim of this paper is to investigate these constraints. In particular, a general approach to derive sum rules and physical limitations is presented along with the necessary proofs.

A system on convolution form is fully described by its impulse response, w. The convolution form is intimately related to the assumptions of linearity, continuity and time-translational invariance. With the added assumptions of causality and passivity, the Fourier transform of w is related to a Herglotz function [22, 27] (sometimes referred to as a Nevanlinna [17], Pick [8], or R-function [20]). The Laplace transform and the related function class of positive real (PR) functions are commonly preferred by some authors [37, 39, 41].

As holomorphic mappings between half-planes, Herglotz functions are closely related to positive harmonic functions and the Hardy space $H^\infty(\mathbb{C}^+)$ via the Cayley transform [9, 25]. Herglotz functions appear in literature concerning continued fractions and the problem of moments [1, 18, 31], but also within functional analysis and spectral theory for self-adjoint operators [2, 17]. There is a powerful representation theorem for Herglotz functions, relating them to positive measures on $\mathbb{R}$. Under certain assumptions on a Herglotz function h it is possible to derive a set of integral identities, relating weighted integrals of h over infinite intervals to its expansion coefficients at the origin and infinity.

The integral identities can be used to derive sum rules for various physical systems, effectively relating dynamic behaviour to static and/or high-frequency properties. This is very beneficial, since static properties are often easier to determine than dynamical behaviour. The representation in itself can also provide information on a system in the form of dispersion relations; consider *e.g.*, the Kramers-Kronig relations [22, 24] discussed in Example 5.4. One way to take advantage of the sum rules is to derive constraints, or physical limitations, by considering finite frequency intervals. In essence, the physical limitations indicate what can and cannot be expected from a system.

Some previous examples of sum rules and physical limitations within electromagnetic theory are in the analysis of matching networks [10], temporal dispersion for metamaterials [12], broadband electromagnetic interaction with objects [33], bandwidth and directivity for antennas of certain sizes [14], extra ordinary transmission through sub-wavelength apertures [15], radar absorbers [29], high-impedance surfaces [4] and frequency selective surfaces [16]. The physical limitations can be very helpful, both from a theoretical point of view where one wishes to understand what factors limit the performance, but also from a designer view-point where the physical limitations can signal if there is room for improvement or not. As the examples show, similar methods to the one presented in this paper have been widely used to derive sum rules for systems on convolution form. For many causal systems, Titchmarsh's theorem can be used to derive dispersion relations in the form of a Hilbert transform pair [21, 22, 27]. However, some more assumptions are needed in order to obtain sum rules, see *e.g.*, [22] and references therein. If, for instance, the transfer function is rational, the Cauchy integral formula may be used, see *e.g.*, [10, 34].

This paper presents an approach to derive sum rules and physical limitations under the assumption that the system under consideration is causal and passive. There does not seem to be a previous account on such an approach. At the core are the integral identities for Herglotz functions, which are proved rigorously in this paper. Many physical systems obey passivity, and so the results presented here are applicable to a wide range of problems. The paper is divided into a number of distinct parts: First, the class of Herglotz functions along with some of its important properties are reviewed in order to pave the way for the integral identities. After this section there is a discussion about passive systems and their connection to Herglotz functions. The proof of the integral identities comes next, and after that follow some examples which serve to illuminate the theory. Last come some concluding remarks.

2 Herglotz functions and integral identities

The aim of this section is to introduce the class of Herglotz functions and recall some well known properties of this class. This naturally leads to the introduction of the integral identities, presented in the end of the section. Start with the definition of a Herglotz function:

Definition 2.1. *A Herglotz function is defined as a holomorphic function* $h : \mathbb{C}^+ \to \mathbb{C}^+ \cup \mathbb{R}$ *where* $\mathbb{C}^+ = \{z : \operatorname{Im} z > 0\}$.

There is a powerful representation theorem for the set of Herglotz functions $\mathcal{H}$ due to Nevanlinna [26], presented in the following form by Cauer [6] (see also [2]):

Theorem 2.1. *A necessary and sufficient condition for a function h to be a Herglotz function is that*

$$h(z) = \beta z + \alpha + \int_{\mathbb{R}} \left(\frac{1}{\xi - z} - \frac{\xi}{1+\xi^2} \right) \mathrm{d}\mu(\xi), \quad \operatorname{Im} z > 0, \tag{2.1}$$

where $\beta \geq 0$, $\alpha \in \mathbb{R}$ and μ is a positive Borel measure such that $\int_{\mathbb{R}} \mathrm{d}\mu(\xi)/(1+\xi^2) < \infty$.[1]

Note the resemblance of (2.1) to the Hilbert transform [21, 25]. The proof of the representation theorem is not included here (it can be found in [2]), but in order to make it believable, (2.1) is cast into the slightly different form

$$h(z) = \beta z + \alpha + \int_{\mathbb{R}} \frac{1+\xi z}{\xi - z} \mathrm{d}\nu(\xi), \quad \operatorname{Im} z > 0, \tag{2.2}$$

where $\mathrm{d}\nu(\xi) = \mathrm{d}\mu(\xi)/(1+\xi^2)$ is a positive and finite measure. The function $F(\xi, z) = (1+\xi z)/(\xi - z)$ is a Herglotz function in z for all $\xi \in \mathbb{R} \cup \{\infty\}$, and sums of Herglotz functions are Herglotz functions. The constant β may be interpreted as $\nu(\{\infty\})$ (the point mass of ν at the point ∞ of the extended real line $\mathbb{R} \cup \{\infty\}$), since $F(\xi, z) \to z$ as $|\xi| \to \infty$. A real constant α may also be added to a Herglotz function, so the function given by (2.2) is a Herglotz function. That (2.1) exhausts the set $\mathcal{H}$ follows *e.g.*, from a representation theorem for positive harmonic functions on the unit disk due to Herglotz [19]. This representation theorem relies on the Riesz representation theorem for continuous, linear functionals on a compact metric space. Note that the only way in which a Herglotz function can be real-valued in $\mathbb{C}^+$ is if $h \equiv \alpha$ for some $\alpha \in \mathbb{R}$.

From the representation (2.1) it follows that $h(z)/z \to \beta$, as $z \hat{\to} \infty$, where $z \hat{\to} \infty$ is a short-hand notation for $|z| \to \infty$ in the Stoltz domain $\theta \leq \arg z \leq \pi - \theta$ for any $\theta \in (0, \pi/2]$ (see Appendix A.1). Hence it makes sense to consider Herglotz functions with the asymptotic expansion

$$h(z) = \sum_{1-2M}^{m=1} b_m z^m + o(z^{1-2M}), \quad \text{as } z \hat{\to} \infty, \tag{2.3}$$

where $b_m \in \mathbb{R}$. Since $b_1 = \beta$, this expansion is always possible for some integer $M \geq 0$. It will simplify notation to define $b_m = 0$ for $m > 1$. The representation also implies that $zh(z) \to -\mu(\{0\})$, as $z \hat{\to} 0$ (once more, see Appendix A.1), and so an asymptotic expansion

$$h(z) = \sum_{n=-1}^{2N-1} a_n z^n + o(z^{2N-1}), \quad \text{as } z \hat{\to} 0, \tag{2.4}$$

[1] The following notation is adopted throughout this paper (*cf.*, [3, 30]): If μ is a positive measure on the Borel subsets $\mathcal{E}$ of $\mathbb{R}$ and $E \in \mathcal{E}$, denote $\mu(E) = \int_E \mathrm{d}\mu(\xi)$. The measure is referred to as μ or $\mathrm{d}\mu$. The Lebesgue integral of f with respect to μ is denoted $\int_{\mathbb{R}} f(\xi)\,\mathrm{d}\mu(\xi)$ whenever f is a complex-valued measurable function on $\mathbb{R}$. The positive measure that maps E to $\int_E u(\xi)\,\mathrm{d}\mu(\xi)$ for some non-negative measurable function u on $\mathbb{R}$ is denoted $u\,\mathrm{d}\mu$.

where $a_{-1} = -\mu(\{0\})$ and all a_n are real, is available for some integer $N \geq 0$. The coefficients a_n are defined to be zero for $n < -1$. It will turn out that it suffices to consider the asymptotic expansions along the imaginary axis, *i.e.*, for $\arg z = \pi/2$ (see Lemma 4.2).

At the core of the approach presented in this paper to derive sum rules for passive systems are the integral identities

$$\lim_{\varepsilon\to 0^+}\lim_{y\to 0^+}\frac{1}{\pi}\int_{\varepsilon<|x|<\varepsilon^{-1}}\frac{\operatorname{Im} h(x+\mathrm{i}y)}{x^p}\,\mathrm{d}x = a_{p-1} - b_{p-1}, \quad p = 2-2M, 3-2M, \dots, 2N. \tag{2.5}$$

Throughout this paper i denotes the imaginary unit ($\mathrm{i}^2 = -1$), and $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are implicit. Note that the origin is no more special than any other point on the real line; a Herglotz function shifted to the left or right is still a Herglotz function. Compositions of Herglotz functions with each other yields new Herglotz functions (barring the trivial case when $h \equiv \alpha$), a property that may be exploited to determine a family of sum rules. See the examples 5.1 and 5.4.

One more point deserves a discussion here: In physical applications it is often desirable to interpret the left-hand side of (2.5) as an integral over the real line. In that case the integral must be interpreted in the distributional sense; the generalized function $h(x) = \lim_{y\to 0^+} h(x+\mathrm{i}y)$, where the right hand side is interpreted as a limit of distributions, is a distribution of slow growth. In a discussion following Lemma 4.1 it is shown that, for almost all $x \in \mathbb{R}$, the limit $\lim_{y\to 0^+} \operatorname{Im} h(x+\mathrm{i}y)$ exists as a finite number. The left-hand side of (2.5) is precisely the integral over the finite part of the limit plus possible contributions from singularities in $\{x : 0 < |x| < \infty\}$, *cf.*, (4.3), Example 5.1 and Example 5.2.

In some special cases the integral identities follow directly from the Cauchy integral formula [10, 34]. This requires some extra assumptions, *e.g.*, that the Herglotz function is the restriction to $\mathbb{C}^+$ of a rational function. Alternative approaches to obtain integral identities from the Hilbert transform under different assumptions is discussed by King [22].

3 Sum rules for passive systems

The integral identities (2.5) offer an approach to construct sum rules and associated physical limitations on various systems. The first step is to ensure that the system can be modelled with a Herglotz function. Secondly, the asymptotic expansions (2.3) and (2.4), here referred to as the high- and low-frequency asymptotic expansions, have to be determined. This step commonly uses physical arguments, and is specific to each application. Finally, the integrals in (2.5) are bounded to construct the physical limitations.

Herglotz functions appear in the context of linear, time translational invariant, continuous, causal and passive systems, see *e.g.*, the paper [39] by Youla et. al., [40] and [41] by Zemanian, and [37] by Wohlers and Beltrami. These treatises are in the context of distributions, while a study in a more general setting is given in [42] and references therein. A short summary of some important results are given in this section. See also the book [27] by Nussenzveig.

Let $\mathcal{D}'$ denote the space of distributions of one variable, and let $\mathcal{D}'_0$ denote distributions with compact support [41]. Consider an operator $\mathcal{R} : D(\mathcal{R}) \subseteq \mathcal{D}' \to \mathcal{D}'$. It is a convolution operator if and only if it is linear, time translational invariant, and continuous [41, Theorem 5.8-2]:

$$u(t) = \mathcal{R}v(t) = w * v(t), \tag{3.1}$$

where t denotes time, $*$ denotes temporal convolution and $w \in \mathcal{D}'$ is the impulse response. The exact definitions of linearity, time translational invariance and continuity can be found in [41]. The output signal u is given by (3.1) at least for all input signals $v \in \mathcal{D}'$ so that convolution with w is defined in the sense of Theorems 5.4-1 and 5.7-1 in [41]. Since $w \in \mathcal{D}'$, $u = w * v$ at least for all $v \in \mathcal{D}'_0$. If, for example, w is in $\mathcal{S}'$, then $u = w * v$ for all $v \in \mathcal{S}$. Here $\mathcal{S}'$ denotes distributions of slow growth and $\mathcal{S}$ denotes smooth functions of rapid descent [41].

The operator is causal if w is not supported in $t < 0$, *i.e.*, $\operatorname{supp} w \subseteq [0, \infty)$. The last crucial property of the operator is that of passivity, which is considered in two different forms. The terminology is borrowed from electric circuit theory. Let v correspond to the electric voltage over some port, and let u correspond to the current into said port. Assume that the voltage and current are almost time-harmonic with an amplitude varying over a timescale much larger than the dominating frequency, so that u and v are complex valued distributions. The power absorbed by the system at the time t is $\operatorname{Re} u^*(t)v(t)$ (if u and v are regular functions), where the superscript * denotes the complex conjugate. The operator $\mathcal{R}$ defined by $u = \mathcal{R}v$ is called the admittance operator. If instead the input signal is $q = (v + u)/2$ and the output is $r = \mathcal{W}q = (v - u)/2$, the corresponding operator $\mathcal{W}$ is the scattering operator, and the absorbed power is $|q(t)|^2 - |r(t)|^2$. Let $\mathcal{D}$ denote the space of smooth functions with compact support and make the following definition [37, 41, 42]:

Definition 3.1. *Let $\mathcal{R}$ be a convolution operator with input v and output $u = \mathcal{R}v$. Define the energy expressions*

$$e_{\text{adm}}(T) = \operatorname{Re} \int_{-\infty}^{T} u^*(t)v(t)\, \mathrm{d}t$$

and

$$e_{\text{scat}}(T) = \int_{-\infty}^{T} |v(t)|^2 - |u(t)|^2\, \mathrm{d}t.$$

The operator is admittance-passive (scatter-passive) if $e_{\text{adm}}(T)$ ($e_{\text{scat}}(T)$) is non-negative for all $T \in \mathbb{R}$ and $v \in \mathcal{D}$.

Note that admittance-passive might as well have been called impedance-passive, if the electric current was assumed to be input and the voltage output in the example from which the name stems.

An operator which is admittance-passive or scatter-passive is called passive in this paper. As it turns out, passivity implies causality for operators on convolution form. Furthermore, in this case the impulse response w must be a distribution of slow growth, *i.e.*, $w \in \mathcal{S}'$ [37, 41], and thus (3.1) is defined for smooth input signals of rapid descent, $v \in \mathcal{S}$. Note that (3.1) is also defined for all input signals v with support bounded on the left, since $\operatorname{supp} w \subseteq [0, \infty)$ [41].

Since the impulse response is in $\mathcal{S}'$, its Fourier transform may be defined as

$$\langle \mathcal{F}w, \varphi \rangle = \langle w, \mathcal{F}\varphi \rangle , \quad \text{for all } \varphi \in \mathcal{S},$$

where $\langle f, \varphi \rangle$ is the value in $\mathbb{C}$ that $f \in \mathcal{S}'$ assigns to $\varphi \in \mathcal{S}$ [41]. The Fourier transform of φ is defined as

$$\mathcal{F}\varphi(\omega) = \int_{\mathbb{R}} \varphi(t) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}t.$$

The Fourier transform of w is the transfer function $\tilde{w}$ of the system, *viz.*,

$$\tilde{w}(\omega) = \mathcal{F}w(\omega). \tag{3.2}$$

The convolution in (3.1) is mapped to multiplication if *e.g.*, $v \in \mathcal{D}'_0$ or $v \in \mathcal{S}$ [41]. In that case the frequency domain system is modeled by

$$\tilde{u}(\omega) = \tilde{w}(\omega)\tilde{v}(\omega),$$

where $\tilde{v} = \mathcal{F}v$ and $\tilde{u} = \mathcal{F}u$ are the input and output signals, respectively.

The transfer function $\tilde{w}(\omega)$ is in $\mathcal{S}'$ for real ω, but since the support of w is bounded on the left the region of convergence for $\tilde{w}$ contains $\mathbb{C}^+$ and $\tilde{w}$ is holomorphic there. The Laplace transform is commonly used in system theory, generating the corresponding transfer function $\tilde{w}_{\text{Laplace}}(s) = \tilde{w}(\mathrm{i}s)$. Scrutinising the transfer function, the following theorem is proved (*cf.*, Theorem 10.4-1 in [41], Theorem 2 in [37] and Theorems 7.4-3 and 8.12-1 in [42]):

Theorem 3.1. *Let $\mathcal{R} = w*$ be a convolution operator and let $\tilde{w}$ be given by* (3.2)*. If $\mathcal{R}$ is admittance-passive, then* $\operatorname{Re} \tilde{w}(\omega) \geq 0$ *for all $\omega \in \mathbb{C}^+$. If $\mathcal{R}$ is scatter-passive, then $|\tilde{w}(\omega)| \leq 1$ for all $\omega \in \mathbb{C}^+$. In both cases $\tilde{w}$ is holomorphic in $\mathbb{C}^+$.*

The converse statement to the theorem can also be made, *i.e.*, that every transfer function on one of the forms described in the theorem generates an admittance-passive or scatter-passive operator, respectively [41, Theorem 10.6-1], [42, Theorems 7.5-1 and 8.12-1].

Evidently, the transfer function of an admittance-passive operator multiplied with the imaginary unit is a Herglotz function, $h = \mathrm{i}\tilde{w}$. For scatter-passive operators a Herglotz function can be constructed from $\tilde{w}$ via the inverse Cayley transform $z \mapsto (\mathrm{i}z + \mathrm{i})/(1 - z)$. Alternatively, factorize $\tilde{w}(\omega) = H(\omega)B(\omega)$, where $H(\omega)$ is a zero free holomorphic function such that $|H(\omega)| \leq 1$ for all $\omega \in \mathbb{C}^+$ and

$$B(\omega) = \left(\frac{\omega - \mathrm{i}}{\omega + \mathrm{i}} \right)^k \prod_{\omega_n \neq \mathrm{i}} \frac{|\omega_n^2 + 1|}{\omega_n^2 + 1} \frac{\omega - \omega_n}{\omega - \omega_n^*} \tag{3.3}$$

is a Blaschke product [9, 25]. Here the zeros ω_n of $\tilde{w}$ are repeated according to their multiplicity and $k \geq 0$ is the order of the possible zero at $\omega = \mathrm{i}$. The convergence factors $|\omega_n^2 + 1|/(\omega_n^2 + 1)$ may be omitted if all $|\omega_n|$ are bounded by the same constant or if $\tilde{w}$ satisfies the symmetry (3.7) discussed below. Since $\tilde{w}$ belongs to the Hardy space $H^\infty(\mathbb{C}^+)$, this factorization is always possible due to a theorem of F. Riesz [9, 25]. Moving on, the function H may be represented as $H(\omega) = \mathrm{e}^{\mathrm{i}h(\omega)}$ since it is holomorphic and zero-free on the simply connected domain $\mathbb{C}^+$. Here the holomorphic function h must have a non-negative imaginary part. Note that the converse to the factorization also holds; a function $\tilde{w}$ is holomorphic and bounded in magnitude by one in $\mathbb{C}^+$ if and only if it is of the form

$$\tilde{w}(\omega) = B(\omega)\mathrm{e}^{\mathrm{i}h(\omega)}, \tag{3.4}$$

where B is a Blaschke product given by (3.3) and h is a Herglotz function.

The formula (3.4) may be inverted:

$$h(\omega) = -\mathrm{i}\log\left(\frac{\tilde{w}(\omega)}{B(\omega)}\right),$$

if the logarithm is defined as

$$\log H(z) = \ln|H(z_0)| + \mathrm{i}\arg H(z_0) + \int_{\gamma_{z_0}^z} \frac{\mathrm{d}H/\,\mathrm{d}\zeta}{H(\zeta)}\,\mathrm{d}\zeta. \tag{3.5}$$

Here $\gamma_{z_0}^z$ is any piecewise C^1 curve from z_0 to z in $\mathbb{C}^+$. The left-hand side of (2.5) takes the form

$$\lim_{\varepsilon\to 0^+}\lim_{y\to 0^+}\int_{\varepsilon<|x|<\varepsilon^{-1}} \frac{\operatorname{Im} h(x+\mathrm{i}y)}{x^p}\,\mathrm{d}x = \lim_{\varepsilon\to 0^+}\lim_{y\to 0^+}\int_{\varepsilon<|x|<\varepsilon^{-1}} \frac{-\ln|\tilde{w}(x+\mathrm{i}y)/B(x+\mathrm{i}y)|}{x^p}\,\mathrm{d}x.$$

The modulus $|B(z)|$ tends to 1 as $z \hat{\to} x$ for almost all $x \in \mathbb{R}$ (the exceptions are the x which are accumulation points of the zeros of $\tilde{w}$ [25]). If the origin is not an accumulation point of the zeros of $\tilde{w}$, the low-frequency asymptotic expansion of h is

$$h(\omega) = -\mathrm{i}\log\tilde{w}(\omega) - \arg B(0) + \mathrm{i}\sum_{m=1}^{\infty}\frac{\omega^m}{m}\sum_{\omega_n}\omega_n^{-m} - \omega_n^{*-m}, \quad \text{as } \omega\hat{\to}0. \tag{3.6}$$

A similar argument may be applied to the high-frequency asymptotic expansion. The asymptotic expansions of $\log\tilde{w}$ must be found by physical arguments, see Example 5.3.

For operators $\mathcal{R}$ mapping real input to real output, the impulse response w has to be real. This implies the symmetry

$$\tilde{w}(\omega) = \tilde{w}^*(-\omega^*), \tag{3.7}$$

which is transferred to the Herglotz function as

$$h(\omega) = -h^*(-\omega^*) \tag{3.8}$$

if it is defined by $h = \mathrm{i}\tilde{w}(\omega)$ (for admittance-passive systems) or by the inverse Cayley transform of $\pm\tilde{w}$ (for scatter-passive systems). The Herglotz function h in (3.4) must be of the form $h = h_1 + \alpha$, where $h_1(\omega) = -h_1^*(-\omega^*)$, and $\alpha \in \mathbb{R}$ is the argument of $\mathrm{e}^{\mathrm{i}h(\omega)}$ for purely imaginary ω. The symmetry restricts the identities (2.5) to even powers and simplifies them to

$$\lim_{\varepsilon\to 0^+}\lim_{y\to 0^+}\frac{2}{\pi}\int_{\varepsilon}^{\varepsilon^{-1}}\frac{\operatorname{Im} h(x+\mathrm{i}y)}{x^{2\hat{p}}}\,\mathrm{d}x = a_{2\hat{p}-1} - b_{2\hat{p}-1}, \quad \hat{p} = 1-M,\dots,N. \tag{3.9}$$

In general, the integral identities (2.5) for even p are the starting point to derive constraints on the system as the non-negative integrand can be bounded by a finite frequency interval.

Summing up, there are two essentially equivalent ways to evaluate if a system can be modeled with a Herglotz function and potentially be constrained according to (2.5): First, just based on a priori knowledge of linearity, continuity and time-translational invariance (*i.e.*, the convolution form (3.1)) together with passivity. This approach can often be applied directly to various physical systems. The second, frequency domain case is often more involved and requires direct verification that $h(\omega)$ is holomorphic and $\operatorname{Im} h(\omega) \geq 0$ for $\operatorname{Im}\omega > 0$. Alternative characterizations in the frequency domain are given in [37].

The high-frequency expansions (2.3) are sometimes hard to evaluate for physical systems. The high-frequency behaviours of $\tilde{w}(\omega)$ and $h(\omega)$ are determined by the behaviour of $w(t)$ for arbitrarily short times. To see this, first assume that w is a regular, integrable function. Then $\tilde{w}$ is defined as

$$\tilde{w}(\omega) = \int_0^\infty w(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \int_0^\varepsilon w(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t + \int_\varepsilon^\infty w(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t.$$

The second term in the right hand side goes to zero as $\omega\hat{\to}\infty$ (but not as $|\omega| \to \infty$ on the real line) for any $\varepsilon > 0$. This verifies the statement for $w \in \mathrm{L}^1$. For a general $w \in \mathcal{S}'$, consider the equivalent definition of $\tilde{w}(\omega)$ for $\operatorname{Im}\omega > 0$ [41]:

$$\tilde{w}(\omega) = \left\langle w(t), \lambda(t)\mathrm{e}^{\mathrm{i}\omega t}\right\rangle = \left\langle w(t), \lambda_1(t)\mathrm{e}^{\mathrm{i}\omega t}\right\rangle + \left\langle w(t), \lambda_2(t)\mathrm{e}^{\mathrm{i}\omega t}\right\rangle.$$

Here $\lambda(t)$ is a smooth function with support bounded on the left, and such that $\lambda(t) \equiv 1$ for $t \geq 0$. It is decomposed into two non-negative smooth functions, $\lambda = \lambda_1 + \lambda_2$, where $\lambda_2 \equiv 0$ for $t \leq \varepsilon$ for some $\varepsilon > 0$. The second term in the right hand side vanishes as $\omega\hat{\to}\infty$. A similar argument may be carried out for the low-frequency expansion (2.3), essentially relating it to the behaviour of $w(t)$ for arbitrarily large t.

4 Proof of the integral identities

The main theorem (Theorem 4.1) of this paper contains the integral identities (2.5). For $p = 2, 3, \dots, 2N$ they rely on two results: The first (Corollary 4.1) states that the left-hand side of (2.5) is equal to moments of the measure $\mathrm{d}\mu(\xi)$. The second (Lemma 4.2) relates the convergence and explicit value of these moments to the expansion (2.4). A change of variables in the left-hand side of (2.5) enables a proof for $p = 2-2M, 3-2M, \dots, 1$.

A Herglotz function $h(z)$ is in general not defined pointwise for $\operatorname{Im} z = 0$, but integrals of the type $\lim_{y\to 0^+}\int_{\mathbb{R}} \varphi(x)\operatorname{Im} h(x+\mathrm{i}y)\,\mathrm{d}x$ are well defined under certain conditions on φ. The following lemma gives such sufficient conditions. They are stronger than needed, but weak enough to lead to the needed Corollary 4.1.

This is a well known result, see *e.g.*, Lemma S1.2.1 in [20] and Theorem 11.9 in [25]. The lemma and proof are included here for clarity.

Lemma 4.1. *Let h denote a Herglotz function. Suppose that the function $\varphi : \mathbb{R} \to \mathbb{R}$ is piecewise C^1, and that there is a constant $D \geq 0$ such that $|\varphi(x)| \leq D/(1+x^2)$ for all $x \in \mathbb{R}$. Then it follows that*

$$\lim_{y\to 0^+} \frac{1}{\pi}\int_{\mathbb{R}} \varphi(x)\operatorname{Im} h(x+\mathrm{i}y)\,\mathrm{d}x = \int_{\mathbb{R}} \breve{\varphi}(\xi)\,\mathrm{d}\mu(\xi), \tag{4.1}$$

where $\mu(\xi)$ is the measure in the representation (2.1) *of h, and*

$$\breve{\varphi}(\xi) = \begin{cases} \varphi(\xi), & \text{if } \varphi \text{ is continuous at } \xi \\ \frac{\varphi(\xi^-)+\varphi(\xi^+)}{2}, & \text{otherwise.} \end{cases} \tag{4.2}$$

Here $\varphi(\xi^\pm) = \lim_{\zeta\to\xi^\pm}\varphi(\zeta)$.

The proof can be found in Appendix A.2. It is readily shown that the limit may be replaced by any non-tangential limit, *i.e.*, the left-hand side of (4.1) may be replaced by $\lim_{u \hat{\to} 0}\int_{\mathbb{R}} \varphi(x)\operatorname{Im} h(x+u)\,\mathrm{d}x$.

Note that the lemma is in some sense an inversion formula; whereas the representation (2.1) gives the Herglotz function h from the measure μ, (4.1) makes possible the retrieval of μ when h is known. In fact, the lemma is the Stieltjes inversion formula in a different form [1, 20, 31]. The inversion is clarified by decomposing the measure as $\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{s}}$, where μ_{a} is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}\xi$ and μ_{s} is singular in the same sense [3]. Recall that $\mathcal{E}$ denotes the set of Borel subsets of $\mathbb{R}$. Then

$$\mu_{\mathrm{a}}(E) = \int_E \mu'_{\mathrm{a}}(\xi)\,\mathrm{d}\xi, \quad \text{for all } E \in \mathcal{E},$$

where the Radon-Nikodym derivative μ'_{a} of μ_{a} with respect to $\mathrm{d}x$ is a finite, locally integrable function, for almost all $x \in \mathbb{R}$ uniquely defined as [3]

$$\mu'_{\mathrm{a}}(x) = \lim_{s\to 0}\frac{\mu_{\mathrm{a}}([x-s, x+s])}{2s}.$$

"Almost all" is with respect to $\mathrm{d}x$. Furthermore [25],

$$\lim_{s\to 0}\frac{\mu_{\mathrm{s}}([x-s, x+s])}{2s} = 0, \quad \text{for almost all } x \in \mathbb{R}.$$

Hence Lemma 4.1 implies that

$$\lim_{z \hat{\to} x} \frac{1}{\pi} \operatorname{Im} h(z) = \lim_{s \to 0} \frac{\mu([x-s, x+s])}{2s}, \quad \text{for almost all } x \in \mathbb{R}.$$

See also [20].

In physical applications it is often desirable to move the limit inside the integral in the left-hand side of (4.1). Clearly, this is possible if $\mu = \mu_{\mathrm{a}}$. Otherwise, set $g(x) = \lim_{y \to 0^+} \operatorname{Im} h(x + \mathrm{i}y)$, whenever the limit exists finitely, to get

$$\lim_{y \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} \varphi(x) \operatorname{Im} h(x + \mathrm{i}y) \, \mathrm{d}x = \frac{1}{\pi} \int_{\mathbb{R}} \varphi(x) g(x) \, \mathrm{d}x + \int_{\mathbb{R}} \check{\varphi}(\xi) \, \mathrm{d}\mu_s(\xi), \tag{4.3}$$

where the second term on the right hand side represents contributions from singularities on the real line. Equivalently, the left-hand side of (4.1) may be interpreted as an integral over the real line in the distributional sense.

The first result needed for the main theorem is this corollary to Lemma 4.1:

Corollary 4.1. *For all Herglotz functions h given by* (2.1) *it holds that*

$$\lim_{\varepsilon \to 0^+} \lim_{\breve{\varepsilon} \to 0^+} \lim_{y \to 0^+} \frac{1}{\pi} \int_{-\breve{\varepsilon}^{-1}}^{-\varepsilon} \frac{\operatorname{Im} h(x + \mathrm{i}y)}{x^p} \, \mathrm{d}x + \lim_{\varepsilon \to 0^+} \lim_{\breve{\varepsilon} \to 0^+} \lim_{y \to 0^+} \frac{1}{\pi} \int_{\varepsilon}^{\breve{\varepsilon}^{-1}} \frac{\operatorname{Im} h(x + \mathrm{i}y)}{x^p} \, \mathrm{d}x$$
$$= \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^p}, \quad p = 0, \pm 1, \pm 2, \ldots$$

Here $\mu_0 = \mu - \mu(\{0\})\delta_0$, i.e., the measure in the representation (2.1) *with the point mass in the origin removed. The terms in the left-hand side are not necessarily finite. The right-hand side is not defined in the case the left-hand side equals $-\infty + \infty$.*

The proof can be found in Appendix A.3.

Before presenting the second result needed for the main theorem, it is noted that h may be decomposed as

$$h(z) = \beta z + \alpha - \frac{\mu(\{0\})}{z} + \int_{\mathbb{R}} \left(\frac{1}{\xi - z} - \frac{\xi}{1 + \xi^2} \right) \mathrm{d}\mu_0(\xi), \tag{4.4}$$

where once again $\mu_0 = \mu - \mu(\{0\})\delta_0$. This decomposition follows directly from the fact that $zh(z) \to -\mu(\{0\})$ as $z \hat{\to} 0$.

Lemma 4.2. *Let h be a Herglotz function given by* (2.1) *and $N \geq 0$ an integer. Then the following statements are equivalent:*

1. *The function h has the asymptotic expansion* (2.4)*, i.e.,*

$$h(z) = \sum_{n=-1}^{2N-1} a_n z^n + o(z^{2N-1}), \quad \text{as } |z| \to 0,$$

for z in the Stoltz domain $\theta \leq \arg z \leq \pi - \theta$ for any $\theta \in (0, \pi/2]$. Here all a_n are real.

2. Statement 1 is true for $\theta = \pi/2$.

3. The measure $\mu_0 = \mu - \mu(\{0\})\delta_0$ *satisfies*

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^{2N}(1+\xi^2)} < \infty.$$

The expansion coefficients in (2.4) *equal:*

$$a_0 = \alpha + \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi(1+\xi^2)}, \tag{4.5}$$

$$a_{p-1} = \delta_{p,2}\beta + \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^p}, \quad p = 2, 3, \ldots, 2N, \tag{4.6}$$

where $\delta_{i,j}$ *denotes the Kronecker delta.*

A similar result is a well-known theorem due to Hamburger and Nevanlinna [1, Theorem 3.2.1], [31, Theorem 2.2]. See also Lemma 6.1 in [17]. Note that the case $N = 0$ is trivial, since then all three statements are true for all Herglotz functions. The proof for $N \geq 1$ can be found in Appendix A.4. The convergence of $\int_{\mathbb{R}} \mathrm{d}\mu_0(\xi)/(|\xi^{2N+1}|(1+\xi^2))$ does guarantee an expansion with real coefficients up to $o(z^{2N})$, but the converse is not true. A counterexample for $N = 0$ is given by the measure $\mathrm{d}\mu_0(\xi) = \mu_0'(\xi)\,\mathrm{d}\xi$ where $\mu_0'(\xi) = -(\ln|\xi|)^{-1}$ when $\xi < 1$ and $\mu_0'(\xi) = 0$ otherwise.

The integral identities for $p = 2, 3, \ldots 2N$ follow directly from Corollary 4.1 and Lemma 4.2 (recall that $b_1 = \beta$ and that $b_{p-1} = 0$ for $p = 3, 4, \ldots$). To prove the identities for $p = 2 - 2M, 3 - 2M, \ldots, 1$, consider the Herglotz function $\breve{h}(z) = h(-1/z)$. With obvious notation, its high- and low-frequency asymptotic expansions are related to those of h as $\breve{b}_n = (-1)^n a_{-n}$ and $\breve{a}_n = (-1)^n b_{-n}$. Evidently, $\breve{M} = N$ and $\breve{N} = M$ applies. Following (4.4), $\breve{h}$ admits the representation

$$\breve{h}(z) = \frac{-\beta}{z} + \alpha + \mu(\{0\})z + \int_{\mathbb{R}} \frac{1+\xi z^{-1}}{1+\xi^2}\,\mathrm{d}\nu_0(\xi), \quad \operatorname{Im} z > 0,$$

where $\mathrm{d}\nu_0(\xi) = \mathrm{d}\mu_0(\xi)/(1+\xi^2)$. It would be desirable to make a change of variables $\xi \mapsto -1/\xi$ in the integral. Therefore, consider the continuous bijection $j : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ defined by $j\xi = -1/\xi$. It is its own inverse, *i.e.*, $j^2\xi = \xi$. Furthermore, it maps Borel sets to Borel sets, which makes the following a valid definition:

Definition 4.1. *Let* $j : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ *be the mapping that takes* ξ *to* $-1/\xi$. *Let* $\mathcal{E}(\mathbb{R}\backslash\{0\})$ *be the Borel sets of* $\mathbb{R}\backslash\{0\}$ *and* $\mathrm{M}(\mathbb{R}\backslash\{0\})$ *be the set of measures on* $\mathcal{E}(\mathbb{R}\backslash\{0\})$. *Define the mapping* $J : \mathrm{M}(\mathbb{R}\backslash\{0\}) \to \mathrm{M}(\mathbb{R}\backslash\{0\})$ *through*

$$J\sigma(E) = \sigma(jE),$$

for all $\sigma \in \mathrm{M}(\mathbb{R}\backslash\{0\})$ *and* $E \in \mathcal{E}(\mathbb{R}\backslash\{0\})$.

From this definition it is clear that $J^2\sigma = \sigma$ and moreover

$$\int_{\mathbb{R}\setminus\{0\}} f(\xi)\,\mathrm{d}\sigma(\xi) = \int_{\mathbb{R}\setminus\{0\}} f(j\xi)\,\mathrm{d}\,(J\sigma)(\xi)$$

for all measurable functions f on $\mathbb{R}\setminus\{0\}$, since it holds if f is a simple measurable function [30]. The representation of $\breve{h}$ can now be rewritten:

$$\breve{h}(z) = \frac{-\beta}{z} + \alpha + \mu(\{0\})z + \int_{\mathbb{R}} \frac{1-\xi z}{1+\xi^2}\,\mathrm{d}\,(J\nu_0)(\xi), \quad \operatorname{Im} z > 0.$$

The function $\breve{h}$ is thus represented by the measure $\mathrm{d}\breve{\nu}_0 = \mathrm{d}\,(J\nu_0)$, or equivalently $\mathrm{d}\breve{\mu}_0 = \xi^2\,\mathrm{d}\,(J\mu_0)$. Therefore

$$\lim_{y\to 0^+} \frac{1}{\pi}\int_{\varepsilon}^{\breve{\varepsilon}^{-1}} \frac{\operatorname{Im} h(x+\mathrm{i}y)}{x^p}\,\mathrm{d}x = \int_{\mathbb{R}} \breve{\varphi}_{p,\varepsilon,\breve{\varepsilon}}(\xi)\,\mathrm{d}\mu_0(\xi) = \int_{\mathbb{R}} \breve{\varphi}_{p,\varepsilon,\breve{\varepsilon}}(-1/\xi)\frac{\mathrm{d}\breve{\mu}_0(\xi)}{\xi^2}$$
$$= \lim_{y\to 0^+} (-1)^p \frac{1}{\pi}\int_{-\varepsilon^{-1}}^{-\breve{\varepsilon}} \frac{\operatorname{Im} \breve{h}(x+\mathrm{i}y)}{x^{2-p}}\,\mathrm{d}x, \quad \text{for } p = 0, \pm 1, \pm 2, \ldots \text{ and } 0 < \varepsilon < \breve{\varepsilon}^{-1}, \tag{4.7}$$

and likewise for the corresponding integral over $(-\breve{\varepsilon}^{-1}, -\varepsilon)$. Here $\breve{\varphi}_{p,\varepsilon,\breve{\varepsilon}}$ is given by (A.2) and (4.2). The proof of the integral identities (2.5) for $p = 2-2M, 3-2M, \ldots, 0$ have now been returned to the case $p = 2, 3, \ldots, 2N$. Here at last is the sought for theorem:

Theorem 4.1 (Main Theorem)**.** *Let h be a Herglotz function. Then it has the asymptotic expansions* (2.3) *and* (2.4) *if and only if the corresponding left-hand sides in* (2.5) *are absolutely convergent. In this case the integral identities* (2.5) *apply.*

The integrals in the left-hand side of (2.5) may be taken over the set $\{x : \varepsilon < |x| < \infty\}$ when $p = 2, 3, \ldots, 2N$ and $\{x : 0 < |x| < \varepsilon^{-1}\}$ when $p = 2-2M, 3-2M, \ldots, 0$, see Appendix A.3. In this case there is an extra term $-\delta_{p,0}\, a_{-1}$ in the right-hand side. This fact is used in the examples below to obtain neater expressions.

Proof. The theorem for $p = 2, 3, \ldots 2N$ follows directly from Corollary 4.1 and Lemma 4.2. For $p = 2-2M, 3-2M, \ldots, 0$ it also requires (4.7) and the relation between the asymptotic expansions of h and $\breve{h}$.

The case $p = 1$ is special as it requires both high- and low-frequency expansions. Assume that the asymptotic expansions (2.3) and (2.4) are valid for $N = M = 1$ and use equation (4.5) for h and $\breve{h}$:

$$a_0 - b_0 = (a_0 - \alpha) - (\breve{a}_0 - \breve{\alpha}) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi(1+\xi^2)} - \int_{\mathbb{R}} \frac{\mathrm{d}\breve{\mu}_0(\xi)}{\xi(1+\xi^2)}$$
$$= \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi(1+\xi^2)} - \int_{\mathbb{R}} \frac{\xi\,\mathrm{d}\mu_0(\xi)}{1+\xi^2} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi}$$
$$= \lim_{\varepsilon\to 0^+}\lim_{\breve{\varepsilon}\to 0^+}\lim_{y\to 0^+} \int_{\varepsilon<|x|<\breve{\varepsilon}^{-1}} \frac{\operatorname{Im} h(x+\mathrm{i}y)}{x}\,\mathrm{d}x.$$

Here all integrals are absolutely convergent. If on the other hand the left-hand sides of (2.5) are absolutely convergent for $p = 0, 1, 2$, then the asymptotic expansions (2.3) and (2.4) clearly hold for $N = 1$ and $M = 1$, respectively. $\square$

5 Examples

5.1 Elementary Herglotz functions

Examples of elementary Herglotz functions are

$$\beta z, \quad C, \quad \frac{-\beta}{z}, \quad \sqrt{z}, \quad \log(z), \quad \mathrm{i}\log(1-\mathrm{i}z),$$

with $\beta \geq 0$, $\operatorname{Im} C \geq 0$, and appropriate branch cuts for $\sqrt{}$ and log.

Herglotz functions are related to the unit ball of the Hardy space $\mathrm{H}^\infty(\mathbb{C}^+)$ via the Cayley transform. An example is $\mathrm{e}^{\mathrm{i}z}$ which shows that

$$h_{\mathrm{e}}(z) = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}z} + \mathrm{i}}{1 - \mathrm{e}^{\mathrm{i}z}}$$

is a Herglotz function. Therefore $\tan z = -1/h_{\mathrm{e}}(2z)$ is a Herglotz function as well. It satisfies the symmetry (3.8) and its asymptotic expansions are $\tan z = \mathrm{i} + o(1)$, as $z \hat{\to} \infty$, and

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \dots, \qquad \text{as } z \to 0,$$

respectively. Note that the integer-order terms in the low-frequency asymptotic expansion are infinite in number since $\tan z$ is holomorphic in a neighbourhood of the origin. Thus there are identities (3.9) for $\hat{p} = 1, 2, \dots$:

$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0} \frac{2}{\pi} \int_\varepsilon^\infty \frac{\operatorname{Im} \tan(x + \mathrm{i}y)}{x^{2\hat{p}}} \, \mathrm{d}x = \begin{cases} 1 & \text{for } \hat{p} = 1 \\ 1/3 & \text{for } \hat{p} = 2 \\ 2/15 & \text{for } \hat{p} = 3 \\ \vdots \end{cases}$$

On the real axis except for $x = n\pi$, where $n = 0, \pm 1, \pm 2, \dots$, $\tan(x)$ is C^∞ and $\operatorname{Im}\tan(x) = 0$. It is not locally integrable around $x = n\pi$, where $\tan z$ has simple poles. There is an essential singularity at ∞, and the limit as $x \to \infty$ of $\tan(x)/x^{2\hat{p}}$ is not defined for any $\hat{p}$. This is thus an illustration of a case where it is difficult to use Cauchy integrals or Hilbert transform techniques to derive integral identities of the form (2.5).

If h_1 and h_2 are Herglotz functions, then so is the composition $h_2 \circ h_1$ (unless $h_1 \equiv \alpha \in \mathbb{R}$). This may be used to derive families of integral identities. Continue the example with $h_1 = \tan z$ and construct the new Herglotz function

$$\mathrm{i}\log(1 - \mathrm{i}\tan z) = \begin{cases} z + \mathcal{O}(1), & \text{as } z \to 0 \\ \mathcal{O}(1), & \text{as } z \hat{\to} \infty, \end{cases}$$

yielding an identity of the type (3.9):

$$\lim_{\varepsilon\to 0^+}\lim_{y\to 0^+}\frac{2}{\pi}\int_{\varepsilon}^{\infty}\frac{\ln|1-\mathrm{i}\tan(x+\mathrm{i}y)|}{x^2}\,\mathrm{d}x = 1.$$

It is also illustrative to consider a case with odd weighting factors in (2.5). The function $\ln(1+\tan(z))$ has the asymptotic expansions

$$\ln(1+\tan(z)) = \begin{cases} z - z^2/2 + 2z^3/3 + \dots, & \text{as } z\to 0\\ \mathcal{O}(1), \quad \text{as } z\hat{\to}\infty. \end{cases}$$

This gives the (2.5)-identities

$$\lim_{\varepsilon\to 0^+}\lim_{y\to 0^+}\frac{1}{\pi}\int_{|x|>\varepsilon}\frac{\arg(1+\tan(x+\mathrm{i}y))}{x^p}\,\mathrm{d}x = \begin{cases} 1 \quad \text{for } p=2\\ -1/2 \quad \text{for } p=3\\ 2/3 \quad \text{for } p=4\\ \vdots \end{cases}$$

where it is observed that the negative part of the integrand dominates for $p=3$. There are other manipulations of Herglotz functions that generate new Herglotz functions as well, *e.g.*, h_1+h_2 and $\sqrt{h_1h_2}$.

5.2 Lossless resonance circuit

Consider a parallel resonance circuit consisting of a lumped inductance, L, and a lumped capacitance, C, see Figure 1. This is an example of an admittance-passive system, where the impedance $Z(s) = sL/(1+s^2LC)$ is the Laplace-transfer function of the system in which the electric current over Z is the input and the voltage is the output. Therefore the transfer function given by (3.2) multiplied by i is a Herglotz function:

$$h(\omega) = \mathrm{i}Z(-\mathrm{i}\omega) = -\frac{\omega_0^2 L}{2}\left(\frac{1}{\omega-\omega_0}+\frac{1}{\omega+\omega_0}\right) = \begin{cases}\sqrt{\frac{L}{C}}\sum_{n=0}^{\infty}\frac{\omega^{2n+1}}{\omega_0^{2n+1}}, & \text{as } \omega\to 0\\ -\sqrt{\frac{L}{C}}\sum_{n=0}^{\infty}\frac{\omega_0^{2n+1}}{\omega^{2n+1}}, & \text{as } \omega\to\infty,\end{cases}$$

where $\omega_0 = 1/\sqrt{LC}$ is the resonance frequency of the LC circuit. In general, the imaginary part of $h(\omega) = \mathrm{i}Z(-\mathrm{i}\omega)$ corresponds to the power absorbed by the impedance Z.

Use of the identities (3.9) gives the sum rules

$$\lim_{\varepsilon\to 0^+}\lim_{\omega''\to 0^+}\frac{2}{\pi}\int_{\varepsilon}^{\varepsilon^{-1}}\frac{\operatorname{Im} h(\omega'+\mathrm{i}\omega'')}{\omega'^{2p}}\,\mathrm{d}\omega' = \sqrt{\frac{L}{C}}\omega_0^{-2p+1}, \quad \text{for } p = 0, \pm1, \pm2, \dots \tag{5.1}$$

Note that on the real axis $\operatorname{Im} h(\omega') = 0$ for $\omega' \neq \pm\omega_0$. All of the contribution to the integral comes from the singularity, which becomes clear if the left-hand side of (5.1) is calculated explicitly. A physical interpretation is that even though the circuit is lossless for any frequency $\omega' \neq \omega_0$, input signals of frequency $\omega' = \omega_0$ are "trapped" in its resonance and thus absorbed by Z.

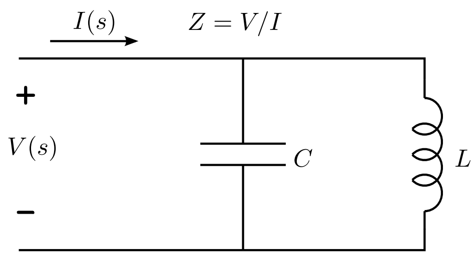

Figure 1: The lossless resonance circuit of Example 5.2.

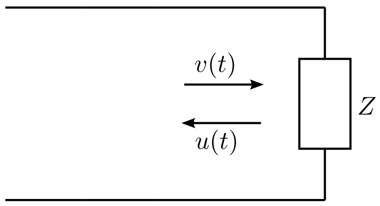

Figure 2: The voltage waves traveling along the transmission line has the amplitudes $v(t)$ and $u(t)$, respectively, measured by the load.

5.3 Reflection coefficient (Fano's matching equations revisited)

Consider a transmission line ended in a load impedance. The transmission line is assumed to be distortionless, *i.e.*, its characteristic impedance is not a function of frequency. Normalise so that the characteristic impedance of the transmission line is 1 and the lumped impedance is $Z(s)$, where $s = -\mathrm{i}\omega$ denotes the Laplace parameter. The load impedance is assumed to be realisable with a finite number of linear passive elements (but otherwise arbitrary), so Z is a rational function.

The reflection coefficient $\rho(s) = (Z(s) - 1)/(Z(s) + 1)$ is of interest, since it determines the power rejected by the load. It is the Laplace-transfer function of the system where the input v and output u are the amplitudes of the voltage waves travelling along the transmission line toward or from the load, respectively. See Figure 2. The Fourier transfer function is $\tilde{w}(\omega) = \rho(-\mathrm{i}\omega)$, satisfying (3.7). This is clearly a scatter-passive system, so $\tilde{w}(\omega)$ is holomorphic and bounded in magnitude by one in $\mathbb{C}^+$.

Assume the asymptotic expansion

$$-\mathrm{i}\log(\tilde{w}(\omega)) = \arg\tilde{w}(0) + c_1\omega + c_3\omega^3 + \ldots + c_{2N-1}\omega^{2N-1} + o(\omega^{2N-1}), \quad \text{as } \omega \to 0, \tag{5.2}$$

where $\arg\tilde{w}(0) = \lim_{\omega \hat{\to} 0}\arg\tilde{w}(\omega)$ and all c_i are real. This is the case *e.g.*, if the impedance Z can be represented as a lossless network terminated in another impedance, Z_2 (*cf.*, Figure 3), and the network has a transmission zero of order N at $\omega = 0$ [10]. The low-frequency asymptotic expansion of the Herglotz-function in

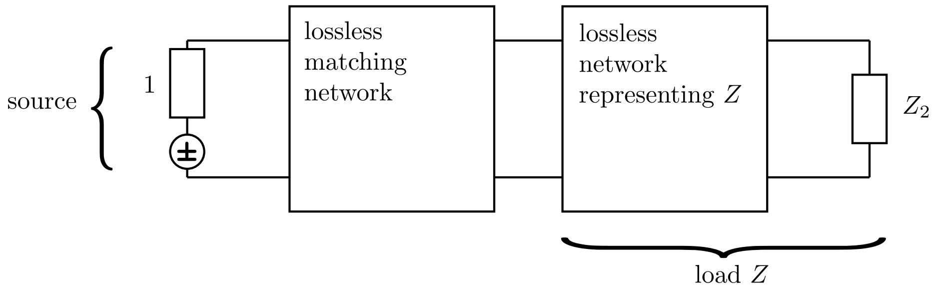

Figure 3: The matching problem as described in [10].

(3.4) is

$$h(\omega) = \arg \tilde{w}(0) + c_1\omega + c_3\omega^3 + \ldots + c_{2N-1}\omega^{2N-1} + o(\omega^{2N-1}) \\ - \arg B(0) - 2 \sum_{m=1,3,\ldots}^{\infty} \frac{\omega^m}{m} \sum_{\omega_n} \operatorname{Im} \omega_n^{-m}, \quad \text{as } \omega \to 0,$$

according to (3.6). In this case only odd terms appear in the sum originating from the Blaschke product due to the symmetry (3.7). The high-frequency asymptotic expansion of h is $o(\omega)$ since $\tilde{w}$ is a rational function. This implies the (3.9)-identities

$$\lim_{\varepsilon\to0^+} \lim_{\omega''\to0^+} \frac{2}{\pi} \int_{\varepsilon}^{\infty} \frac{-\ln|\tilde{w}(\omega' + \mathrm{i}\omega'')|}{\omega'^{2\hat{p}}} \,\mathrm{d}\omega' \\ = c_{2\hat{p}-1} - \frac{2}{2\hat{p}-1} \sum_{\omega_n} \operatorname{Im} \omega_n^{1-2\hat{p}}, \quad \text{for } \hat{p} = 1, 2, \ldots, N.$$

If ρ has no zeros at the imaginary axis, the limit as $\omega'' \to 0^+$ may be moved inside the integral. These are the original Fano matching equations, derived with the Cauchy integral formula in [10]. In said paper they are used to derive the best possible match of a source to a load over an open frequency interval, and how the lossless matching network should be constructed to obtain this best match. See Figure 3. When ρ is not a rational function (consider *e.g.*, the scattering of electromagnetic waves by a permittive object), the Cauchy integral formula-approach falls short. Theorem 4.1 guarantees integral identities as long as asymptotic expansions of the type (5.2) are valid as $\omega \hat{\to} 0$ and/or $\omega \hat{\to} \infty$, respectively. It should be mentioned that Fano's results have been treated more generally also in *e.g.*, [5].

5.4 Kramers-Kronig relations and ϵ near-zero materials

Suppose there is an isotropic constitutive relation on convolution form relating the electric field $\boldsymbol{E} = E\hat{\boldsymbol{e}}$ to the electric displacement $\boldsymbol{D} = D\hat{\boldsymbol{e}}$ [24]:

$$D(t) = \epsilon_0 \chi * E(t). \tag{5.3}$$

The permittivity of free space is denoted ϵ_0, and a possible instantaneous response is included in $\chi(t)$ as a term $\epsilon_\infty\delta(t)$, where $\epsilon_\infty \geq 0$. Let the input be $v(t) = \epsilon_0 E(t)$ and the output be $u(t) = \partial D/\partial t$. The impulse response of this system is $w(t) = \partial \chi/\partial t$. The system is admittance-passive if the material is passive, since that means that the energy expression [24]

$$e(T) = \int_{-\infty}^{T} E(t) \frac{\partial D}{\partial t} \,\mathrm{d}t$$

is non-negative for all $E \in \mathcal{D}$ and $T \in \mathbb{R}$. The Herglotz function given by $h = \mathrm{i}\tilde{w}$ is $h(\omega) = \omega\epsilon(\omega)$, where $\epsilon(\omega) = \mathcal{F}\chi(\omega)$. It satisfies the symmetry (3.8), since $w(t)$ is assumed to be real.

Lemma 4.1 may be applied to the representation (2.1), since $|1/(\xi - z) - \xi/(1 + \xi^2)| \leq D_z/(1 + \xi^2)$ for any fixed $z \in \mathbb{C}^+$. This gives

$$\omega\epsilon(\omega) = \omega\epsilon_\infty + \lim_{\psi\to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} \left[\frac{1}{\xi - \omega} - \frac{\xi}{1+\xi^2} \right] \Big[\psi \operatorname{Re}\epsilon(\xi + \mathrm{i}\psi) + \xi \operatorname{Im}\epsilon(\xi + \mathrm{i}\psi) \Big] \,\mathrm{d}\xi, \quad (5.4)$$

for $\operatorname{Im}\omega > 0$. This is one of the two Kramers-Kronig relations [22, 24] in a general form, where no assumptions other than those of convolution form and passivity has been made for the constitutive relation in the time-domain. It may be simplified if $\epsilon(\omega') = \lim_{\omega''\to 0^+} \epsilon(\omega' + \mathrm{i}\omega'')$ is sufficiently well-behaved. Here the notation $\omega' = \operatorname{Re}\omega$ and $\omega'' = \operatorname{Im}\omega$ has been used. If for instance $\epsilon(\omega')$ is a continuous and bounded function, the limit may be moved inside the integral in (5.4):

$$\omega\epsilon(\omega) = \omega\epsilon_\infty + \frac{1}{\pi} \int_{\mathbb{R}} \left[\frac{1}{\xi - \omega} - \frac{\xi}{1+\xi^2} \right] \xi \operatorname{Im}\epsilon(\xi) \,\mathrm{d}\xi, \quad \operatorname{Im}\omega > 0.$$

Assuming that $\operatorname{Im}\epsilon(\omega') = \mathcal{O}(1/\omega')$ as $\omega' \to \pm\infty$ and employing the fact that $\operatorname{Im}\epsilon(\omega')$ is odd gives (after division with ω)

$$\epsilon(\omega) = \epsilon_\infty + \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\xi - \omega} \operatorname{Im}\epsilon(\xi) \,\mathrm{d}\xi, \quad \operatorname{Im}\omega > 0.$$

Letting $\omega'' \to 0$ and using the distributional limit $\lim_{\omega''\to 0}(\xi - \omega' - \mathrm{i}\omega'')^{-1} = \mathcal{P}(\xi - \omega')^{-1} + \mathrm{i}\pi\delta(\xi - \omega')$, where $\mathcal{P}$ is the Cauchy principal value, yields

$$\epsilon(\omega') = \epsilon_\infty + \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|\xi-\omega'|>\varepsilon} \frac{\operatorname{Im}\epsilon(\xi)}{\xi - \omega'} \,\mathrm{d}\xi + \mathrm{i}\operatorname{Im}\epsilon(\omega').$$

The real part of this equation is the Kramers-Kronig relation (5.4) as presented in *e.g.*, [24]:

$$\operatorname{Re}\epsilon(\omega') = \epsilon_\infty + \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|\xi-\omega'|>\varepsilon} \frac{\operatorname{Im}\epsilon(\xi)}{\xi - \omega'} \,\mathrm{d}\xi.$$

The assumption that $\epsilon(\omega')$ is continuous rules out the possibility of static conductivity, which however can be included with a small modification of the arguments. Assuming that $h(\omega) = \omega\epsilon(0) + o(\omega)$, as $\omega \hat{\to} 0$, there is a sum rule of the type (3.9) for $\hat{p} = 1$ (also presented in *e.g.*, [24]):

$$\lim_{\varepsilon\to 0^+} \frac{2}{\pi} \int_{\varepsilon}^{\infty} \frac{\operatorname{Im}\epsilon(\omega')}{\omega'} \,\mathrm{d}\omega' = \epsilon(0) - \epsilon_\infty.$$

It shows that the losses are related to the difference between the static and instantaneous responses of the medium. The Kramers-Kronig relations and their connection to Herglotz functions are also discussed in [22, 27, 36, 38].

In applications such as high-impedance surfaces and waveguides, it is desirable to have so called ϵ near-zero materials [32], *i.e.*, materials with $\epsilon(\omega') \approx 0$ in a frequency interval around some center frequency ω_0. Define the Herglotz function

$$h_1(\omega) = \frac{\omega}{\omega_0}\epsilon(\omega) = \begin{cases} o(\omega^{-1}), & \text{as } \omega \hat{\to} 0 \\ \frac{\omega}{\omega_0}\epsilon_\infty + o(\omega), & \text{as } \omega \hat{\to} \infty. \end{cases} \tag{5.5}$$

Compositions of Herglotz functions may be used to derive limitations different from those that h_1 would produce on its own. In the present case the area of interest is the frequency region where $h_1(\omega) \approx 0$. A promising function is

$$h_\Delta(z) = \frac{1}{\pi}\int_{-\Delta}^{\Delta} \frac{1}{\xi - z}\,\mathrm{d}\xi = \frac{1}{\pi}\ln\frac{z-\Delta}{z+\Delta} = \begin{cases} \mathrm{i} + o(1), & \text{as } \omega \to 0 \\ \frac{-2\Delta}{\pi z} + o(z^{-1}), & \text{as } \omega \to \infty, \end{cases} \tag{5.6}$$

designed such that $\operatorname{Im} h_\Delta(z) \approx 1$ for $\operatorname{Im} z \approx 0$ and $|\operatorname{Re} z| \leq \Delta$, see Figure 4. Here the logarithm has its branch cut along the negative imaginary axis. The asymptotic expansions of the composition are

$$h_\Delta(h_1(\omega)) = \begin{cases} \mathcal{O}(1), & \text{as } \omega \hat{\to} 0 \\ \frac{-2\omega_0\Delta}{\omega\pi\epsilon_\infty} + o(\omega^{-1}), & \text{as } \omega \hat{\to} \infty, \end{cases}$$

yielding the following sum rule for $\hat{p} = 0$:

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\lim_{\omega''\to 0^+}\int_0^{\varepsilon^{-1}} \operatorname{Im} h_\Delta(h_1(\omega' + \mathrm{i}\omega''))\,\mathrm{d}\omega' \\ &\quad = \lim_{\varepsilon\to 0^+}\lim_{\omega''\to 0^+}\int_0^{\varepsilon^{-1}} \arg\left(\frac{(\omega'+\mathrm{i}\omega'')\epsilon_\infty - \Delta\omega_0}{(\omega'+\mathrm{i}\omega'')\epsilon_\infty - \Delta\omega_0}\right)\mathrm{d}\omega' = \frac{\omega_0\Delta}{\epsilon_\infty}. \end{aligned} \tag{5.7}$$

An illustration of $\lim_{\omega''\to 0} \operatorname{Im} h_\Delta(h_1(\omega' + \mathrm{i}\omega''))$ for a permittivity function ϵ described by a Drude model can be found in Figure 5.

Let the frequency interval be $\mathcal{B} = [\omega_0(1-B_\mathrm{F}/2), \omega_0(1+B_\mathrm{F}/2)]$, where B_F denotes the fractional bandwidth. Assume that $h_1(\omega') = \lim_{\omega''\to 0^+} h_1(\omega' + \mathrm{i}\omega'')$ exists finitely in this interval and let $\Delta = \sup_{\omega'\in\mathcal{B}} |h_1(\omega')|$. Then $\inf_{\omega'\in\mathcal{B}} \lim_{\omega''\to 0^+} \operatorname{Im} h_\Delta(h_1(\omega' + \mathrm{i}\omega'')) \geq 1/2$ which yields the bound

$$\sup_{\omega'\in\mathcal{B}} |h_1(\omega')| \geq \frac{B_\mathrm{F}}{2}\epsilon_\infty$$

or

$$\sup_{\omega'\in\mathcal{B}} |\epsilon(\omega')| \geq \frac{B_\mathrm{F}}{2+B_\mathrm{F}}\epsilon_\infty.$$

This shows that ϵ near-zero materials are dispersive and that the deviation from zero is proportional to the fractional bandwidth when $B_\mathrm{F} \ll 1$.

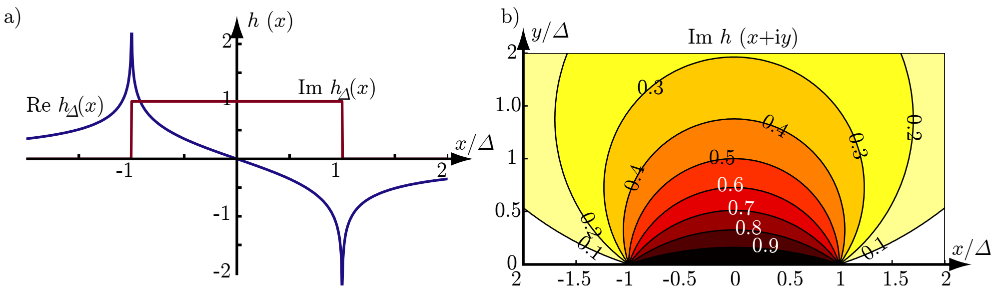

Figure 4: The function $h_\Delta(x+\mathrm{i}y)$ given by (5.6) illustrated by its limit as $y \to 0$ to the left and the contours of $\operatorname{Im} h_\Delta(x+\mathrm{i}y)$ to the right.

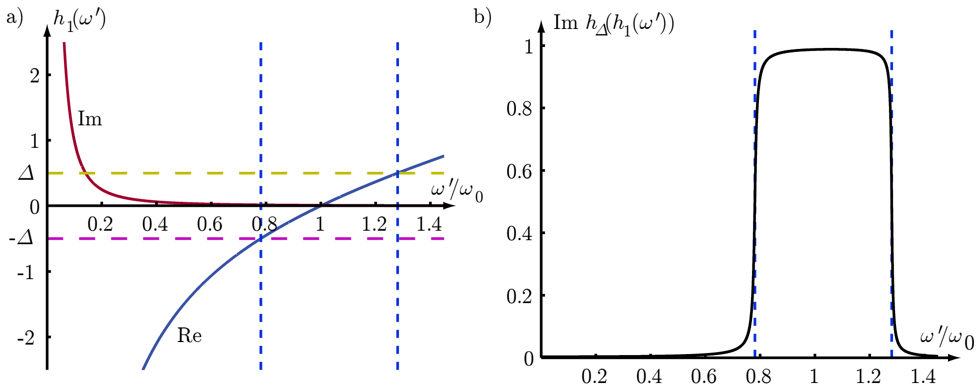

Figure 5: The left figure depicts the real and imaginary part of $h_1(\omega') = \lim_{\omega''\to 0} h_1(\omega' + \mathrm{i}\omega'')$, where h_1 is given by (5.5) and the permittivity is described by the Drude model $\epsilon(\omega) = 1 - (\omega/\omega_0\,(\omega/\omega_0 - 0.01\mathrm{i}))^{-1}$. The right figure depicts the integrand $\operatorname{Im} h_\Delta(h_1(\omega')) = \lim_{\omega''\to 0} \operatorname{Im} h_\Delta(h_1(\omega' + \mathrm{i}\omega''))$ in (5.7) for this choice of $\epsilon(\omega)$ and $\Delta = 1/2$.

5.5 Extinction cross section

This example revisits a set of sum rules for the extinction cross sections of certain passive scattering objects. The sum rules were first presented for linearly polarized waves in [33], and later generalized to elliptical polarizations in [13]. A time-domain approach to derive them was adopted in [11]. Here they are reviewed in the special case of a spherically symmetric scatterer; the material properties of the scatterer considered is only dependent on the distance r from the origin in the center of the sphere. Furthermore, the isotropic constitutive relation for the electric flux density in the object is on convolution form as described in (5.3), and the material is passive. For simplicity, the sphere is assumed to be non-magnetic and surrounded by free space.

Let a plane electromagnetic wave, propagating in the $\hat{\boldsymbol{k}}$-direction, impinge on the sphere. The electric field of such a plane wave in the time domain is $\boldsymbol{E}_\mathrm{i}(t,\boldsymbol{r}) = \boldsymbol{E}_0(t - \boldsymbol{r}\cdot\hat{\boldsymbol{k}}/\mathrm{c})$. Here $\boldsymbol{r}$ denotes the spatial coordinate, $\hat{\boldsymbol{k}}$ is of unit length, and c denotes the speed of light in free space. The electric field in the frequency domain may be written $\widetilde{\boldsymbol{E}}_{\mathrm{i}}(k) = \mathrm{e}^{\mathrm{i}\boldsymbol{r}\cdot k\hat{\boldsymbol{k}}}\widetilde{\boldsymbol{E}}_0(k)$, where the wavenumber $k = \omega/\mathrm{c}$ is used instead of the angular frequency ω.

The extinction cross section $\sigma_{\mathrm{e}}(k)$ is a measure of the amount of energy in the incoming wave that is scattered or absorbed when the wave interacts with the sphere:

$$e_{\mathrm{e}}(\infty) = \frac{\mathrm{c}}{2\pi\eta_0}\int_{-\infty}^{\infty} \sigma_{\mathrm{e}}(k)|\widetilde{\boldsymbol{E}}_{\mathrm{i}}(k)|^2 \,\mathrm{d}k.$$

Here η_0 is the wave impedance of free space. The extinct energy, and hence also the extinction cross section, must be non-negative when the material of the sphere is passive. In fact, $\sigma_{\mathrm{e}}(k)$ is given by the imaginary part of a Herglotz function $h(k)$ due to the optical theorem [11, 13, 33]:

$$\sigma_{\mathrm{e}}(k) = \operatorname{Im} h(k).$$

In turn, the Herglotz function is given by

$$h(k) = \frac{4\pi}{k}\tilde{S}(k;0),$$

where $\tilde{S}(k;0)$ describes the scattered field in the forward direction. This Herglotz function satisfies the symmetry (3.8).

For most materials, it can be argued that $h(k) = \mathcal{O}(1)$ as $k \hat{\to} \infty$ [11, 33]. If the sphere is coated with metal (or some other material with static conductivity), then the low-frequency behaviour of $h(k)$ is described by

$$h(k) = 4\pi a^3 k + \mathcal{O}(k^2), \quad \text{as } k \to 0,$$

where a is the outer radius. Note that the dominating term does not depend on the type of metal used. Consequently, the following sum rule applies for the extinction cross section of a sphere coated with metal:

$$\lim_{\varepsilon\to 0^+}\lim_{k''\to 0^+}\int_{\varepsilon}^{\infty} \frac{\sigma_{\mathrm{e}}(k' + \mathrm{i}k'')}{k'^2}\,\mathrm{d}k' = 2\pi^2 a^3.$$

Alternatively, express the extinction cross section as a function of the wavelength, $\lambda = 2\pi/k$:

$$\lim_{\varepsilon\to 0^+}\lim_{\lambda''\to 0^+}\int_0^{\varepsilon^{-1}} \sigma_{\mathrm{e},\lambda}(\lambda' - \mathrm{i}\lambda'')\,\mathrm{d}\lambda' = 4\pi^3 a^3. \tag{5.8}$$

To exemplify the sum rule (5.8), consider the spherical nanoshells depicted in Figure 6. A nanoshell is a dielectric core covered by a thin coat of metal, used for instance for biomedical imaging or treatment of tumours. Depending on the application, the core radius, shell thickness, and materials are varied to make the nanoshells scatter or absorb different parts of the visible light and near-infrared (NIR) spectra. In [7, 28], the nanoshells are spherical cores of silicon dioxide (SiO_2) covered with gold. The radius of the core is typically around 60 nm, and the gold shell is $5 - 20$ nm thick. The extinction cross sections for four such spheres are plotted in Figure 6. Following the sum rule (5.8), the integrated extinction for any nanoshells is $4\pi^3 a^3$. This is confirmed by a numerical integration.

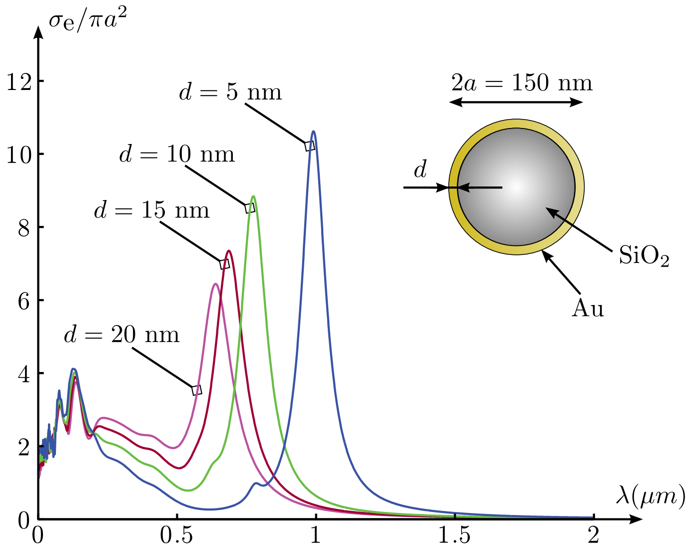

Figure 6: The normalised extinction cross section for four nanoshells, consisting of spherical silicon dioxide (SiO_2) cores with coats of gold. The outer radius is $a = 75\,\text{nm}$ and the shell thicknesses is $d = 5, 10, 15$ and $20\,\text{nm}$, respectively. The extinction cross section σ_e was calculated from a closed form expression, using a Matlab-script for a Lorentz-Drude model for gold by Ung et al. [35]. The silicon dioxide core is modeled as being lossless with a constant complex permittivity $\epsilon(\omega) \equiv 2.25$, which is a good model at least for wavelengths 0.4–1.1 μm [23]. Following the sum rule (5.8), the integrated extinction for all four nanoshells is $4\pi^3 a^3$, which is confirmed by a numerical integration.

6 Conclusions

Many physical systems are modeled as a rule that assigns an output signal to every input signal. It is often natural to let the space of admissible input signals be some subset of the space of distributions, since generalized functions such as the delta function should be allowed. Under the general assumptions of linearity, continuity and time-translational invariance, such a system is on convolution form, and thus fully described by its impulse response. The assumption of passivity (and thereby causality, as described in Section 3), imply that the transfer function is related to a Herglotz function [37, 39, 41]. In many areas it is convenient to analyse systems in the frequency domain, where the transfer function plays the role of the impulse response.

A set of integral identities for Herglotz functions is presented and proved in this paper, showing that weighted integrals of Herglotz functions over infinite intervals are determined by their high- and low-frequency asymptotic expansions. The identities rely on a well-known representation theorem for Herglotz functions [2], and furthermore makes use of results from the classical problem of moments [1, 31].

The integral identities make possible a general approach to derive sum rules for passive systems. The first step is to use the assumptions listed above to assure that the transfer function is related to a Hergloz function, h. Secondly, the low- and/or high-frequency asymptotic expansions of h must be determined. Finally, physical limitations may be derived by considering finite frequency intervals. The sum rules effectively relate dynamic behaviour to static and/or high frequency properties, which must be found by physical arguments. However, since static properties are often easier to determine than dynamical behaviour in various applications, this is beneficial. The physical limitations indicate what can and cannot be expected from certain physical systems.

Sum rules, or more general dispersion relations, and physical limitations, have been widely used in *e.g.*, electromagnetic theory. Two famous examples are the Kramers-Kronig relations for the frequency dependence of the electric permittivity [22, 24], discussed in Example 5.4, and Fano's matching equations [10], considered in Example 5.3. There are more recent examples as well, see *e.g.*, [4, 12, 14–16, 29, 33].

For many causal systems on convolution form, dispersion relations in the form of a Hilbert transform pair follow from Titchmarsh's theorem [21, 22, 27]. Sometimes, sum rules can be derived from the dispersion relations [22]. Many previous papers use the Cauchy integral formula, see *e.g.*, [10, 34]. This approach demands *e.g.*, that the transfer function $\tilde{w}$ is rational. The present paper seems to be the first to describe and rigorously prove a general approach to obtain sum rules for systems on convolution form under the assumption of passivity. It should be stressed that since the different approaches works under different assumptions, they are complementary rather than in competition. One advantage of the Herglotz function-approach presented in this paper is that a wide range of physical systems obey passivity. Another advantage is that it gives an insight into how compositions of Herglotz functions may be used to derive new physical limitations, see Example 5.4.

Acknowledgments

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Appendix A Proofs

A.1 Calculation of the limits $\lim_{z \hat{\to} \infty} h(z)/z$ and $\lim_{z \hat{\to} 0} zh(z)$

For all z in the Stoltz domain $\theta \leq \arg z \leq \pi - \theta$, $|\xi - z|$ is greater than or equal to both $|z| \sin\theta$ and $|\xi| \sin\theta$. See Figure 7. Thus

$$\frac{|1+\xi z|}{|z(\xi - z)|} \leq \frac{1 + 1/|z|^2}{\sin\theta},$$

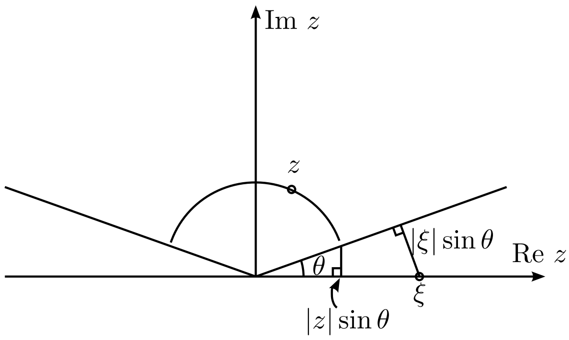

Figure 7: The Stoltz domain, $\{z : \theta \leq \arg z \leq \pi - \theta\}$ for some $\theta \in (0, \pi/2]$.

and (2.2) implies that

$$\lim_{z \hat{\to} \infty} \frac{h(z)}{z} = \beta + \lim_{z \hat{\to} \infty} \int_{\mathbb{R}} \frac{1 + \xi z}{z(\xi - z)} \, \mathrm{d}\nu(\xi) = \beta,$$

where Theorem A.2 has been used to move the limit inside the integral. Likewise, $|z(1 + \xi z)|/|\xi - z| \leq (1 + |z|^2)/\sin\theta$, which together with Theorem A.2 gives

$$\lim_{z \hat{\to} 0} z h(z) = \lim_{z \hat{\to} 0} \int_{\mathbb{R}} \frac{z(1 + \xi z)}{\xi - z} \, \mathrm{d}\nu(\xi) = -\nu(\{0\}) = -\mu(\{0\}).$$

A.2 Proof of Lemma 4.1

The left-hand side of (4.1) is

$$\lim_{y \to 0^+} \int_{\mathbb{R}} \varphi(x) \left(\beta y + \int_{\mathbb{R}} \frac{y}{(x - \xi)^2 + y^2} \, \mathrm{d}\mu(\xi) \right) \mathrm{d}x$$

$$= \lim_{y \to 0^+} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \frac{y}{(x - \xi)^2 + y^2} \, \mathrm{d}x \, \mathrm{d}\mu(\xi).$$

Here Fubini's Theorem [30, pp. 164–165] has been used to change the order of integration.

Theorem A.2 is used to show that the order of the limit and the integrals may be interchanged. First set

$$f_y(\xi) = \int_{\mathbb{R}} \varphi(x) \frac{y}{(x - \xi)^2 + y^2} \, \mathrm{d}x.$$

To find an integrable majorant $g \in \mathrm{L}^1(\mu)$ such that $|f_y(\xi)| \leq g(\xi)$ for all $\xi \in \mathbb{R}$ and $y \geq 0$, handle the cases $|\xi| < 2$ and $|\xi| \geq 2$ separately. For $|\xi| < 2$, the boundedness of φ guarantees that

$$|f_y(\xi)| \leq \int_{\mathbb{R}} D \frac{y}{(x - \xi)^2 + y^2} \, \mathrm{d}x = D\pi.$$

For $|\xi| \geq 2$, divide the integral into $|x-\xi| < 1$ and $|x-\xi| \geq 1$:

$$\left| \int_{|x-\xi|<1} \varphi(x) \frac{y}{(x-\xi)^2 + y^2} \, \mathrm{d}x \right| \leq \frac{2D}{\xi^2+1} \int_{\mathbb{R}} \frac{y}{(x-\xi)^2+y^2} \, \mathrm{d}x = \frac{2\pi D}{\xi^2+1}$$

and

$$\begin{aligned} \left| \int_{|x-\xi|\geq 1} \varphi(x) \frac{y}{(x-\xi)^2+y^2} \, \mathrm{d}x \right| &\leq \int_{|x-\xi|\geq 1} \frac{D}{1+x^2} \frac{y}{(x-\xi)^2} \, \mathrm{d}x \\ &= Dy \left[\frac{\xi}{(\xi^2+1)^2} \ln \left| \frac{(\xi-1)^2+1}{(\xi+1)^2+1} \right| + \frac{2}{1+\xi^2} + \frac{\xi^2-1}{(\xi^2+1)^2} \frac{\pi}{2} \right] \leq \frac{D_1 y}{\xi^2+1}. \end{aligned}$$

Summing up, for all y less than some arbitrary constant there is a constant $D_2 \geq 0$ such that

$$|f_y(\xi)| \leq g(\xi) = \frac{D_2}{\xi^2+1},$$

which is an integrable majorant. Since $\lim_{y\to 0^+} f_y(\xi)$ exists for all $\xi \in \mathbb{R}$ (shown below), the conditions of Theorem A.2 are fulfilled, and the limit may be moved inside the first integral.

Now let

$$f_{y,\xi}(x) = (\varphi(x) - \varphi(\xi)) \frac{y}{(x-\xi)^2+y^2}.$$

First suppose that ξ is not a point of discontinuity for $\varphi(\xi)$, so that there is some $K > 0$ such that $\varphi(x)$ is continuous for $x \in [\xi - K, \xi + K]$. The constant K may be chosen so that φ is continuously differentiable in said interval, except possibly at the point $x = \xi$. For $x \in [\xi - K, \xi + K]$,

$$|f_{y,\xi}(x)| \leq \max_{|\zeta-\xi|\leq K} |\varphi'(\zeta)| |x-\xi| \frac{y}{(x-\xi)^2+y^2} \leq D_3,$$

for some constant $D_3 \geq 0$. Here it has been used that $|\varphi'(x)|$ is bounded in $[\xi - K, \xi + K]$, and that $|sy/(s^2+y^2)|$ is bounded. An integrable majorant for $f_{y,\xi}(x)$ is

$$|f_{y,\xi}(x)| \leq g_\xi(x) = \begin{cases} D_3, & \text{for } |x-\xi| \leq K \\ \frac{2D}{(x-\xi)^2}, & \text{otherwise,} \end{cases} \qquad \text{for all } y \leq 1.$$

Furthermore, the limit $\lim_{y\to 0^+} f_{y,\xi}(x)$ exists and is zero for all $x \in \mathbb{R}$. Thus Theorem A.2 applies and states that

$$\lim_{y\to 0^+} \int_{\mathbb{R}} (\varphi(x) - \varphi(\xi)) \frac{y}{(x-\xi)^2+y^2} \, \mathrm{d}x = 0,$$

which is equivalent to

$$\lim_{y\to 0^+} \int_{\mathbb{R}} \varphi(x) \frac{y}{(x-\xi)^2+y^2} \, \mathrm{d}x = \pi\varphi(\xi).$$

This proves the lemma for continuous φ.

Now suppose that ξ is a point where $\varphi(\xi)$ has a discontinuity. Divide $\varphi(x)$ into two parts:

$$\varphi(x) = \underbrace{\frac{1}{2}\left(\varphi(x) + \varphi(2\xi - x)\right)}_{\varphi_{\text{even}}(x)} + \underbrace{\frac{1}{2}\left(\varphi(x) - \varphi(2\xi - x)\right)}_{\varphi_{\text{odd}}(x)},$$

where φ_{even} is even in x with respect to an origin at the point $x = \xi$, and likewise φ_{odd} is odd in the same sense. Therefore

$$\int_{\mathbb{R}} \varphi_{\text{odd}}(x) \frac{y}{(x-\xi)^2 + y^2}\,\mathrm{d}x = 0, \quad \text{for all } y \geq 0. \tag{A.1}$$

Since the discontinuities of φ are isolated points, φ_{even} is continuous in a neighbourhood of ξ and continuously differentiable except possibly at the point $x = \xi$. Furthermore, $\varphi_{\text{even}}(\xi) = \breve{\varphi}(\xi)$. The same reasoning as for continuous φ results in

$$\lim_{y\to 0^+} \int_{\mathbb{R}} \varphi_{\text{even}}(x) \frac{y}{(x-\xi)^2 + y^2}\,\mathrm{d}x = \pi\breve{\varphi}(\xi).$$

Together with (A.1) this concludes the proof of the lemma for φ that are not continuous everywhere.

A.3 Proof of Corollary 4.1

Let $p = 0, \pm 1, \pm 2, \ldots$ and set

$$\varphi_{p,\varepsilon,\breve{\varepsilon}}(x) = \begin{cases} 0, & x < \varepsilon \\ x^{-p}, & \varepsilon < x < \breve{\varepsilon}^{-1} \\ 0, & x > \breve{\varepsilon}^{-1}. \end{cases} \tag{A.2}$$

This function satisfies the conditions of Lemma 4.1 for each fixed pair $\varepsilon > 0, \breve{\varepsilon} > 0$. Thus

$$\lim_{y\to 0^+} \frac{1}{\pi} \int_{\varepsilon}^{\breve{\varepsilon}^{-1}} \frac{\operatorname{Im} h(x + \mathrm{i}y)}{x^p}\,\mathrm{d}x = \int_{\mathbb{R}} \breve{\varphi}_{p,\varepsilon,\breve{\varepsilon}}(\xi)\,\mathrm{d}\mu(\xi),$$

where $\breve{\varphi}_{p,\varepsilon,\breve{\varepsilon}}(\xi)$ is given by (4.2). The function $\breve{\varphi}_{p,\varepsilon,\breve{\varepsilon}}$ is monotonically increasing as $\varepsilon \to 0^+$ and/or $\breve{\varepsilon} \to 0^+$. The limit is:

$$\lim_{\varepsilon\to 0^+} \lim_{\breve{\varepsilon}\to 0^+} \breve{\varphi}_{p,\varepsilon,\breve{\varepsilon}}(\xi) = \begin{cases} 0, & \xi \leq 0 \\ \xi^{-p}, & \xi > 0. \end{cases}$$

Implement Theorem A.1 to get

$$\lim_{\varepsilon\to 0^+} \lim_{\breve{\varepsilon}\to 0^+} \int_{\mathbb{R}} \breve{\varphi}_{p,\varepsilon,\breve{\varepsilon}}(\xi)\,\mathrm{d}\mu(\xi) = \int_{\xi>0} \frac{\mathrm{d}\mu(\xi)}{\xi^p}, \quad p = 0, \pm 1, \pm 2, \ldots$$

The integral over $(-\breve{\varepsilon}^{-1}, -\varepsilon)$ is treated in the same manner. This proves the lemma, seeing that

$$\int_{\xi<0} \frac{\mathrm{d}\mu(\xi)}{\xi^p} + \int_{\xi>0} \frac{\mathrm{d}\mu(\xi)}{\xi^p} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^p},$$

unless the left-hand side is $-\infty+\infty$. In this case the right-hand side is not defined.

For $p = 2, 3, \ldots$, the order of the limits $\breve{\varepsilon} \to 0^+$ and $y \to 0^+$ may be interchanged. Likewise, for $p = 0, -1, -2, \ldots$ the order of the limits $\varepsilon \to 0^+$ and $y \to 0^+$ may be interchanged. In that case there is an extra term $\delta_{p,0}\ \mu(\{0\})$ in the right-hand side. This is readily proved by considering the functions $\lim_{\breve{\varepsilon}\to 0^+} \varphi_{p,\varepsilon,\breve{\varepsilon}}(x)$ and $\lim_{\varepsilon\to 0^+} \varphi_{p,\varepsilon,\breve{\varepsilon}}(x)$, respectively.

A.4 Proof of Lemma 4.2

Evidently, statement *1* always implies *2*. Here it will be shown that *2* implies *3* and that *3* implies *1*. Start with the case $N = 1$ and assume that *3* holds. Consider the Herglotz function $h_0(z) = h(z) + \mu\{0\}/z$, represented by the measure μ_0. Set

$$a_0 = \lim_{z \hat{\to} 0} h_0(z) = \alpha + \lim_{z \hat{\to} 0} \int_{\mathbb{R}} \frac{1 + \xi z}{(\xi - z)(1 + \xi^2)}\, \mathrm{d}\mu_0(\xi) = \alpha + \int_{\mathbb{R}} \frac{1}{\xi(1 + \xi^2)}\, \mathrm{d}\mu_0(\xi).$$

Here Theorem A.2 could be used to move the limit under the integral sign, since for z restricted to the Stoltz domain $\theta \le \arg z \le \pi - \theta$ it holds that $|\xi - z| \ge |\xi| \sin\theta$ (see Appendix A.1) and $\int_{\mathbb{R}} \xi^{-2}\, \mathrm{d}\mu_0(\xi)$ is finite by assumption. Use this expression for a_0:

$$\lim_{z \hat{\to} 0} \frac{h_0(z) - a_0}{z} = \beta + \lim_{z \hat{\to} 0} \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{(\xi - z)\xi} = \beta + \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^2} = a_1,$$

where Theorem A.2 was used once more. Summing up, statement *1* is true.

Now assume that statement *2* is valid (still $N = 1$), *i.e.*,

$$h_0(\mathrm{i}y) = a_0 + a_1 \mathrm{i}y + o(y), \quad \text{as } y \to 0^+,$$

where $a_0, a_1 \in \mathbb{R}$. From this condition it follows that

$$\lim_{y \to 0^+} \frac{h_0(\mathrm{i}y) - h_0^*(\mathrm{i}y)}{2\mathrm{i}y} = \lim_{y \to 0^+} \left(a_1 + \frac{o(y)}{\mathrm{i}y} \right) = a_1.$$

But on the other hand,

$$\lim_{y \to 0^+} \frac{h_0(\mathrm{i}y) - h_0^*(\mathrm{i}y)}{2\mathrm{i}y} = \beta + \lim_{y \to 0^+} \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^2 + y^2} = \beta + \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^2}.$$

The exchange of the limit and integral is motivated by Theorem A.1. Ergo,

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^2} = a_1 - \beta < \infty,$$

and thus statement *3* is true.

The equivalence of the statements for all $N = 0, 1, 2, \ldots$ is proved by induction. For this reason, suppose that the equivalence has been proven for some $N \ge 1$, and that statement *3* holds for $N + 1$:

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^{2N+2}} < \infty.$$

Consider the function

$$h_1(z) = \frac{h_0(z) - a_0 - a_1 z}{z^2}.$$

This function may be expressed as:

$$h_1(z) = \frac{1}{z^2}\Bigg[\beta z + \alpha + \int_{\mathbb{R}} \left(\frac{1}{\xi - z} - \frac{\xi}{1+\xi^2}\right) \mathrm{d}\mu_0(\xi) \\ - \left(\alpha + \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi(1+\xi^2)}\right) - z\left(\beta + \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^2}\right)\Bigg] = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_1(\xi)}{\xi - z},$$

where $\mathrm{d}\mu_1(\xi) = \mathrm{d}\mu_0(\xi)/\xi^2$. Hence h_1 is a Herglotz function, and furthermore

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_1(\xi)}{\xi^{2N}} < \infty,$$

so h_1 has the asymptotic expansion

$$h_1(z) = \sum_{n=0}^{2N-1} a_{n+2} z^n + o(z^{2N-1}) \quad \text{as } z \hat{\to} 0,$$

where all a_n are real. This proves statement *1* for $N+1$.

On the other hand, assume that statement *2* holds for $N+1$, where $N \geq 1$. Consider the function h_1 once more. The induction assumption ensures that

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_1(\xi)}{\xi^{2N}} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^{2N+2}} < \infty,$$

which proves that statement *3* is true for $N+1$.

Finally, note that from the representation of h_1 it is clear that

$$a_3 = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_1(\xi)}{\xi^2} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^4}.$$

Furthermore,

$$a_2 = \lim_{z \hat{\to} 0} h_1(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_1(\xi)}{\xi} = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_0(\xi)}{\xi^3}.$$

This procedure may be continued for $a_4, a_5, \ldots, a_{2N-1}$ to prove (4.6), concluding the proof of the lemma.

A.5 Auxiliary theorems

The following theorem can be found in *e.g.*, [30], page 21:

Theorem A.1 (Lebesgue's Monotone Convergence Theorem). *Let $\{f_n\}$ be a sequence of real-valued measurable functions on X, and suppose that*

$$0 \leq f_1(x) \leq f_2(x) \leq \ldots \leq \infty, \quad \textit{for all } x \in X$$

and

$$f_n(x) \to f(x), \quad \text{as } n \to \infty \text{ for all } x \in X.$$

Then f is measurable, and

$$\lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu(x) = \int_X f(x)\,\mathrm{d}\mu(x).$$

The next theorem is also available in *e.g.*, [30], page 26:

Theorem A.2 (Lebesgue's Dominated Convergence Theorem)**.** *Suppose $\{f_n\}$ is a sequence of complex-valued measurable functions on X such that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$. If there is a function $g \in \mathrm{L}^1(\mu)$ such that

$$|f_n(x)| \le g(x), \quad \text{for all } n = 1, 2, \ldots \text{ and } x \in X,$$

then $f \in \mathrm{L}^1(\mu)$,

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)|\,\mathrm{d}\mu(x) = 0$$

and

$$\lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}\mu(x) = \int_X f(x)\,\mathrm{d}\mu(x).$$

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