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# An Expurgation Upper Bound on the Probability of Correct Path Loss for List Decoding of Time-invariant Convolutional Codes

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**Abstract** — In this paper list decoding of convolutional codes is considered. List decoding is a very powerful, low complexity, non-backtracking decoding method that does not fully exploit the error correcting capability of the code. A correct path loss is a serious kind of error event that is typical for list decoding. An expurgated upper bound on the probability of correct path loss is derived for the ensembles of systematic and nonsystematic time-varying convolutional codes.

## I. INTRODUCTION

In Viterbi decoding we first choose a suitable code and then design the decoder in order to "squeeze all juice" out of the chosen code. In sequential decoding we choose a code whose encoder memory is long enough to warrant essentially error free decoding.

In *list decoding* ( $M$ -algorithm) we first limit the resources of the *decoder*, then we choose an encoding matrix with a state space that is larger than the *decoder* state space. Thus, assuming the same decoder complexity, we use a more powerful code with list decoding than with Viterbi decoding. A list decoder is a very powerful non-backtracking decoding method that does not fully exploit the error correcting capability of the code.

List decoding is a breadth-first search of the code tree. At each depth only the  $L$  most promising subpaths are extended, not all, as is the case

with Viterbi decoding. These subpaths form a *list* of size  $L$ . Since the search is breadth-first, all subpaths on the list are of the same length and finding the  $L$  best extensions reduces to choosing the  $L$  extensions with the largest values of the cumulative Viterbi metric.

## II. THE EVENT ERROR PROBABILITY AND THE SYSTEMATIC VS. NONSYSTEMATIC ENCODERS QUESTION

Suppose that both a nonsystematic polynomial encoder of memory  $m$  and a first-memory-length equivalent systematic polynomial encoder, in general also of memory  $m$ , are used together with a list decoder.

For a range of interesting list sizes  $L$  the list decoder will operate mostly in the identical parts of the code trees encoded by the two encoders. The event error probabilities measured at the root will then be almost identical for both encoders. This is easily confirmed by simulations [1].

## III. THE CORRECT PATH LOSS PROBLEM AND THE SYSTEMATIC VS. NONSYSTEMATIC ENCODERS QUESTION

Since only the  $L$  best extensions are kept it can happen that the correct path is lost. This is a very severe event that causes many bit errors. If the decoder cannot recover a lost correct path it is of course a "catastrophe", i.e., a situation similar to the catastrophic error propagation that can occur

when a catastrophic encoding matrix is used to encode the information sequence.

The list decoder's ability to recover a lost correct path depends heavily on the type of *encoder* that is used. A systematic feed-forward encoder supports a spontaneous recovery while a nonsystematic encoder does not [1].

A suggestion of why systematic feed-forward encoders offer rapid recovery of a lost correct path may be found by considering the trellises of rate  $R = 1/2$  random systematic feed-forward and nonsystematic encoders. Suppose the correct path is the all-zero one and no errors occur for a time, and consider an arbitrary trellis node. The 0-branch (the one that inserts a zero into the encoder shift register) is the one leading back to the correct path. For a systematic feed-forward encoder, the distance increment of this "correct" branch is 0.5 on the average, while the incorrect branch has increment 1.5. For a nonsystematic encoder, these average increments are both 1, and give no particular direction of the search back to the correct path.

#### IV. BIT ERROR PROBABILITY AND THE SYSTEMATIC VS. NONSYSTEMATIC ENCODERS QUESTION

The fact a systematic feed-forward encoder supports a spontaneous recovery of a lost correct path implies a superior bit error probability performance. We will illustrate this by comparing the bit error probability for list decoders with various list sizes when they are used to decode sequences received over a BSC and encoded with both systematic feed-forward and nonsystematic encoders that are equivalent over the first memory length (Fig. 1). Both encoders have the same distance profile. The free distance of the systematic encoder is by far the least, yet its bit error probability is more than ten times better! The only advantage of the nonsystematic encoder is its larger free distance. Yet this extra distance has almost no effect on neither the burst nor the bit error probability. Nor does it change the list size  $L$  needed to correct  $e$  errors, as long as  $e$  falls within the powers of the systematic encoder.

In conclusion, using systematic feed-forward convolutional encoders essentially solves the correct path loss problem with list decoders. Since both systematic feed-forward and nonsystematic encoders have the same error rate in the absence of correct path loss, systematic feed-forward encoders are clearly superior to nonsystematic ones.

#### V. LIST MINIMUM WEIGHT AND LIST WEIGHT

Consider a list decoder with a fixed list size  $L$ . For every depth  $t = 0, 1, \dots$  and every received sequence  $\mathbf{r}_{[0,t]} \in \mathbb{F}_2^{(1+t)c}$  let  $\rho_L(\mathbf{r}_{[0,t]})$  denote the largest radius of a sphere with center  $\mathbf{r}_{[0,t]}$  such that the number of codewords in the sphere is less than or equal to  $L$ . The smallest such radius is of particular significance:

**Definition:** For a list decoder with a given list size  $L$  the *list minimum weight*  $w_{L-\min}$  is

$$w_{L-\min} = \min_t \min_{\mathbf{r}_{[0,t]}} \rho_L(\mathbf{r}_{[0,t]}),$$

where  $\mathbf{r}_{[0,t]}$  is the initial part of the received sequence  $\mathbf{r}$ .  $\square$

Given a list decoder of list size  $L$  and a received sequence with at most  $w_{L-\min}$  errors. Then the correct path will not be forced outside the  $L$  survivors.

Unfortunately,  $w_{L-\min}$  is hard to estimate. This leads us to restrict the minimization to those received sequences that are codewords:

**Definition:** For a given list size  $L$  the *list weight*  $w_{L-\text{list}}$  of the convolutional code  $C$  is

$$w_{L-\text{list}} = \min_t \min_{\mathbf{v}_{[0,t]}} \rho_L(\mathbf{v}_{[0,t]}),$$

where  $\mathbf{v}_{[0,t]}$  is the initial part of the codeword  $\mathbf{v} \in C$ .  $\square$

The list minimum weight  $w_{L-\min}$  is upper and lower bounded by  $w_{L-\text{list}}$  according to

$$\left\lfloor \frac{1}{2} w_{L-\text{list}} \right\rfloor \leq w_{L-\min} \leq w_{L-\text{list}}.$$

Given a list decoder of list size  $L$  and a received sequence with at most  $\lfloor \frac{1}{2} w_{L-\text{list}} \rfloor$  errors. Then the

correct path will not be forced outside the  $L$  survivors. If the number of errors exceeds  $\lfloor \frac{1}{2} w_{L\text{-list}} \rfloor$ , then it depends on the code  $\mathcal{C}$  and on the received sequence  $\mathbf{r}$  whether the correct path is forced outside the list.

We shall now lowerbound the list weight and show that the required list size grows exponentially with the number of errors to be corrected!

To prove a random coding lower bound on  $w_{L\text{-list}}$  we consider the ensemble of infinite memory, time-invariant, binary convolutional codes with encoding matrix

$$\mathbf{G} = \begin{pmatrix} G_0 & G_1 & \dots & & \\ & G_0 & G_1 & \dots & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

in which each digit in each of the  $b \times c$  submatrices  $G_i$ ,  $i = 0, 1, \dots$ , is chosen independently with probability  $1/2$  to be 0 or 1. Hence, the ensemble is path-independent, i.e., all code symbols on a subpath diverging from the allzero path are mutually independent. Furthermore, these code symbols are also equally probable binary digits.

The following lemma and theorem establish lower bounds on the list weight  $w_{L\text{-list}}$  that are similar to Costello's lower bound on the free distance of a convolutional code:

**Lemma 1** *The fraction of binary, rate  $R = b/c$ , infinite memory, time-invariant convolutional codes with polynomial encoding matrices having  $L$ -list weight  $w_{L\text{-list}}$  satisfying the inequality*

$$w_{L\text{-list}} > \frac{\log_2 L}{-\log_2(2^{1-R} - 1)} + \frac{\log_2((2^R - 1)(1 - \pi))}{-\log_2(2^{1-R} - 1)} - 1,$$

exceeds  $\pi$ , where  $0 \leq \pi < 1$ .  $\square$

By letting  $\pi = 0$  in Lemma 1, we obtain the following

**Theorem 2** *There exists a binary, rate  $R = b/c$ , infinite memory, time-invariant convolutional code having a list weight  $w_{L\text{-list}}$  satisfying the inequality*

$$w_{L\text{-list}} > \frac{\log_2 L}{-\log_2(2^{1-R} - 1)} + O(1),$$

where

$$O(1) = \frac{\log_2((2^R - 1)(2^{1-R} - 1))}{-\log_2(2^{1-R} - 1)}.$$

$\square$

Next we consider the ensemble of infinite memory, time-invariant, binary convolutional codes with systematic, polynomial encoding matrices  $\mathbf{G}$ , in which each  $(b \times c)$  submatrix  $G_i$ ,  $i = 0, 1, \dots$ , is systematic, i.e.,

$$G_i = \begin{pmatrix} 1 & \dots & 0 & \vdots & g_{1,b+1} & \dots & g_{1,c} \\ 0 & \dots & 0 & \vdots & g_{2,b+1} & \dots & g_{2,c} \\ \vdots & & \vdots & \vdots & & & \vdots \\ 0 & \dots & 1 & \vdots & g_{b,b+1} & \dots & g_{b,c} \end{pmatrix}$$

and each digit  $g_{ij}$ ,  $i = 1, 2, \dots, b$ ,  $j = b + 1, b + 2, \dots, c$ , is chosen independently with probability  $1/2$  to be 0 or 1.

**Lemma 3** *The fraction of binary, rate  $R = b/c$ , infinite memory, time-invariant convolutional codes with systematic, polynomial encoding matrices, having a list weight  $w_{L\text{-list}}$ , and satisfying the inequality in Lemma 1 exceeds  $\pi$ .  $\square$*

From Lemma 3 follows immediately

**Theorem 4** *There exists a binary, rate  $R = b/c$ , infinite memory, time-invariant convolutional code with systematic, polynomial encoding matrix, having a list weight satisfying the inequality in Theorem 2.  $\square$*

## VI. EXPURGATION UPPER BOUND ON THE PROBABILITY OF CORRECT PATH LOSS FOR LIST DECODING OF TIME-INVARIANT CONVOLUTIONAL CODES

The correct path loss on the  $t$ th step of a list decoding algorithm is a random event  $\mathcal{E}_t$  which consists of deleting at the  $t$ th step the correct codeword from the list of the  $L$  most likely codewords. In this section we shall give an expurgation upper bound on the probability of this event. Our bound is valid for transmission rates  $R$  less than the computational cut-off rate  $R_0$ , which for the

binary symmetric channel (BSC) with crossover probability  $\epsilon$  is given by

$$R_0 = 1 - \log_2 \left( 1 + 2\sqrt{\epsilon(1-\epsilon)} \right).$$

By using an expurgation technique we can prove the following

**Theorem 5 (Expurgation bound)** *For a list decoder of list size  $L$  and the BSC with crossover probability  $\epsilon$  there exist infinite memory, time-invariant, binary convolutional codes of rate  $R < R_0$  with systematic and nonsystematic, polynomial encoding matrices such that the probability of correct path loss is upperbounded by the following expurgated bound:*

$$P(\mathcal{E}_t) \leq L^{-\frac{\log_2(2\sqrt{\epsilon(1-\epsilon)})}{\log_2(2^{1-R}-1)}} \cdot O(1),$$

where  $O(1) = \frac{1}{1-2^{R-R_0}}$ .  $\square$

When the rate  $R \rightarrow R_0$  the exponent of the  $L$ -dependent factor of the upper bound approaches  $-1$ , while the second factor approaches  $+\infty$ .

We would like to stress that the very same bound is valid for both systematic and nonsystematic polynomial encoding matrices.

## VII. CONCLUSION

When using list decoding to decode convolutional codes we have noticed that

- systematic feed-forward and nonsystematic encoders give virtually the same event error probability.
- when we consider the bit error probability as our performance measure, systematic feed-forward encoders are by far superior to nonsystematic ones.
- we have the same upper bound on the probability of correct path loss for both types of encoders.

Hence, when list decoding is used, the encoder should be systematic and feed-forward.

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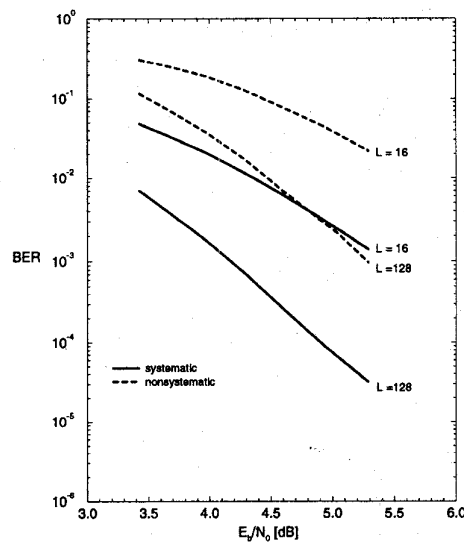


Figure 1: Bit error probabilities for various types of encoders all of memory  $m = 20$ .