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Hankel-type Model Reduction Based on Frequency Response Matching

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Abstract— **In this paper, a stability preserving model reduction algorithm for single-input single-output linear time invariant systems is presented. It performs a data fitting in the frequency domain using semidefinite programming methods. Computing the frequency response of a model can be done efficiently even for large scale models making this approach applicable to those. The relaxation used to obtain a semidefinite program is similar to one used in Hankel model reduction. Therefore accuracy of approximation is also similar to Hankel model reduction one. The approach can be easily extended to frequency-weighted and parameter dependent model reduction problems.**

I. Introduction

Model order reduction has received considerable attention is the past and there exists a number of established techniques to obtain low-order approximations. Most of the existing methods fall into two categories: singular value decomposition (SVD)-based and Krylov-based methods. Due to high computational cost for large scale systems, the SVD-based methods, which incorporate balanced truncation and Hankel model reduction, are mostly used for approximation of low and medium scale models. Balanced truncation ([1]) proposes a simple, yet a very powerful algorithm with a stability guarantee for the reduced model and the approximation error bounds. Hankel model reduction ([2]) is much more complicated than the latter, on the other hand it solves a suboptimal problem with the error bounds tighter than the balanced truncation ones. Both methods rely on solutions to Lyapunov equations to calculate the approximation, which makes them numerically heavy. The Krylov-based methods ([3], [4]) rely on moment matching techniques and therefore provide much cheaper solutions, however, there exist issues with stability of the approximation. These issues were solved in Krylov/SVD-based methods ([5]) for an extra computational cost.

All the described methods calculate the approximation from state-space representations of the full models. Instead, one can use the frequency domain data i.e., the frequency response samples, to obtain an approximation. Computing the frequency response for particular applications (e.g. modeling of electro-magnetic structures) can be even cheaper, than inverting the state-space matrix A, as shown in [6], [7], [8]. The approximation from the frequency data is related to the celebrated Nevanlinna-Pick interpolation problem, an extension of which to Hardy spaces can be found in [9], [10] with a recent progress in [11], [12]. In [13] another approach was developed to obtain an approximation. It is based on frequency data fitting and a convex relaxation, which is similar to one used in Hankel reduction. In this paper, a generalization of [13] is proposed, which has smaller lower and upper error bounds than the predecessor. The proposed method, as well as [13], can be regarded as an extension of Hankel model reduction to large scale model approximation. The semidefinite optimization approach is valuable due to simplicity of possible extensions, e.g. frequency-weighted and parameter-dependent model reduction. A multi-input-multi-output extension can be achieved using methods similar to [14], [15]

The first step towards the relaxation is a reformulation of the original problem, which will be called a positive real formulation. Basically a positive real constraint is introduced instead of a stability one. A similar approach was employed in [16] as a convex parameterization of robustly stabilizing controllers. In [17] a positive real condition on system parameters was used to synthesise stabilizing controllers. In [18] the latter framework was extended to account for performance. The proposed method, as well, can be extended to controller design problems and numerical simulations show that it is less restrictive than [18] (see, Sec. V for details).

The paper is organized as follows. In Sec. II the positive real reformulation is presented. In Sec. III the proposed Hankel-type relaxation or Optimization-based Hankel Model Reduction (OHMR) method is described. Sec. IV a computationally tractable algorithm and implementation issues are discussed. Finally, examples are found in Sec. V.

Notation

$\mathcal{H}_\infty$ stands for the space of discrete-time stable transfer functions. Operation $^\sim$ denotes a complex conjugate on the unit circle i.e., $G^\sim(e^{\jmath\omega}) = G^T(e^{-\jmath\omega})$, where $\jmath$ is a complex identity. $G(\omega)$ stands for the frequency response of $G(e^{\jmath\omega})$ to $\omega \in [0, \pi]$. The infinity norm is computed as $\|G\|_\infty = \sup_\omega |G(\omega)|$, where $G(\omega)$ is a scalar-valued function. The Hankel norm of a transfer function is denoted as $\|\cdot\|_H$ (for the definition see, [19]).

By optimal Hankel model reduction (HMR) in this paper is assumed the following algorithm. First, compute the Hankel approximation of G. Denote it as $G_H = p_H/q_H$, where p_H and q_H are polynomials. Then subject the approximation to a further optimization:

$$\gamma_H = \min_p \|G - p/q_H\|_\infty \tag{1}$$

The QCO method from [13] forms the following algorithm. Solve the relaxed problem,

$$\gamma_2 = \min_{\gamma>0,a,b} \gamma \quad \text{subject to } |G(\omega)a(\omega) - b(\omega)| \leq \gamma a(\omega) \quad \forall\omega \in [0,\pi] \tag{2}$$

where $b = \sum_{i=-k}^{k} b_i e^{ij\omega}$, $a = 1 + \sum_{i=1}^{k} a_i(e^{ij\omega} + e^{-ij\omega})$ and γ_2 is a sub-optimal approximation level. Then given a, solve a spectral factorization problem $a = qq^{\sim}$, where q has only stable zeros and poles. Finally using the computed q solve,

$$\min_p \|G(\omega) - p(\omega)/q(\omega)\|_\infty \tag{3}$$

II. Preliminaries

The main focus of this paper is reduction of discrete time models in the frequency domain. However, the algorithms can be easily extended to the continuous time case. It is assumed that the full model G is an asymptotically stable, rational, scalar transfer function. Now the model reduction problem can be easily formulated as a minimization one:

$$\min_{p,q} \|G - p/q\|_\infty$$

where $p = \sum_{i=0}^{k} p_i z^{-i}$, $q = \sum_{i=0}^{k} q_i z^{-i}$, q is a minimum phase transfer function and p/q is a sought-for approximation. It is also assumed that the order of G is bigger than k. The constraints on p and q will be enforced on the unit circle $\{z||z| = 1\}$, therefore often z will be substituted by $e^{j\omega}$ with a slight abuse of notation for p and q. Minimizing the $\mathcal{H}_\infty$ norm can be rewritten as a minimization of approximation level γ with a norm constraint enforced for all ω in $[0,\pi]$:

$$\gamma_4 = \min_{p,q} \gamma \quad \text{subject to } |G(\omega)q(\omega) - p(\omega)| \leq \gamma|q(\omega)| \quad \forall\omega \in [0,\pi] \quad q \text{ is minimum phase} \tag{4}$$

The minimum phase condition here is equivalent to stability of the reduced model p/q, which has the order less or equal to k. This problem is known to be non-convex and some successful (quasi-)convex relaxations have been proposed (e.g., [13]).

Another way to obtain a convex problem is using a restriction on decision variables as in [20]. Instead of the minimum phase constraint on q, consider a positive real one i.e., $\mathrm{Re}\,(q)$ is positive for all ω. This constraint is convex and can be expressed as a semidefinite one, unlike the minimum phase condition. Now the only obstacle in obtaining a quasi-convex program is the right-hand side of the norm constraint $|G(\omega)q(\omega) - p(\omega)| \leq \gamma|q(\omega)|$. If q would be positive then the minimization would correspond to a second order cone program, which is convex. In [20] it was proposed to substitute $|q|$ with $\mathrm{Re}\,(q)$, since it has been already parameterized as a positive pseudo-polynomial. Finally the norm constraint is reformulated as:

$$|G(\omega)q(\omega) - p(\omega)| \leq \gamma \mathrm{Re}\,(q(\omega))$$

In [20] it was shown that the positive real constraint on the denominator q may be very restrictive. In fact, as long as the full system has poles near the unit circle (which is often the case) the approximation will be too conservative. To address the restrictiveness introduce a new frequency dependent variable $\varphi(\omega)$ (not equal to zero almost everywhere) into the minimization. This program will be referred to as a positive real formulation:

$$\gamma_5 = \min_{\gamma>0,p,q,\varphi} \gamma \quad \text{subject to} \quad |G(\omega)q(\omega)\varphi^{\sim}(\omega) - p(\omega)\varphi^{\sim}(\omega)| < \gamma \mathrm{Re}\,(q(\omega)\varphi^{\sim}(\omega)) \ \forall\omega \quad q \text{ is minimum phase} \tag{5}$$

Surprisingly it is equivalent to the original formulation of the problem (4),

Lemma 2.1: The following statements are fulfilled:

a) Problems (4) and (5) are equivalent i.e., optimal values γ_4 and γ_5 are equal.
b) An optimal φ is such that $\varphi, \varphi^{-1} \in \mathcal{H}_\infty$, moreover, it is a polynomial in z^{-1} of the same order as q.

Proof: The program (5) is obtained by restriction of the general model reduction problem and it may be easily shown that from the constraints in (5) the constraints in (4) follow for any φ. Therefore we have $\gamma_4 \geq \gamma_5$.

To prove the converse, assume p_*/q_* is an optimal solution to the model reduction problem (4) with the optimal approximation level $\gamma_* = \gamma_4$. If we choose $\varphi_* = q_*$, it is easy to verify that $p_*, q_*, \gamma_*, \varphi_*$ satisfy the constraints of (5). Thus $\gamma_4 \leq \gamma_5$.

We have constructed $\varphi_* = q_*$ which is an optimal solution to the program (5) . By noting that $q_*, q_*^{-1} \in \mathcal{H}_\infty$ the second statement follows. ■

Remark 2.1: As a consequence of the statement b) in Lemma 2.1 φ may be parameterized in (5) as $\varphi = \sum_{i=0}^{k} \varphi_i z^{-i}$ without loss of generality.

Two quasi-convex algorithms exploiting this formulation are presented in the sequel. The first one is a relaxation presented in details in the following section. The second one, is choosing φ in advance (e.g. using an approximation of G of the same order) and solving (5) with fixed φ. This approach is referred to as a Positive Real Denominator (PRD) method and discussed in details in Sec. IV.

III. Optimization-based Hankel-type Model Reduction (OHMR)

Consider the program (5) and a straightforward convex relaxation of the structure i.e., introduce new variables $a :\triangleq q\varphi^{\sim}$ and $b :\triangleq p\varphi^{\sim}$. This yields an algorithm:

$$\gamma_6 = \min_{\gamma>0,a,b} \gamma \quad \text{subject to} \quad |G(\omega)a(\omega) - b(\omega)| \leq \gamma \mathrm{Re}\,(a(\omega)) \quad \forall\omega \tag{6}$$

where $a = \sum_{i=-k}^{k} a_i e^{ij\omega}$ and $b = \sum_{i=-k}^{k} b_i e^{ij\omega}$ (by definition of a, b, p, q and Remark 2.1) The non-convex condition *q is minimum-phase*, which corresponds to *a has exactly k stable zeros*, is very hard to parameterize in

a convex manner in a and b. Remarkably this constraint becomes redundant using the following statement,

Lemma 3.1: Consider a function $a = \sum_{i=-k}^{k} a_i z^i$ and a closed encircling the origin contour $\partial\mathbb{D}$. Assume, also that $a_{-k} \neq 0$. If $\mathrm{Re}\,(a(\partial\mathbb{D})) > 0$ then the pseudo-polynomial a has exactly k zeros in $\mathbb{D}$ and no zeros on $\partial\mathbb{D}$.

Proof: The function $a(z)$ does not have zeros or poles on the contour (since $\mathrm{Re}\,(a(\partial\mathbb{D})) > 0$) and it is analytic in $\mathbb{D}$, except for a set of isolated points. Thus by Cauchy's argument principle $N_z - N_p = N_o$ where N_z is the number of zeros in $\mathbb{D}$, N_p is the number of poles in $\mathbb{D}$ and N_o is a winding number of $a(\partial\mathbb{D})$ (number of times $a(\partial\mathbb{D})$ encircles the origin). Since $\mathrm{Re}\,(a(\partial\mathbb{D})) > 0$ for all the frequencies ω, the curve $a(\partial\mathbb{D})$ lies only in the right half plane and thus $N_o = 0$. Since $N_p = k$ the result follows. ■

The condition $a_{-k} \neq 0$ is not convex. It prevents the situation, when the best approximation of order not bigger, than k, has actually order less than k. However, such a situation is highly unlikely since an extra degree of freedom disappears and therefore the condition can be omitted.

After solving (6), the denominator q is obtained by solving the equation:

$$a = q\varphi^{\sim} \tag{7}$$

where φ, q have only stable zeroes. q can be computed by a simple zero extraction, however, there are more numerically robust ways to do it ([21]).

Now it is clear that the transfer function b/a has stable and antistable parts both of order k. Since the optimization is also performed over both, this approach may be regarded as a suboptimal model reduction in Hankel norm i.e., suboptimal Hankel approximation. Finally the stable approximation of G is obtained by another minimization

$$\min_p \|G - p/q\|_\infty \tag{8}$$

The described algorithm still has an infinite number of constraints, since the conditions are enforced for all frequencies $\omega \in [0, \pi]$. However, it was assumed, that the function G is rational, therefore it may be sufficient to impose the constraints in the finite number of points ω_i. A detailed description of this approach may be found in Sec. IV.

A theoretical relaxation gap for the algorithm can be calculated as well,

Theorem 3.1: Consider the model reduction problem (6,7,8) with the full sampling (the constraints are enforced for all frequencies ω). γ_6, a, and b are obtained from (6), q and p are obtained from (7) and (8) respectively. Let also γ_4 be the optimal approximation level from (4). Then the following error bounds hold:

$$\sigma_{k+1}(G) \leq \gamma_6 \leq \gamma_4 \leq \|G - p/q\|_\infty \tag{9}$$

$$\|G - p/q\|_\infty \leq (k+1)\gamma_6 \tag{10}$$

Proof: Inequalities in (9) are mostly trivial except $\gamma_6 \leq \gamma_4$, which is proved by construction. Indeed, recall that (4) is general and (5) is positive real formulations of model reduction Furthermore, $\gamma_4 = \gamma_5$ by Lemma 2.1. Since (6) is a relaxation of (5) $\gamma_6 \leq \gamma_5$ and the inequality follows.

Now, prove the upper bound (10). Note that $\|G - b/a\|_\infty \leq \gamma_6$, where $a = q\varphi^{\sim}$ and $\varphi^{\sim}$ has only zeros outside the unit circle. Since b/a has both antistable and stable modes, there exist a unique decomposition,

$$\frac{b}{a} = \frac{z^{-k}t}{q} + \frac{z^{-k}r}{\varphi^{\sim}}$$

where t and r are polynomials in z of order k and $k-1$ respectively. Now $\left\|G - z^{-k}t/q - z^{-k}r/\varphi^{\sim}\right\|_\infty \leq \gamma_6$ implies that $\left\|z^{-k}r/\varphi^{\sim}\right\|_H \leq \gamma_6$ by the famous Adamian-Arov-Krein theorem (see for example [19]). The same theorem also states that there exist such a D that $\left\|z^{-k}r/\varphi^{\sim} + D\right\|_\infty \leq k\gamma_6$. Now combining this bound with the triangle inequality yields:

$$\left\|G - \frac{z^{-k}t + Dq}{q}\right\|_\infty \leq (k+1)\gamma_6$$

Since the numerator p is obtained by means of optimization the upper bound follows. ■

Note, if one fixes a to be a symmetric pseudo-polynomial, then the proposed approach reduces to the QCO method from [13]. Therefore the OHMR method would have a smaller lower and upper bounds than the QCO one.

IV. Implementation

In order to obtain a tractable optimization problem the constraints are imposed only on a finite frequency grid $\{\omega_i\}_{i=1}^N \in [0, \pi]$. The number of points should be at least $O(k^2)$, where k is the order of approximation, to avoid over-fit. This approach may create unstable approximations, therefore the positivity constraint $\mathrm{Re}\,(a) > 0$, which guarantees stability of the reduced models, is enforced for all frequencies. It may be done efficiently using the KYP lemma, e.g using the formulation from [22].

The algorithm is as follows: solve,

$$\min_{a,b,\gamma} \gamma \qquad \text{subject to} \tag{11}$$

$$|G(\omega_i)a(\omega_i) - b(\omega_i)| \leq \gamma \mathrm{Re}\,(a(\omega_i)) \quad i = 1, \ldots, N \tag{12}$$

$$\mathrm{Re}\,(a) > 0 \quad \forall\, \omega \in [0, \pi] \tag{13}$$

Also fix $a_0 = 1$ for normalization. Given a, perform stable-antistable factorization,

$$a = q\varphi^{\sim} \tag{14}$$

where φ, q have only stable zeros and poles. Finally using the obtained q solve,

$$\min_p \max_{i=1,\ldots,N} |G(\omega_i) - p(\omega_i)/q(\omega_i)| \tag{15}$$

Surely, the norm constraint (12) could also be enforced for all frequencies using the same techniques as in the case of (13). However, then the size of the norm constraint LMI would be four times the order of the full model G, which makes it computationally more expensive than HMR. Thus there would be no advantages with respect to the latter and therefore this approach is not considered.

A. Positive Real Denominator (PRD) Method

Consider (5) and assume that φ is fixed, predefined and parameterized according to Remark 2.1. Define p, q and φ :

$$p = \sum_{i=0}^{k} p_i e^{-ij\omega} \quad q = 1 + \sum_{i=1}^{k} q_i e^{-ij\omega} \quad \varphi = \sum_{i=0}^{k} \varphi_i e^{-ij\omega}$$

q_0 is fixed to 1 for normalization. Given the described above parameterization, the minimization is set up as,

$$\min_{p,q,\gamma} \gamma \quad \text{subject to } \mathrm{Re}\,(q\varphi^{\sim}) > 0 \quad \forall \omega \in [0, \pi] \tag{16}$$

$$|(G(\omega_i)q(\omega_i) - p(\omega_i))\varphi^{\sim}(\omega_i)| \leq \gamma \mathrm{Re}\,(q(\omega_i)\varphi^{\sim}(\omega_i)) \tag{17}$$

$$\omega_i \in [0, \pi] \quad \forall i = 1, \ldots, N \tag{18}$$

Note, that the condition $\mathrm{Re}\,(q\varphi^{\sim}) > 0$ implies that q is minimum phase as long as φ is. The choice of φ is guided by a specific G. For example, φ may be chosen as the denominator of any approximation of G of order k. Therefore this approach can be seen as a local improvement of any approximation method, where the neighbourhood of improvement is specified by a choice of φ.

V. Examples

These examples are set to estimate the actual relaxation gap in the Hankel-type approximation and a possible improvement by the PRD approach to any reduction procedure.

Some of the models in the considered examples are continuous-time systems. Since the proposed approaches (as well as the QCO method) deal with reduction of discrete-time models, the systems should first be discretized. The discretization is performed, while warping around a particular frequency ω_0 in the process:

$$s = \lambda \frac{z-1}{z+1}, \text{ where } \lambda = \frac{\omega_0}{\tan(\omega_0 T_s/2)}$$

If the Nyquist sampling time is bigger than T_s, then no dynamics are lost. Parameter ω_0 is a tuning parameter for numerical conditioning. For example, if the biggest resonant peak occurs around a frequency ω_0, then prewarping around this frequency will create a better numerically conditioned problem.

The Hankel model reduction is implemented by a MATLAB™ routine HANKELMR using the procedure (1) afterwards. OHMR and QCO are implemented using a cutting plane algorithm (for more details, see [23]) except for Example 5 where the interior point solver SeDuMi [24] and the parser YALMIP [25] were used. For the optimization algorithms a uniform on the interval $[0, \pi]$ frequency grid is considered with denser regions around the peaks in magnitude of the frequency response (if any exist).

Example 1: Reduction of an all-pass system. This toy example was created to show a better numerical robustness of the proposed Hankel-type approximation approach in comparison with the QCO method.

First, two all-pass models are specified as:

$$G_i = \prod_{j=1}^{12} \frac{1 - z\xi_j^{\sim}}{z - \xi_j}, \text{where } |\xi_j| = 0.96 \quad \forall i, j$$

and the arguments for the complex conjugate poles are chosen as:

$$G_1 : \arg(\xi_j) = \pm[0.11, 0.13, 0.14, 3.1, 3.11, 3.14]$$
$$G_2 : \arg(\xi_j) = \pm[0.11, 0.13, 0.14, 1.57, 1.57, 1.57]$$

The full models then are taken as $H_i = G_i G_0$, where G_0 is a transfer function with poorly observable and poorly controllable dynamics and $\mathcal{H}_\infty$ norm around 1. Basically G_0 should be reduced in the approximation procedure. The order of G_0 is set to 300, where the poles were chosen randomly and the zeros are the slightly perturbed poles. This does not affect the approximation, since the impact of G_0 on the magnitude and phase of the full models H_i is negligible. The models are available at request. The approximation errors are presented in Table I.

TABLE I

Approximation errors in Example 1

Models	H_1	H_2
Number of points in the grid	300	300
$\sigma_{13}(H_i)$	$1.6 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$
HMR	$1.6 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$
QCO	$7.4 \cdot 10^{-2}$	$8.5 \cdot 10^{-2}$
OHMR	$5.1 \cdot 10^{-3}$	$1.9 \cdot 10^{-3}$
PRD as HMR improvement	$1.6 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$
PRD as QCO improvement	$1.6 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$
PRD as OHMR improvement	$1.6 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$

Example 2: Deformable Mirror Modeling. The following model was studied in [26] and obtained by means of a finite element modeling approach that resulted in a system of second-order differential equations:

$$I\ddot{x} + \alpha\Lambda\dot{x} + \Lambda^2 x = Bu \qquad y = B^T x \tag{19}$$

where matrices Λ, B and scalar α are known and the state-space model has 2000 states.

TABLE II

Approximation errors in percent in Example 2

Reduction order k	8	16
Number of points in the grid	400	800
$\sigma_{k+1}(G)$	1.07%	0.17%
HMR	2.07%	0.28%
QCO	1.93%	0.35%
OHMR	1.93%	0.26%
PRD with a structure	2.18%	0.47%

For the reduction purposes we examine a one input one output model, i.e. B is a single column. The frequency responses of the full model and its 8-th order approximations are shown in Fig. 1. The $\mathcal{H}_\infty$ norms of the approximation errors for orders 8 and 16 are also presented in Table II. Hankel model reduction in this examples showed worse performance both in accuracy and computational speed. It is most likely that the accuracy was affected by the high order of the full model. Note also, that in Fig. 1 8-th order OHMR approximation (thick black line) follows the phase of the full model (thick gray line) much better than the HMR approximation with the corresponding order (dashed blue line).

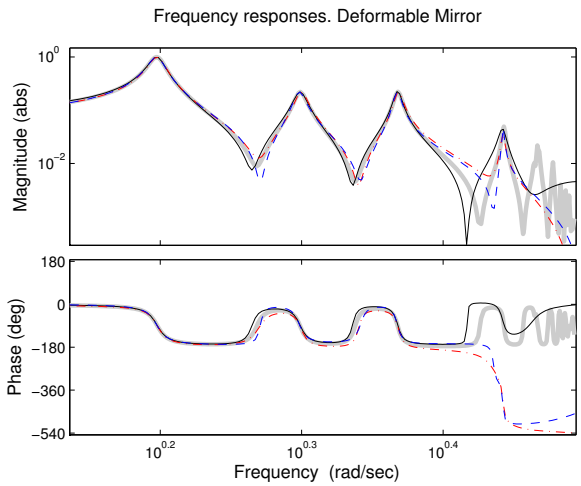

Fig. 1. Deformable mirror modeling frequency responses. Black solid line - OHMR approximation of order 8, red dash-dotted line - PRD approximation with a structure of order 8, blue dashed line - HMR approximation of order 8, thick gray line - the full model G.

By PRD with a structure is assumed the algorithm with two zeros of approximation fixed at $\omega = \pi$. This was implemented, since any system with a structure as (19) would have such zeros in any discrete-time realization.

Example 3: Transmission Line Modeling. This example was described in [27] and the references within. A transmission line is a 2-input-2-output model, which is described by the following system of differential equations:

$$E\dot{x} = Ax + Bu \qquad\qquad y = Cx$$

where matrix E is positive semidefinite, however, badly conditioned. The order of the original model is 398. Here only the $(1,1)$-entry of the transfer function was reduced.

The $\mathcal{H}_\infty$ norm of the approximation errors are given in Table III. The badly conditioned E matrix affects the Hankel approximation providing much worse results for the order 8.

TABLE III
APPROXIMATION ERRORS IN PERCENT IN EXAMPLE 3

Reduction order k	8	18
Number of points in the grid	700	700
$\sigma_{k+1}(G)$	10.83%	0.31%
HMR	15.71%	0.47%
QCO	11.81%	0.41%
OHMR	11.28%	0.41%

Example 4: Eady example. This is a model of the atmospheric storm track taken from [27]. The system is described by a set of partial-differential equations. A finite dimensional approximation of order 598 is reduced as an example here. The results of approximation in the frequency range of interest to orders 3 and 9 see in Fig. 2. The HMR approximation of order 9 is not depicted, since it coincides with OHMR approximation of the same order.

Example 5: Controller Reduction. Consider the complementary sensitivity function $T = GK(1+GK)^{-1}$, where

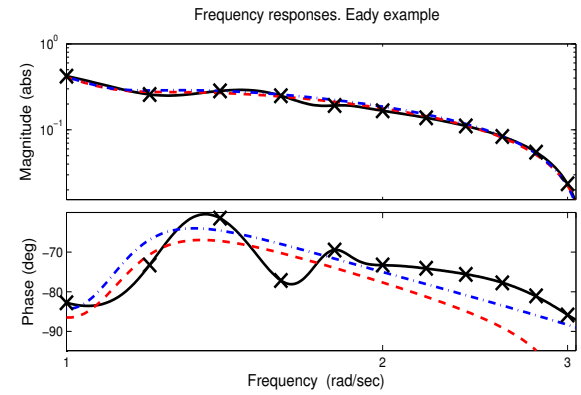

Fig. 2. Frequency responses in Eady Example. Black solid line - reduced with OHMR to order 9, red dashed line - reduced with OHMR to order 3, blue dash-dotted line - reduced with HMR to order 3 and the black crosses represent the original data.

K is a controller and G is a plant. A different closed-loop function can be used, e.g. the gang of four or the sensitivity function. In fact, as long as the closed loop is a rational function of K of degree 1, the proposed approach can be applied. The complementary sensitivity function is chosen for simplicity.

The 152-nd order controller K was designed in [28] using Youla parameterization. The controller K itself is stable and so is the third order plant $G = b/a$, let also the reduced controller $\widehat{K} = p/q$, where p, q, b and a are the polynomials in $e^{-\jmath\omega}$ and

$$c(p,q) = p(\omega)b(\omega)$$
$$d(p,q) = q(\omega)a(\omega) + p(\omega)b(\omega)$$

To obtain a stabilizing controller solve:

$$\gamma_c = \min_{\gamma>0,p,q} \gamma \quad \text{subject to}$$
$$|(T(\omega)d(p,q) - c(p,q))\varphi^{\sim}(\omega)| < \gamma \mathrm{Re}\,(d(p,q)\varphi^{\sim}(\omega)) \ \forall\omega$$

The obtained p and q yield a closed loop error bound:

$$\|T(G,K) - T(G,\widehat{K})\| \le \gamma_c$$

Note also, that the described approach to controller reduction preserves the closed loop stability. Indeed, the function $d(p,q)$ is minimum-phase, b, a are the coprime factors of G and p, q are the coprime factors of $\widehat{K}$. Therefore the closed-loop system is internally stable in definitions of [19].

The performance of various methods for different order reduction is presented in Fig. 3. For the frequency weighted balanced truncation and OHMR algorithm the weight is chosen as a minimum-phase spectral factor of $G(1+KG)^{-1}(G(1+KG)^{-1})^{\sim}$. Besides the frequency weighted techniques, a described above one (PRD) is applied as well as the method from [18]. Note, that [18] is based on very similar ideas and also an auxiliary variable φ, which is called a *central polynomial*, is introduced. Notably, in both approaches φ has the same interpretation - closed loop poles of the approximated closed loop. Therefore the same φ is

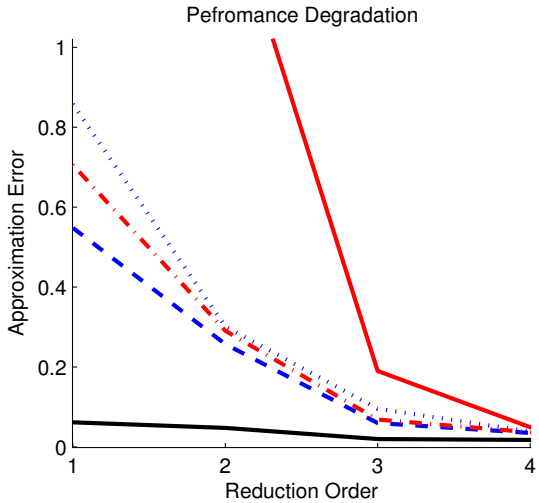

Fig. 3. Performance of various methods in Example 5. Solid red line - frequency weighted balanced truncation, dotted blue - frequency weighted OHMR, dashed blue - PRD, dash-dotted red - method from [18], solid black - lower error bound.

used for both methods. φ is computed as a denominator of the closed loop transfer function $T(G, K_o)$, where K_o is obtained using the FW OHMR method.

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VI. Conclusion

In this paper, an approach to model reduction of linear time invariant systems has been presented. The method requires only the frequency response samples to obtain an approximation and guarantees stability of one in $\mathcal{H}_\infty$ sense. The minimization is performed in a Hankel-type norm, therefore the accuracy of the algorithm is expected to be close to optimal Hankel one.

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