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Semidefinite Hankel-type Model Reduction Based on Frequency Response Matching

Aivar Sootla

Abstract

This paper is dedicated to model order reduction of linear time-invariant systems. The main contribution of this paper is the derivation of two scalable stability-preserving model reduction algorithms. Both algorithms constitute a development of a recently proposed model reduction method. The algorithms perform a curve fitting procedure using frequency response samples of a model and semidefinite programming methods. Computation of these samples can be done efficiently even for large scale models. Both algorithms are obtained from a reformulation of the model reduction problem. One proposes a semidefinite relaxation, while the other is an iterative semidefinite approach. The relaxation approach is similar to Hankel model reduction, which is a well-known and established method in the control literature. Due to this resemblance, the accuracy of approximation is also similar to the one of Hankel model reduction. An appealing quality of the proposed algorithms is the ability to easily perform extensions, e.g., introduce frequency-weighting, positive-real and bounded-real constraints.

Index Terms

Reduced order modeling, Model/controller reduction, Optimization, Semidefinite programming

I. PROBLEM FORMULATION AND INTRODUCTION

The main subject of this publication is \mathbb{H}_∞ model reduction. The focus is kept on discrete-time models, nevertheless, the algorithms can be applied to the continuous time case, as discussed in Section V. Throughout the paper, it is assumed that the full order model G is an asymptotically

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stable, scalar-valued (SISO), proper, rational transfer function. The reduction problem for such models can be formulated as:

$$\min_{p, q} \|G(z) - p(z)/q(z)\|_{\mathbb{H}_\infty}$$

where $p(z) = \sum_{i=0}^k p_i z^{-i}$ and $q(z) = \sum_{i=0}^k q_i z^{-i}$ (i.e., they are FIR filters of order k), p_i, q_i are real scalars, q has a stable inverse and p/q is a reduced order model. It is also assumed that the order of G is much larger than k . The problem is non-convex, therefore suboptimal methods are typically used to address it. Most of the existing methods fall into two categories: singular value decomposition based (for example, balanced truncation and Hankel model reduction) and Krylov based methods. Balanced truncation ([1]) proposes a simple, yet, very powerful algorithm with a stability guarantee for the reduced model and the approximation error bounds. Hankel model reduction ([2]) solves an optimization problem in the Hankel norm:

$$\min_{p, q} \|G(z) - p(z)/q(z)\|_H$$

The problem can be rewritten as minimization in the \mathbb{L}_∞ norm, where an additional variable is introduced - an anti-stable transfer function Δ of order less or equal to $n + k - 1$ (anti-stable means that all the poles lie outside the unit circle, [3]).

$$\min_{p, q, \Delta} \|G(z) - p(z)/q(z) - \Delta\|_\infty$$

This problem is more complicated than balanced truncation, but it has tighter error bounds. The problem can be solved analytically, however, the cost is not suitable for large scale systems. On the other hand, the Krylov based methods (cf. [4], [5]) rely on moment matching techniques. These methods match the derivatives of the transfer functions at pre-defined frequencies, without their computation. They provide numerically much cheaper solutions, however, without explicit error bounds in \mathbb{H}_∞ . It is important to remark that the Krylov based method in [6] provides a solution which satisfies first order optimality conditions in the \mathbb{H}_2 norm.

The described above methods compute a reduced order model based on a state-space representation of the full order one. As an alternative, one can use frequency domain data, i.e., frequency response samples. Computing a frequency response for particular applications (e.g., modeling of electro-magnetic structures) can be even cheaper, than inverting a state-space matrix A , as shown in [7], [8], [9]. One of the main tools for frequency domain approximation is the interpolation techniques ([10], [11], [12]). Another tool is convex optimization as in the method proposed

in [13], [14]. It is not an interpolation technique, but, the objective is to minimize the distance between the frequency response samples of the full and reduced order models. Therefore, there is a bigger degree of flexibility in comparison to interpolation techniques. In [14] it is proposed to solve:

$$\min_{p, q, r} \left\| G(z) - p(z)/q(z) - z^{k-1}r(z)/q^{\sim}(z) \right\|_{\infty}$$

where $q^{\sim}(z) = q(1/z)$ and r is an FIR filter of order $k-1$. In practice the norm is replaced with a constraint on $|G(e^{j\omega}) - p(e^{j\omega})/q(e^{j\omega}) - e^{(k-1)j\omega}r(e^{j\omega})/q^{\sim}(e^{j\omega})|$ for every frequency on a finite grid. It can be shown that this problem is finite dimensional and quasi-convex, therefore it can be addressed with reasonable scalability properties. More importantly the cost of the quasi-convex program does not depend on the order of the original model G . It can be even argued that [14] is a scalable version of Hankel model reduction, since [14] is Hankel model reduction with extra constraints on Δ . A more appealing quality of [14] is the ability to add extra constraints *without* affecting the complexity of the algorithm.

This paper constitutes a development of the method introduced in [14]. Two algorithms are presented which result from a reformulation of the single-input-single-output model reduction problem. The new formulation will be called *Hankel-type* to stress the connection to Hankel model reduction. One of the obtained algorithms employs a relaxation, which is related to both [14] and Hankel model reduction ([2]). The other introduced algorithm uses an iterative procedure, where a semidefinite program is solved on every iteration. This iterative algorithm is related to the ‘‘central polynomial’’ framework ([15], [16]) used for low-order controller design.

It cannot be claimed that the proposed algorithms always provide a better model match than [14]. Nevertheless, their use has a number of advantages:

- While comparing the relaxation algorithm to [14] a better numerical robustness is noticed, which is illustrated in Example 1. Secondly, results of the parameterized model reduction extension show a considerable improvement in the quality of approximation (see, [17, Chapter 3]).
- The presented iterative algorithm is a powerful tool when systems with a structure are considered, i.e., decentralized structure, plant-controller systems, etc. The algorithm is extended to such problems in [17, Chapters 4 and 5].
- For particular models, it is possible that the actual approximation error of the relaxation

algorithm is larger than the one of [14]. Nonetheless, the presented iterative algorithm is able to significantly reduce the loss of approximation quality, if it occurs.

The proofs of some facts are omitted due to space limitations. The proofs are found in [17, Chapter 2], where an extension to multi-variable (or MIMO) systems, as well as extensions to the frequency-weighted, passive model reduction, are also outlined. The rest of the paper is organized as follows: the Hankel-type formulation of model order reduction problem is presented in Section II. Section III describes the proposed relaxation. Different subsections are devoted to a system theoretic interpretation, a discussion on relationship to [14] and implementation details of the algorithm. The iterative algorithm is discussed in Section IV. Numerical examples are found in Section V.

II. HANKEL-TYPE FORMULATION OF MODEL REDUCTION

First, minimizing the \mathbb{H}_∞ norm is rewritten as a minimization of an approximation level γ subject to the stability constraint and the norm constraints enforced only on the unit circle $\partial\mathbb{D} = \{z \mid |z| = 1\}$. Therefore, z is often substituted by $e^{j\omega}$. The resulting program reads:

$$\begin{aligned} \gamma_{\text{mor}} = \min_{p, q} \gamma \quad & \text{subject to} \\ |G(e^{j\omega})q(e^{j\omega}) - p(e^{j\omega})| < \gamma |q(e^{j\omega})| \quad & \forall \omega \in [0, \pi] \\ q(z) \text{ has a stable inverse} \end{aligned} \quad (1)$$

where $p(z) = \sum_{i=0}^k p_i z^{-i}$ and $q(z) = \sum_{i=0}^k q_i z^{-i}$ and p_i, q_i are real scalars. Shortly, it will be shown that (1) is equivalent to:

$$\begin{aligned} \gamma_{\text{htf}} = \min_{\gamma > 0, p, q, \varphi} \gamma \quad & \text{subject to} \\ |(Gq - p)\varphi^\sim| < \gamma \text{Re}(q\varphi^\sim) \quad & \forall \omega \in [0, \pi] \\ \varphi(z) \text{ has a stable inverse} \end{aligned} \quad (2)$$

where the arguments of the functions ($e^{j\omega}$) are skipped for brevity, the notation \cdot^\sim stands for $\varphi^\sim(z) = \varphi(1/z)$ and $\varphi(z)$ is an FIR filter of order k with real coefficients. The program (2) is called a *Hankel-type formulation* of model order reduction. The name of the formulation is explained in detail in Section III. The main benefit of this formulation is the absence of the

absolute value function on the right hand side, which gives two possibilities to obtain convex programs: a relaxation and an iterative approach, where φ is iterated over.

Lemma 1: The optimal values γ_{mor} and γ_{htf} are equal. Moreover, (q_*, p_*) is the optimal solution to (2) for some φ_* , if and only if (q_*, p_*) is the optimal solution to (1).

Proof. Firstly note, that the inequality $\text{Re}(q\varphi^\sim) > 0$ implies that $\text{Re}(q\varphi^{-1}) > 0$ and $\text{Re}(\varphi q^{-1}) > 0$. Moreover, if φ^{-1} is stable, then the transfer function $q\varphi^{-1}$ is positive real and q^{-1} is stable as well. Therefore the constraint “ φ^{-1} is stable” in (2) can be replaced with “ q^{-1} is stable” and “ $\varphi \neq 0$ for all ω ”. Now, since $\text{Re}(q\varphi^\sim) \leq |q\varphi^\sim|$ for all q, φ and the frequencies ω in $[0, \pi]$, we have:

$$\begin{aligned} \gamma_{\text{htf}} &\geq \min_{\gamma > 0, p, q, \varphi} \gamma \quad \text{subject to} \\ |(Gq - p)\varphi^\sim| &< \gamma |q\varphi^\sim| \quad \forall \omega \in [0, \pi] \\ q(z) &\text{ has a stable inverse and } \varphi \neq 0 \quad \forall \omega \end{aligned}$$

We can divide both sides of the norm constraint with $|\varphi|$, and the minimization program reduces to (1). Therefore, we have $\gamma_{\text{mor}} \leq \gamma_{\text{htf}}$. To prove the converse, assume $p_*q_*^{-1}$ is the optimal solution to the model reduction problem (1) with the optimal approximation level γ_{mor} . If we choose $\varphi_* = q_*$, it is easy to verify that $p_*, q_*, \gamma_{\text{mor}}, \varphi_*$ satisfy the constraints of (2). Thus $\gamma_{\text{mor}} \geq \gamma_{\text{htf}}$.

Now, prove the second statement. If (p_*, q_*, φ_*) is the optimal solution to (2), then:

$$|Gq_* - p_*| < \gamma_{\text{htf}} \frac{\text{Re}(q_*\varphi_*^\sim)}{|\varphi_*^\sim|} \leq \gamma_{\text{htf}} |q_*| \quad \forall \omega \in [0, \pi]$$

Since $\gamma_{\text{htf}} = \gamma_{\text{mor}}$, the pair (p_*, q_*) is also the optimal solution to (1). The converse is easily shown by choosing $\varphi_* = q_*$. ■

Lemma 1 provides an optimal choice of the auxiliary variable φ , which is simply equal to q . On the other hand, the variable $|q|$ is replaced with $\text{Re}(q \cdot \varphi^\sim / |\varphi|)$. This implies that the complex vector q is rotated in a way such that $\text{Re}(q\varphi^\sim)$ becomes positive, where $\varphi^\sim / |\varphi|$ is the angle of such a rotation. In the optimality, this angle is equal to $-\arg(q)$, which leaves only the positive part in the expression $q\varphi^\sim$.

III. SEMIDEFINITE HANKEL-TYPE MODEL REDUCTION

First, consider the relaxation of the Hankel-type formulation (2). Let $a = \sum_{i=-k}^k a_i e^{-ij\omega}$ and $b = \sum_{i=-k}^k b_i e^{-ij\omega}$, where a_i, b_i are real scalars and introduce new constraints in the program (2)

as below. Note that p , q and φ are also decision variables in the new program, however, they are skipped in order to avoid confusion in future references.

$$\min_{\gamma > 0, a, b} \quad \gamma \quad \text{subject to} \quad (3)$$

$$|Ga - b| < \gamma \operatorname{Re}(a) \quad \forall \omega \in [0, \pi] \quad (4)$$

$$b = p\varphi^{\sim} \quad \forall z \in \mathbb{C} \quad (5)$$

$$a = q\varphi^{\sim} \quad \forall z \in \mathbb{C}, \quad \varphi \text{ has a stable inverse} \quad (6)$$

Removing the constraint (5) constitutes a relaxation. Later, it will be shown that the relaxed optimal solution will always lie within certain bounds of the original optimal solution. On the other hand, the constraints (6) can be removed *without* affecting the solution. These constraints are equivalent to *a has exactly k stable zeros*, which is implied by positivity of $\operatorname{Re}(a)$:

Lemma 2: Consider a function $a = \sum_{i=-k}^k a_i z^{-i}$, the unit disc $\mathbb{D} = \{z \mid |z| < 1\}$ and the unit circle $\partial\mathbb{D}$. Assume that a_k is not equal to zero. If $\operatorname{Re}(a(\partial\mathbb{D})) > 0$ then the pseudo-polynomial a has k zeros in \mathbb{D} .

Proof. The function $a(z)$ does not have zeros or poles on the unit circle (since $\operatorname{Re}(a(\partial\mathbb{D})) > 0$). It is also analytic in \mathbb{D} except for a set of isolated points. Thus by Cauchy's argument principle $N_z - k = N_o$, where N_z is the number of zeros in \mathbb{D} , k is the number of poles in \mathbb{D} (which is guaranteed, if $a_k \neq 0$) and N_o is the number of times $a(\partial\mathbb{D})$ encircles the origin. Since $\operatorname{Re}(a(\partial\mathbb{D})) > 0$ for all the frequencies ω in $[0, \pi]$, the curve $a(\partial\mathbb{D})$ lies only in the right half plane and thus $N_o = 0$ and $N_z = k$. ■

Remark 1: The condition $a_k \neq 0$ is not restrictive. The pseudo-polynomials with $a_k = 0$ constitute a measure zero subspace of the pseudo-polynomials with $a_k \neq 0$. Therefore, numerical solutions of semidefinite optimization will have $a_k \neq 0$ almost surely.

Given a_* a solution of (3) subject only to (4), the denominator q_* is obtained by solving the equation:

$$a_* = q_* \varphi_*^{\sim} \quad (7)$$

where φ_* , q_* have only stable zeroes and are the solutions to the non-symmetric spectral factorization problem (see, [18]). The reduced order model is simply p_*/q_* , where the numerator p_* is obtained from:

$$p_* = \underset{p}{\operatorname{argmin}} \|G - p/q_*\|_{\mathbb{H}_\infty} \quad (8)$$

Algorithm 1 Semidefinite Hankel-type Model Reduction

Inputs: Frequency grid $\Omega = \{\omega_i\}_{i=1}^N \in [0, \pi]$, reduction order k .

Outputs: Reduced order model p_*/q_* , suboptimal approximation level γ_{shmr}^N

- 1) Compute $G(e^{j\omega})$ for all $\omega \in \Omega$
- 2) Solve

$$\gamma_{\text{shmr}}^N = \min_{\gamma > 0, a_i \in \mathbb{R}, b_i \in \mathbb{R}} \gamma \quad \text{subject to} \quad (9)$$

$$|Ga - b| < \gamma \text{Re}(a) \quad \forall \omega \in \Omega \quad (10)$$

$$\text{Re}(a) > 0 \quad \forall \omega \in [0, \pi] \quad (11)$$

$$a = \sum_{i=-k}^k a_i e^{-ij\omega} \quad b = \sum_{i=-k}^k b_i e^{-ij\omega} \quad (12)$$

- 3) Given a solution a_* , solve $a_* = q_* \varphi_*^\sim$, where φ_* , q_* have only stable zeros and poles
- 4) Solve for $p = \sum_{i=0}^k p_i e^{-ij\omega}$ and $p_i \in \mathbb{R}$

$$p_* = \underset{p}{\text{argmin}} \max_{\omega \in \Omega} |G - p/q_*| \quad (13)$$

Finally, the semidefinite Hankel-type model reduction reads as solving (3) subject only to (4), and then solving (7), (8) consecutively.

A. Tractable Algorithm and its Computational Complexity

The programs (3) subject to (4) and (8) have an infinite number of constraints, one for each frequency ω in $[0, \pi]$. Therefore, these are not tractable problems. However, since G is a rational transfer function, its frequency response cannot change infinitely fast. It means that it is sufficient to impose the constraints (4) on a finite number of frequencies $\{\omega_i\}_{i=1}^N \in [0, \pi]$. This is outlined in Algorithm 1, where γ_{shmr}^N is defined. The frequency gridding is also a relaxation, as discussed in [13]. Let $\{\omega_i\}_{i=1}^N$ be an N element subset of a countable set $\{\omega_i\}_{i=1}^\infty$, which is dense in $[0, \pi]$, and γ_{shmr}^c be the solution to (3) subject only to (4). Then by construction $\gamma_{\text{shmr}}^N \leq \gamma_{\text{shmr}}^c$ for any positive integer N . Moreover, $\lim_{N \rightarrow \infty} \gamma_{\text{shmr}}^N = \gamma_{\text{shmr}}^c$, therefore, with a large enough N the theoretical value γ_{shmr}^c is approximated by γ_{shmr}^N . To avoid over-fit, the number of points N in the grid should be at least $O(k^2)$, where k is the order of the reduced model. This gridding approach

may create unstable approximations, therefore the positivity constraint (11) is enforced for all the frequencies ω in $[0, \pi]$ using the Kalman-Yakubovitch-Popov lemma (see, e.g. [19]). Note that use of this lemma will add an LMI constraint with k rows and $O(k^2)$ decision variables.

There are two main contributors to complexity of the algorithm: the computation of frequency response samples and the solution of (9-10) and (13). Note that spectral factorization in the step 3 of Algorithm 1 is performed over the variables describing the reduced model, therefore, the cost is not substantial and equal to $O(k^3)$. Computing the frequency response samples costs in general $O(n^3)$ and can be lowered to $O(n^2)$ or $O(n \log(n))$ in certain relevant cases ([7], [8], [9]), where n is the order of G . The computational costs of (9-10) and (13) when solved with a cutting plane method are $O(k^4)$ (see, [14]). Based on numerical simulations, the computationally heaviest part for large scale systems ($n > 10000$) is the computation of frequency response samples. If $\Omega = [0, \pi]$, Algorithm 1 is not tractable (the cost of optimization depends on n), but can be theoretically solved, while enforcing all constraints with the KYP lemma.

B. Error Bounds and System Theoretic Interpretation of the Relaxation

The presented relaxation is interesting due to its connection to Hankel model reduction. Recall that a and b are pseudo-polynomials in z with degrees spanning from $-k$ to k . Rewrite the constraints in (3) with a norm constraint and note that $|a/\text{Re}(a)|$ is larger or equal to one for all the frequencies ω , since $|a| \geq |\text{Re}(a)|$. This yields:

$$\min_{\text{Re}(a)>0, b} \left\| \left(G - \frac{b}{a} \right) \frac{a}{\text{Re}(a)} \right\|_{\infty} \geq \min_{\text{Re}(a)>0, b} \left\| G - \frac{b}{a} \right\|_{\infty}$$

In the right part of the inequality, we have an optimization program over a transfer function b/a , which has k stable poles and it has an anti-stable part. This resembles optimization in the Hankel norm, however, in the presented optimization program there are extra constraints on the decision variables. Foremost, it is a positive real constraint on a . All in all, we can deduce that:

$$\min_{\text{Re}(a)>0, b} \left\| \left(G - \frac{b}{a} \right) \frac{a}{\text{Re}(a)} \right\|_{\infty} \geq \min_{p, q} \left\| G - \frac{p}{q} \right\|_H$$

It means that Hankel model reduction is a relaxation of our algorithm, without a weight in the objective ($a/\text{Re}(a)$). Now we are ready to formulate the main theorem of the section.

Theorem 1: Let γ_{shmr}^c and p_*/q_* denote the output of Algorithm 1 with inputs $\Omega = [0, \pi]$ and k , γ_{mor} denote the optimal approximation level from (1) and $\sigma_{k+1}(G)$ denote $k + 1$ -st largest

Hankel singular value of G . Then the following error bounds hold:

$$\sigma_{k+1}(G) \leq \gamma_{\text{shmr}}^c \leq \gamma_{\text{mor}} \leq \|G - p_*/q_*\|_{\mathbb{H}_\infty} \quad (14)$$

$$\|G - p_*/q_*\|_{\mathbb{H}_\infty} \leq (k+1)\gamma_{\text{shmr}}^c \quad (15)$$

The proof of the theorem is based on [13] and is found in [17]. The obtained upper bounds are conservative, since no information about the full order model G is used. However, an upper bound is required as a guarantee, that the employed relaxation will not create unreasonable approximations. The quality of a relaxation is estimated on numerical examples.

The quasi-convex optimization (QCO) approach [14] is Algorithm 1 with an extra symmetry constraint $a(z) = a(1/z)$. It is straightforward to show that if $a(z) = a(1/z)$, then $\text{Im}(a(e^{j\omega}))$ is equal to zero and $\text{Re}(a(e^{j\omega}))$ is equal to a . Thus, the following chain of inequalities is available, where γ_{qco}^c is the solution to the relaxed problem in [14], defined similarly to γ_{shmr}^c :

$$\begin{aligned} (k+1)\gamma_{\text{qco}}^c &\geq (k+1)\gamma_{\text{shmr}}^c \geq \gamma_{\text{mor}} \geq \underbrace{\min_{a>0,b} \left\| G - \frac{b}{a} \right\|_{\infty}}_{=\gamma_{\text{qco}}^c} \geq \\ &\underbrace{\min_{\text{Re}(a)>0,b} \left\| \left(G - \frac{b}{a} \right) \frac{a}{\text{Re}(a)} \right\|_{\infty}}_{=\gamma_{\text{shmr}}^c} \geq \underbrace{\min_{p/q \in \mathbb{H}_\infty} \left\| G - \frac{p}{q} \right\|_H}_{=\sigma_{k+1}(G)} \end{aligned}$$

IV. AN ITERATIVE APPROACH TO HANKEL-TYPE FORMULATION

The iterative algorithm is obtained from (2) simply by choosing an FIR filter ψ with a stable inverse in advance and solving (2) with a fixed $\varphi = \psi$:

$$\begin{aligned} \min_{\gamma>0,p,q} \gamma \quad \text{subject to} \quad & (16) \\ |(Gq - p)\psi| < \gamma \text{Re}(q\tilde{\psi}) \quad \forall \omega \in [0, \pi] \end{aligned}$$

This program slightly differs from (2). However, $|(Gq - p)\psi| = |(Gq - p)\psi^\sim|$ and $\text{Re}(q\tilde{\psi}) = \text{Re}(q\psi^\sim)$. This form is taken to simplify the notation. Note that the stability constraint on q is satisfied given a stable ψ^{-1} . It is possible to iterate this program by setting $\psi = q$ on the next step. The proposed approach is summarized in Algorithm 2, where the sequences ψ^j and γ^j are defined. Its implementation is similar to the one of Algorithm 1. The same remarks about the number of samples and computational complexity are valid for Algorithm 2, as well. Convergence of the algorithm is shown in the following lemma, which is proved in [17].

Algorithm 2 Iterative Approach to Model Reduction

Inputs: Frequency grid $\Omega = \{\omega_i\}_{i=1}^N \in [0, \pi]$, $\psi^1(z) = \sum_{i=0}^k \psi_i^1 z^{-i}$, where $(\psi^1)^{-1}$ is stable

Outputs: Sequences γ^j , ψ^j , reduced models p^j/q^j

Compute $G(e^{j\omega})$ for all $\omega \in \Omega$, let $j = 1$, $\gamma^0 = \max_{\omega \in \Omega} |G(e^{j\omega})|$ and

$$q = 1 + \sum_{i=1}^k q_i e^{-ij\omega} \quad \text{and} \quad p = \sum_{i=0}^k p_i e^{-ij\omega}$$

repeat

 Given ψ^j , solve

$$\gamma^j = \min_{\gamma > 0, p, q} \gamma \quad \text{subject to} \quad \operatorname{Re}(q \sim \psi^j) > 0 \quad \forall \omega \in [0, \pi]$$

$$|(Gq - p) \psi^j| < \gamma \operatorname{Re}(q \sim \psi^j) \quad \forall \omega \in \Omega$$

 Let q^j and p^j denote the solution. Set $\psi^{j+1} = q^j$ and $j = j + 1$.

until $|\gamma^{j-1} - \gamma^{j-2}| \leq \varepsilon$

Lemma 3: If the constraints are enforced for all the frequencies ω in the interval $[0, \pi]$, then $\{\gamma^j\}_{j=1}^\infty$ is a bounded, non-increasing sequence and there exist such a γ^∞ that $\lim_{j \rightarrow \infty} \gamma^j = \gamma^\infty$. Note, convergence in variables p^j , q^j and ψ^j is not shown in this paper. Nevertheless, in all numerical examples the convergence is achieved, if the stopping criterion is changed to $\|\psi^j - \psi^{j-1}\|_\infty$, for example. The results for the converged approximation levels γ^∞ are the same in both cases.

The properties of the limit function p^∞/q^∞ is an intriguing topic, however, such analysis is difficult since the behavior of the limit is dependent on the initial point. If minimized over ψ as well, the program (16) becomes non-convex and non-smooth, and can have numerous local minima. Any iterative approach to solve the problem does not, generally, converge to the globally optimal solution.

V. EXAMPLES

Hankel model reduction and balanced truncation are implemented by MATLABTM routines HANKELMR and BALANCMR respectively. Algorithm 1 and [14] are implemented using a cutting plane method (for more details, see [14]). Continuous-time models are discretized first, while

pre-warping around a specified frequency ω_p . Discretization is performed by mapping continuous-time frequencies ω_c to discrete-time ones as $\omega_d = 2 \operatorname{atan}(\omega_c/\omega_p)$ (or a MATLAB™ code `C2D($G, \pi/\omega_p, \text{'PREWARP'}, \omega_p$)`). Note that frequency responses of the discrete-time model at ω_d and of the continuous time one at ω_c are the same. The parameter ω_p is a tuning parameter for numerical conditioning. For example, if the biggest resonant peak occurs around a frequency ω_p , then pre-warping around ω_p creates a better numerically conditioned problem.

Example 1 (Reconstruction of all-pass systems): This toy example is created to show better numerical robustness of Algorithm 1 in comparison to [14]. Specify two all-pass models (i.e., their \mathbb{H}_∞ norms are equal to one) as:

$$G_i = \prod_{j=1}^{12} \frac{1 - z \xi_j^\sim}{z - \xi_j}, \text{ where } |\xi_j| = 0.96 \quad \forall i, j$$

and the arguments for the complex conjugate poles are:

$$G_1 : \arg(\xi_j) = \pm[0.11, 0.13, 0.14, 3.1, 3.11, 3.14]$$

$$G_2 : \arg(\xi_j) = \pm[0.11, 0.13, 0.14, 1.57, 1.57, 1.57]$$

Our goal is to reconstruct the models G_1 and G_2 from the frequency response data using [14] and Algorithm 1. It means that the reduction order k is equal to the orders of G_i , which are equal to 12. Theoretically both methods are able to do this with the approximation error 0 for any frequency grid Ω . Algorithm 1 provided reduced order models with approximation errors less than 0.01 (recall that $\|G_i\|_{\mathbb{H}_\infty} = 1$). However, [14] failed in both cases providing approximation errors around 0.10. The reason for such a behavior is the large ratio between maximal and minimal coefficients in the denominators of the models G_i . Since the goal is the reconstruction of G_i , the variables q should be equal to the denominators of G_i . The algorithm from [14] is obliged to preserve the relationship $a = q\varphi^\sim$, therefore the ratio between maximal and minimal coefficients of q is squared in a . On the other hand, Algorithm 1 is able to choose φ freely in $a = q\varphi^\sim$, which ensures better numerical robustness.

Such numerical issues occur due to the choice of the polynomial basis for parametrization of a and b , i.e. the functions z^{-i} . One possible treatment for these problems is choosing a different basis (see, for details [20]). Note that theoretical results do not depend on the choice of a basis.

Example 2 (Transmission Line Modeling): This example is described in [21] and the references therein. This is a 2-input-2-output transmission line model with 256 states. In this example,

TABLE I
RELATIVE APPROXIMATION ERRORS $100 \cdot \frac{\|G - \hat{G}\|_{\mathbb{H}_\infty}}{\max_{\omega \in \Omega} |G(e^{j\omega})|}$ IN EXAMPLE 2

Reduction order k	8	13	18
$\sigma_{k+1}(G)$	10.83	3.16	0.31
HMR	21.72	6.35	0.63
BT	21.72	6.41	0.63
[6]	21.21	6.23	0.62
HMR+Algorithm 2	12.07	3.92	0.43
[14]	11.61	3.92	0.43
Algorithm 1	11.27	3.47	0.42

only the $\{2, 2\}$ -entry of the transfer function is being reduced. The main goal of this example is to evaluate the best possible performance in terms of accuracy of Algorithms 1 and 2. For this reason the number of computed frequency samples is large (around 900) and the frequencies are spread in the interval $[0, +\infty)$. The algorithms are compared to [14], [6], balanced truncation (BT) and Hankel model reduction (HMR). Recall that [6] computes a model, which satisfies the first order optimality conditions in \mathbb{H}_2 , if this model is stable. In order to apply [6] one needs to specify initial shift points, which are chosen randomly according to [6]. Therefore [6] is run numerous times and the best results are presented in Table I. The notation *HMR+Algorithm 2* means that algorithms are run consecutively. That is, the initial ϕ^1 for Algorithm 2 is computed based on the Hankel approximation.

As shown in Table I, HMR, BT and [6] perform similarly to each other for all the orders. The optimization-based approaches ([14], Algorithm 1 and HMR+Algorithm 2) show similar to each other performance. Note that the behavior of HMR+Algorithm 2 is more consistent with [14] and Algorithm 1, than HMR. Moreover, Algorithm 2 is able to significantly reduce the HMR approximation errors. It is also reasonable to apply Algorithm 2 with $\psi^1 = q_*$ from Algorithm 1. There is a gain in the approximation quality in this case as well, however, our goal is to evaluate the algorithms separately. An advantage of the proposed algorithms, which explains such a behavior, is a better choice of the numerator. This can be better understood by referring to Figure 1. The order 8 Hankel approximation (blue dashed line) cuts off the high-frequency peaks of the full order model (thin black line). Similar behavior is exhibited by the 8-th order models obtained by balanced truncation and [6], which are not depicted due to

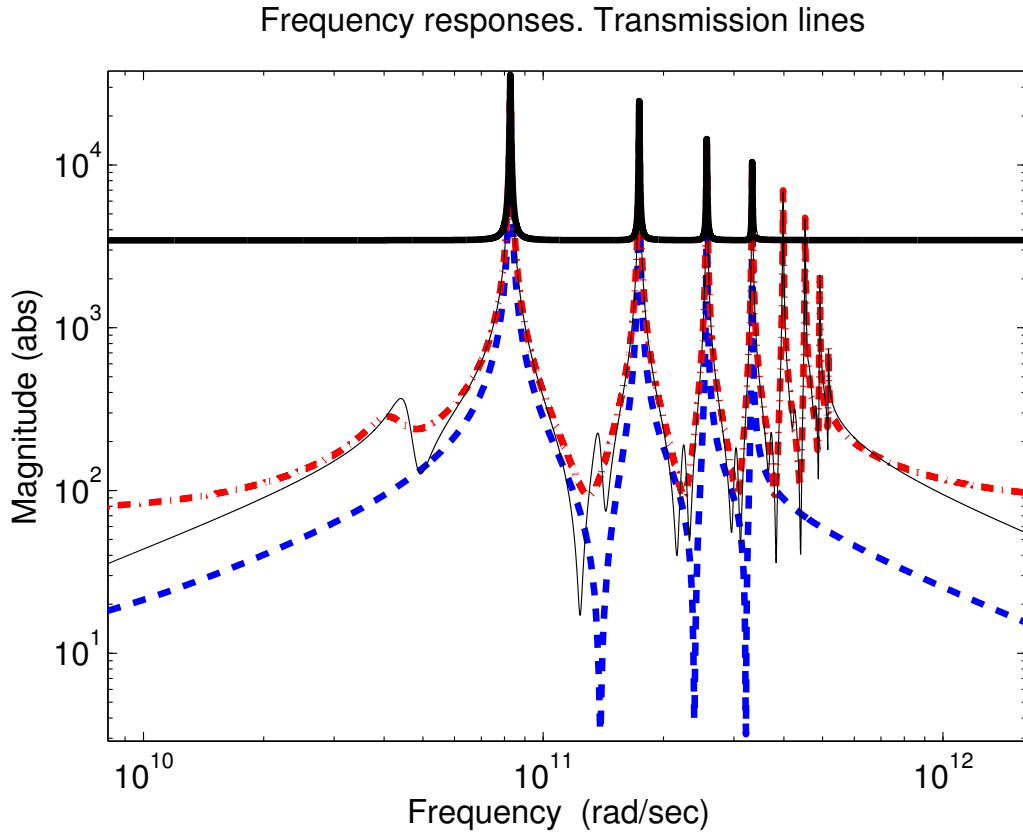


Fig. 1. Bode magnitude plots in Example 2. Thin black line - the full order model, blue dashed line - HMR approximation of order 8, thick black and red dash-dotted lines are obtained with Algorithm 1 and are approximations of order 8 and 18 respectively.

overlapping. On the other hand, the optimization techniques succeed in improving the \mathbb{H}_∞ error by choosing a nearly constant gain for high frequencies. The 8-th order approximation resulting from Algorithm 1 is depicted in Figure 1 (thick black line). To conclude, the minimization objectives in the presented algorithms reflect the \mathbb{H}_∞ error better than the reduction criteria in the discussed state-space approaches. If one desires to cut off the high-frequency peaks as state-space methods do, then this can be incorporated into model reduction objective using frequency weights or simply by choosing appropriately the frequency grid Ω .

An important question is choosing the order of the reduced model. Guided by this example and Table I a lower bound on the order can be deduced by inspecting the frequency response of the full order model. In Figure 1 the Bode magnitude plot has 8 peaks with large magnitude,

which are contributed by at least 16 complex conjugate poles. Therefore, the reduced order models would provide a decent match if the order is at least 16.

VI. CONCLUSION AND DISCUSSION

The examples show that the presented algorithms can provide more accurate models in \mathbb{H}_∞ than other model reduction techniques. They are more flexible in terms of adding constraints on reduced order model and/or modifying the objective function of model reduction problem. However, in the current state the presented methods require more computational effort to obtain reduced order models than Krylov methods. At the same time the optimization programs themselves do not depend on the order of the original model. Therefore, for large scale systems the computational time for Krylov methods and the presented algorithms is comparable. Other issues of the optimization programs are of numerical nature. These are associated with the choice of a basis and a frequency grid. Both have heuristic solutions in the literature, which can improve numerical robustness of the optimization program. For example, randomized methods can be used to choose the grid, when a larger number of samples is generated around peaks in magnitude.

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