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Abstract

We derive four variational principles for the electric and magnetic polarizabilities for a structure consisting of anisotropic media with perfect electric conductor inclusions. From these principles we derive monotonicity results and upper and lower bounds on the electric and magnetic polarizabilities. When computing the polarizabilities numerically, the bounds can be used as error bounds. The variational principles demonstrate important differences between electrostatics and magnetostatics when PEC bodies are present.

1 Introduction

Variational principles can be viewed as a physical way of interpreting mathematical equations. Instead of giving the relevant physical law as, for instance, a partial differential equation, a variational principle typically defines an energy functional, where the correct physical behavior is obtained for the trial function giving the minimum value of the functional. Typically, the functional can be interpreted as the energy of the system.

In our case, we are interested in calculating the electric and magnetic polarizability of a system consisting of anisotropic permittivity and permeability, possibly containing inclusions of metal modelled as a perfect electric conductor (PEC). This can be used in various applications, for instance Rayleigh scattering [10, 11, 13, 14] and homogenization theory [7, 9, 16]. Recently, a series of papers have proposed the use of electric and magnetic polarizability to give physical bounds on electromagnetic interaction over all frequencies for antennas, materials and general scatterers [6, 19–22].

Electrostatics is one of the prime examples of the Laplace equation, and has thus been studied thoroughly. Magnetostatics is somewhat younger, but due to its large financial impact on for instance power transformers and hard disk drives, it has also received significant attention [1–3]. A classical problem linking directly to ours is to compute low frequency circuit parameters such as capacitance and inductance [12, 24, 26, 27]. It is interesting to note that the problem of magnetic polarizability of a PEC body, is mathematically equivalent to the problem of computing the virtual mass [15, p. 31] of a body in a uniformly flowing fluid [17, 18, 23].

Even though variational principles are typically associated with a self-adjoint operator, we note that in some cases there exist techniques for reformulating the problem so that variational principles can be found for, for instance, complex valued non-hermitian matrices describing material properties [4]. However, in this paper we assume all material properties can be modelled using symmetric, real-valued matrices.

This paper is organized as follows. In Section 2, we state the geometry of our problem. In Section 3 we summarize the variational principles, which are given a full derivation from Maxwell's equations in Appendix A. In Section 4, it is shown that these principles imply monotonicity results for the polarizabilities. The variational principles are interpreted as giving upper and lower bounds for the polarizabilities in

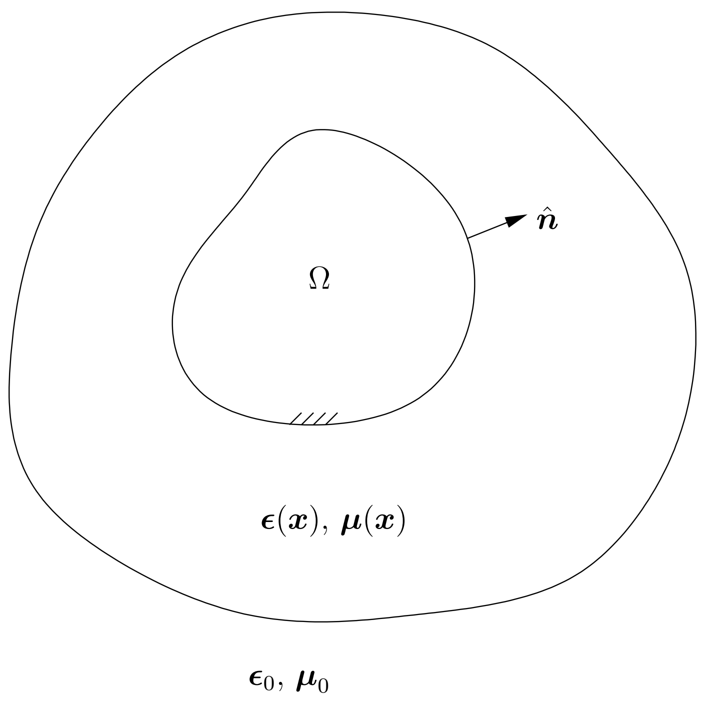

Figure 1: Typical geometry of the considered structure.

Section 5, and a numerical illustration is given in Section 6. Finally, some conclusions are given in Section 7.

2 Geometry and statement of the problem

We consider the situation of a structure as in Figure 1 with anisotropic permittivity and permeability matrices $\boldsymbol{\epsilon}(\boldsymbol{x})$ and $\boldsymbol{\mu}(\boldsymbol{x})$ and possibly PEC inclusions in a region Ω with volume V_Ω, such that $\mathbb{R}^3 \setminus \Omega$ is simply connected. The structure is surrounded by a vacuous medium with permittivity $\boldsymbol{\epsilon}_0 = \epsilon_0 \mathbf{I}$ and permeability $\boldsymbol{\mu}_0 = \mu_0 \mathbf{I}$. The structure is subjected to a homogeneous electric field and a homogeneous magnetic field. The induced redistribution of charges and currents in the structure gives rise to an electric and magnetic dipole moment according to

$$\boldsymbol{p} = \int (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0)\boldsymbol{E}\,\mathrm{d}V + \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}} \cdot \boldsymbol{D}\,\mathrm{d}S \tag{2.1}$$

$$\boldsymbol{m} = \int (\boldsymbol{\mu}_0^{-1} - \boldsymbol{\mu}^{-1})\boldsymbol{B}\,\mathrm{d}V + \frac{1}{2}\oint_{\partial\Omega} \boldsymbol{x} \times (\hat{\boldsymbol{n}} \times \boldsymbol{H})\,\mathrm{d}S \tag{2.2}$$

where $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ are the full electric field strength, electric flux density, magnetic flux density, and magnetic field strength, respectively. Even though we do not write it out explicitly, the volume integrals do not encompass the PEC region Ω, since both the electric and magnetic fields are zero inside, with boundary conditions $\hat{\boldsymbol{n}} \times \boldsymbol{E} = \mathbf{0}$ and $\hat{\boldsymbol{n}} \cdot \boldsymbol{B} = 0$ [8, p. 204]. Thus, the integral $\int (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0)\boldsymbol{E}\,\mathrm{d}V$ should be interpreted as $\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}\,\mathrm{d}V$. This helps us considering structures with and without PEC bodies without complicating the notation.

The electric and magnetic polarizabilities are defined by

$$\boldsymbol{p} = \epsilon_0\boldsymbol{\gamma}_{\mathrm{e}}\boldsymbol{E}_0 = \boldsymbol{\gamma}_{\mathrm{e}}\boldsymbol{D}_0 \tag{2.3}$$

$$\boldsymbol{m} = \mu_0^{-1}\boldsymbol{\gamma}_{\mathrm{m}}\boldsymbol{B}_0 = \boldsymbol{\gamma}_{\mathrm{m}}\boldsymbol{H}_0 \tag{2.4}$$

where we used that in the surrounding medium, we cannot distinguish between the applied electric field strength $\boldsymbol{E}_0$ or the flux density $\boldsymbol{D}_0$, since these are related by $\boldsymbol{D}_0 = \epsilon_0\boldsymbol{E}_0$. The same reasoning applies for the magnetic fields.

Even though the problem is primarily stated for finite structures in three-dimensional space, the final formulation of the variational principles admits also periodic solutions, where for instance the finite structure in Figure 1 is repeated periodically without the PEC portion of the copies touching each other.

3 Summary of variational principles

In this section, we summarize the variational formulations derived in Appendix A. The derivation is based on starting from the static Maxwell's equations, representing the fields using either scalar or vector potentials, and constructing natural quadratic forms. The variational principles along with the associated classes of admissible potentials are as follows. Using scalar potential for the electric field:

$$\begin{aligned} J_{\mathrm{e}}(\varphi,\boldsymbol{E}_0) = &\int \nabla\varphi\cdot\boldsymbol{\epsilon}\nabla\varphi\,\mathrm{d}V - 2\int\nabla\varphi\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V \\ &+ \boldsymbol{E}_0\cdot\left[\int(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\,\mathrm{d}V + V_\Omega\boldsymbol{\epsilon}_0\right]\cdot\boldsymbol{E}_0 \end{aligned} \tag{3.1}$$

$$\mathcal{A}_\varphi = \{\varphi\in H_1(\mathbb{R}^3\setminus\Omega);\ \hat{\boldsymbol{n}}\times(\boldsymbol{E}_0-\nabla\varphi) = \boldsymbol{0}\quad\text{on}\quad\partial\Omega\} \tag{3.2}$$

Using vector potential for the electric field:

$$\begin{aligned} K_{\mathrm{e}}(\boldsymbol{F},\boldsymbol{D}_0) = &\int(\nabla\times\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}\nabla\times\boldsymbol{F}\,\mathrm{d}V - 2\int(\nabla\times\boldsymbol{F})\cdot(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1})\boldsymbol{D}_0\,\mathrm{d}V \\ &- 2\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})\,\mathrm{d}S \\ &+ \boldsymbol{D}_0\cdot\left[-\int(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1})\,\mathrm{d}V + V_\Omega\boldsymbol{\epsilon}_0^{-1}\right]\boldsymbol{D}_0 \end{aligned} \tag{3.3}$$

$$\mathcal{A}_{\boldsymbol{F}} = \{\boldsymbol{F}\in H_1(\mathrm{curl},\mathbb{R}^3\setminus\Omega);\ \hat{\boldsymbol{n}}\times\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}) = \boldsymbol{0}\quad\text{on}\quad\partial\Omega\} \tag{3.4}$$

Using scalar potential for the magnetic field:

$$\begin{aligned} J_{\mathrm{m}}(\psi,\boldsymbol{H}_0) = &\int \nabla\psi\cdot\boldsymbol{\mu}\nabla\psi\,\mathrm{d}V - 2\int\nabla\psi\cdot(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\boldsymbol{H}_0\,\mathrm{d}V \\ &+ 2\boldsymbol{H}_0\cdot\frac{\mu_0}{2}\oint_{\partial\Omega}\boldsymbol{x}\times(\hat{\boldsymbol{n}}\times(\boldsymbol{H}_0-\nabla\psi))\,\mathrm{d}S \\ &+ \boldsymbol{H}_0\cdot\left[\int(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\,\mathrm{d}V + V_\Omega\boldsymbol{\mu}_0\right]\boldsymbol{H}_0 \end{aligned} \tag{3.5}$$

$$\mathcal{A}_\psi = \{\psi\in H_1(\mathbb{R}^3\setminus\Omega);\ \hat{\boldsymbol{n}}\cdot\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi)=0\quad\text{on}\quad\partial\Omega\} \tag{3.6}$$

Using vector potential for the magnetic field:

$$\begin{aligned} K_{\mathrm{m}}(\boldsymbol{A},\boldsymbol{B}_0) = &\int(\nabla\times\boldsymbol{A})\cdot\boldsymbol{\mu}^{-1}\nabla\times\boldsymbol{A}\,\mathrm{d}V - 2\int(\nabla\times\boldsymbol{A})\cdot(\boldsymbol{\mu}_0^{-1}-\boldsymbol{\mu}^{-1})\boldsymbol{B}_0\,\mathrm{d}V \\ &+ \boldsymbol{B}_0\cdot\left[-\int(\boldsymbol{\mu}_0^{-1}-\boldsymbol{\mu}^{-1})\,\mathrm{d}V + V_\Omega\boldsymbol{\mu}_0^{-1}\right]\boldsymbol{B}_0 \end{aligned} \tag{3.7}$$

$$\mathcal{A}_{\boldsymbol{A}} = \{\boldsymbol{A}\in H_1(\mathrm{curl},\mathbb{R}^3\setminus\Omega);\ \hat{\boldsymbol{n}}\cdot(\boldsymbol{B}_0+\nabla\times\boldsymbol{A})=0\quad\text{on}\quad\partial\Omega\} \tag{3.8}$$

The minimum values of these functionals are given by

$$\min_{\varphi\in\mathcal{A}_\varphi} J_{\mathrm{e}}(\varphi,\boldsymbol{E}_0) = J_{\mathrm{e}}(\varphi_0,\boldsymbol{E}_0) = \boldsymbol{E}_0\cdot\boldsymbol{p} = \epsilon_0\boldsymbol{E}_0\cdot\boldsymbol{\gamma}_{\mathrm{e}}\boldsymbol{E}_0 \tag{3.9}$$

$$\min_{\boldsymbol{F}\in\mathcal{A}_{\boldsymbol{F}}} K_{\mathrm{e}}(\boldsymbol{F},\boldsymbol{D}_0) = K_{\mathrm{e}}(\boldsymbol{F}_0,\boldsymbol{D}_0) = -\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{p} = -\epsilon_0^{-1}\boldsymbol{D}_0\cdot\boldsymbol{\gamma}_{\mathrm{e}}\boldsymbol{D}_0 \tag{3.10}$$

$$\min_{\psi\in\mathcal{A}_\psi} J_{\mathrm{m}}(\psi,\boldsymbol{H}_0) = J_{\mathrm{m}}(\psi_0,\boldsymbol{H}_0) = \boldsymbol{H}_0\cdot\boldsymbol{\mu}_0\boldsymbol{m} = \mu_0\boldsymbol{H}_0\cdot\boldsymbol{\gamma}_{\mathrm{m}}\boldsymbol{H}_0 \tag{3.11}$$

$$\min_{\boldsymbol{A}\in\mathcal{A}_{\boldsymbol{A}}} K_{\mathrm{m}}(\boldsymbol{A},\boldsymbol{B}_0) = K_{\mathrm{m}}(\boldsymbol{A}_0,\boldsymbol{B}_0) = -\boldsymbol{B}_0\cdot\boldsymbol{m} = -\mu_0^{-1}\boldsymbol{B}_0\cdot\boldsymbol{\gamma}_{\mathrm{m}}\boldsymbol{B}_0 \tag{3.12}$$

where the minimizing potentials $(\varphi_0,\boldsymbol{F}_0,\psi_0,\boldsymbol{A}_0)$ satisfy the electrostatic and magnetostatic equations

$$\nabla\cdot\boldsymbol{D} = \nabla\cdot[\boldsymbol{\epsilon}(\boldsymbol{E}_0-\nabla\varphi_0)] = 0 \tag{3.13}$$

$$\nabla\times\boldsymbol{E} = \nabla\times[\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)] = \boldsymbol{0} \tag{3.14}$$

$$\nabla\cdot\boldsymbol{B} = \nabla\cdot[\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi_0)] = 0 \tag{3.15}$$

$$\nabla\times\boldsymbol{H} = \nabla\times[\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A}_0)] = \boldsymbol{0} \tag{3.16}$$

We call J_{e} and K_{m} the direct functionals, since they are associated with the "natural" potentials for electric and magnetic fields, whereas K_{e} and J_{m} are called the dual functionals. An interesting difference between the direct functionals and the dual functionals, is that the dual functionals include a term which is the scalar product of the applied field and (twice) the induced dipole moment in the PEC body. In the electric case this term is minimized when the dipole moment is parallel with the applied field, whereas in the magnetic case the term is minimized when the dipole moment is antiparallel to the applied field. This expresses a fundamental difference in sign for the PEC polarizability for electric and magnetic fields. This is further explored in the following section.

4 Monotonicity of the polarizabilities

Consider a situation with two different PEC bodies, Ω and Ω', where $\Omega' \subseteq \Omega$ and $\delta\Omega = \Omega \setminus \Omega'$. Associated with Ω and Ω' are also the permittivity functions $\boldsymbol{\epsilon}(\boldsymbol{x})$ and $\boldsymbol{\epsilon}'(\boldsymbol{x})$ and permeability functions $\boldsymbol{\mu}(\boldsymbol{x})$ and $\boldsymbol{\mu}'(\boldsymbol{x})$, respectively. The spaces of admissible test functions are slightly different, since the boundary conditions are not the same. However, for each $\varphi \in \mathcal{A}_\varphi$, we can choose a $\varphi' \in \mathcal{A}'_{\varphi'}$ such that $\varphi' = \varphi$ in the exterior of Ω and $\varphi' = \boldsymbol{x} \cdot \boldsymbol{E}_0$ in $\delta\Omega$, *i.e.*, $\boldsymbol{E}_0 - \nabla \cdot \varphi' = \boldsymbol{0}$ in $\delta\Omega$. Also, for each $\boldsymbol{A} \in \mathcal{A}_{\boldsymbol{A}}$, we can choose a $\boldsymbol{A}' \in \mathcal{A}'_{\boldsymbol{A}'}$ such that $\boldsymbol{A}' = \boldsymbol{A}$ in the exterior of Ω and $\boldsymbol{A}' = \frac{1}{2}\boldsymbol{x} \times \boldsymbol{B}_0$ in $\delta\Omega$, *i.e.*, $\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}' = \boldsymbol{0}$ in $\delta\Omega$. This construction can only be applied to the test functions for the direct functionals J_e and K_m, and not the dual functionals K_e and J_m. This is because the boundary conditions for the dual functionals are expressed in spatial derivatives of the potentials, and making the corresponding construction for the duals functionals would lead to discontinuous potentials. Using the results from (A.22) and (A.37) we then have (for arbitrary $\varphi \in \mathcal{A}_\varphi$ and $\boldsymbol{A} \in \mathcal{A}_{\boldsymbol{A}}$)

$$J_\mathrm{e}(\varphi, \boldsymbol{E}_0) - J'_\mathrm{e}(\varphi', \boldsymbol{E}_0) = \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{E}_0 - \nabla\varphi) \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}')(\boldsymbol{E}_0 - \nabla\varphi)\,\mathrm{d}V \tag{4.1}$$

$$K_\mathrm{m}(\boldsymbol{A}, \boldsymbol{B}_0) - K'_\mathrm{m}(\boldsymbol{A}', \boldsymbol{B}_0) = \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})(\boldsymbol{\mu}^{-1} - (\boldsymbol{\mu}')^{-1})(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})\,\mathrm{d}V \tag{4.2}$$

Using the minimizing potentials φ_0 and $\boldsymbol{A}_0$ for the unprimed functionals, we obtain the inequalities, valid for any $\Omega' \subseteq \Omega$,

$$\epsilon_0 \boldsymbol{E}_0 \cdot (\boldsymbol{\gamma}_\mathrm{e} - \boldsymbol{\gamma}'_\mathrm{e})\boldsymbol{E}_0 \geq \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{E}_0 - \nabla\varphi_0) \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}')(\boldsymbol{E}_0 - \nabla\varphi_0)\,\mathrm{d}V \tag{4.3}$$

$$\mu_0^{-1} \boldsymbol{B}_0 \cdot (-\boldsymbol{\gamma}_\mathrm{m} + \boldsymbol{\gamma}'_\mathrm{m})\boldsymbol{B}_0 \geq \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}_0)(\boldsymbol{\mu}^{-1} - (\boldsymbol{\mu}')^{-1})(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}_0)\,\mathrm{d}V \tag{4.4}$$

It is seen that these inequalities imply $\boldsymbol{\gamma}_\mathrm{e} \geq \boldsymbol{\gamma}'_\mathrm{e}$ if $\boldsymbol{\epsilon} \geq \boldsymbol{\epsilon}'$, and $\boldsymbol{\gamma}'_\mathrm{m} \geq \boldsymbol{\gamma}_\mathrm{m}$ if $\boldsymbol{\mu}' \geq \boldsymbol{\mu}$. This proves that both the electric and the magnetic polarizability are nondecreasing when the material parameters increase in the region exterior to Ω. If the material parameters are equal in the region exterior to Ω, *i.e.*, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'$ and $\boldsymbol{\mu} = \boldsymbol{\mu}'$, it is seen that $\boldsymbol{\gamma}_\mathrm{e} \geq \boldsymbol{\gamma}'_\mathrm{e}$ and $\boldsymbol{\gamma}_\mathrm{m} \leq \boldsymbol{\gamma}'_\mathrm{m}$ for arbitrary $\Omega' \subseteq \Omega$. Thus, when the volume of PEC increases, the electric polarizability is nondecreasing, but the magnetic polarizability is nonincreasing. A corresponding result is shown for isotropic dielectric bodies in [11].

When the structure consists of only PEC in vacuum, *i.e.*, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_0$ and $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ everywhere, the minimum properties (3.9) and (3.12) imply

$$\epsilon_0 \boldsymbol{E}_0 \cdot \boldsymbol{\gamma}_\mathrm{e} \boldsymbol{E}_0 = \epsilon_0 \int |\nabla\varphi_0|^2\,\mathrm{d}V + \epsilon_0 V_\Omega |\boldsymbol{E}_0|^2 \tag{4.5}$$

$$-\mu_0^{-1} \boldsymbol{B}_0 \cdot \boldsymbol{\gamma}_\mathrm{m} \boldsymbol{B}_0 = \mu_0^{-1} \int |\nabla \times \boldsymbol{A}_0|^2\,\mathrm{d}V + \mu_0^{-1} V_\Omega |\boldsymbol{B}_0|^2 \tag{4.6}$$

Since the right hand sides of these equations are positive, it is readily seen that the electric polarizability for PEC bodies in vacuum is positive, whereas the magnetic polarizability is negative. It is also seen that the amplitude of the polarizability in each case is always larger than the volume of the PEC body. When embedding PEC bodies in a magnetic material, there is an interplay where the PEC properties promote negative polarizability, whereas the material properties promote positive polarizability if $\boldsymbol{\mu} \geq \boldsymbol{\mu}_0$. A precise example is given by a PEC sphere of radius a, surrounded by a spherical layer with isotropic permeability μ and outer radius b. It can be shown that the total magnetic polarizability of this structure is zero if

$$\left(\frac{a}{b}\right)^3 = \frac{2(\mu/\mu_0 - 1)}{2\mu/\mu_0 + 1} = 1 - \frac{3}{2\mu/\mu_0 + 1} \tag{4.7}$$

If $(a/b)^3$ is larger than this value, the polarizability is negative, and if it is smaller, the polarizability is positive. In homogenization theory, inclusions with zero polarizability are called neutral [16, pp. 134–139], and are typically constructed from layered spheres as this one.

5 Upper and lower bounds on the polarizabilities

Using the variational formulations, we can find upper and lower bounds for the polarizabilities by inserting any set of admissible trial potentials $(\varphi, \boldsymbol{F}, \psi, \boldsymbol{A})$ in the inequalities

$$-K_{\mathrm{e}}(\boldsymbol{F}, \boldsymbol{D}_0) \leq \epsilon_0 \boldsymbol{E}_0 \cdot \boldsymbol{\gamma}_{\mathrm{e}} \boldsymbol{E}_0 \leq J_{\mathrm{e}}(\varphi, \boldsymbol{E}_0) \tag{5.1}$$

$$-K_{\mathrm{m}}(\boldsymbol{A}, \boldsymbol{B}_0) \leq \mu_0 \boldsymbol{H}_0 \cdot \boldsymbol{\gamma}_{\mathrm{m}} \boldsymbol{H}_0 \leq J_{\mathrm{m}}(\psi, \boldsymbol{H}_0) \tag{5.2}$$

where the applied fields are related by $\boldsymbol{D}_0 = \epsilon_0 \boldsymbol{E}_0$ and $\boldsymbol{B}_0 = \mu_0 \boldsymbol{H}_0$. Using for instance the finite element method (FEM) for solving the field equations, we can compute each functional and consider the numerical potentials as trial fields. Each set of numerical potentials $(\varphi_{\mathrm{num}}, \boldsymbol{F}_{\mathrm{num}}, \psi_{\mathrm{num}}, \boldsymbol{A}_{\mathrm{num}})$ can then be inserted in the inequalities (5.1) and (5.2), which provides a strict error bound for the numerical computation of the polarizabilities. A corresponding interpretation of variational bounds in homogenization theory can be found in [5].

When there are no PEC bodies, the zero potentials are admissible in the inequalities (5.1) and (5.2), implying

$$\boldsymbol{D}_0 \cdot \int (\boldsymbol{\epsilon}_0^{-1} - \boldsymbol{\epsilon}^{-1}) \,\mathrm{d}V \boldsymbol{D}_0 \leq \epsilon_0 \boldsymbol{E}_0 \cdot \boldsymbol{\gamma}_{\mathrm{e}} \boldsymbol{E}_0 \leq \boldsymbol{E}_0 \cdot \int (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) \,\mathrm{d}V \boldsymbol{E}_0 \tag{5.3}$$

$$\boldsymbol{B}_0 \cdot \int (\boldsymbol{\mu}_0^{-1} - \boldsymbol{\mu}^{-1}) \,\mathrm{d}V \boldsymbol{B}_0 \leq \mu_0 \boldsymbol{H}_0 \cdot \boldsymbol{\gamma}_{\mathrm{m}} \boldsymbol{H}_0 \leq \boldsymbol{H}_0 \cdot \int (\boldsymbol{\mu} - \boldsymbol{\mu}_0) \,\mathrm{d}V \boldsymbol{H}_0 \tag{5.4}$$

This states that the polarizabilities are bounded by the harmonic and arithmetic mean of the material parameters. In homogenization theory, this is known as the Wiener bounds [25].

From the monotonicity results in the previous section, it can be concluded that if we have a set of PEC regions included in each other, $\Omega' \subseteq \Omega \subseteq \Omega''$, then we have

$$\boldsymbol{\gamma}'_{\mathrm{e}} \leq \boldsymbol{\gamma}_{\mathrm{e}} \leq \boldsymbol{\gamma}''_{\mathrm{e}} \tag{5.5}$$

$$-\boldsymbol{\gamma}'_{\mathrm{m}} \leq -\boldsymbol{\gamma}_{\mathrm{m}} \leq -\boldsymbol{\gamma}''_{\mathrm{m}} \tag{5.6}$$

if the material parameters $\boldsymbol{\epsilon}(\boldsymbol{x})$ and $\boldsymbol{\mu}(\boldsymbol{x})$ are identical in each case. If the polarizability can be computed for the regions Ω' and Ω'', this leads to bounds for the unprimed polarizability. For instance, for PEC spheres in vacuum, it is easy to show that

$$\boldsymbol{\gamma}_{\mathrm{e}} = 4\pi a^3\mathbf{I} = 3V\mathbf{I}, \quad \boldsymbol{\gamma}_{\mathrm{m}} = -2\pi a^3\mathbf{I} = -\frac{3}{2}V\mathbf{I} \tag{5.7}$$

where V is the volume of the sphere. For an arbitrary PEC region Ω in vacuum, we can then formulate the bounds [18]

$$3V'\mathbf{I} \leq \boldsymbol{\gamma}_{\mathrm{e}} \leq 3V''\mathbf{I} \tag{5.8}$$

$$3V'/2\mathbf{I} \leq -\boldsymbol{\gamma}_{\mathrm{m}} \leq 3V''/2\mathbf{I} \tag{5.9}$$

where V' is the volume of the largest sphere contained in the body, and V'' is the volume of the smallest sphere containing the body. This result can be generalized to shapes like ellipsoids.

6 Numerical example

To demonstrate the upper and lower bounds provided by the variational principles, we consider the case of a PEC sphere in vacuum. For an axially symmetric structure, we can reduce the problem to two dimensions using cylindrical coordinates. In this case, the vector potentials are reduced to a single ϕ-component. The scalar and vector potentials satisfy the ϕ-independent Laplace equation with associated boundary conditions (assuming all exciting fields are directed along the z-direction)

$$\frac{1}{r_{\mathrm{c}}}\frac{\partial}{\partial r_{\mathrm{c}}}\left(r_{\mathrm{c}}\frac{\partial \varphi}{\partial r_{\mathrm{c}}}\right) + \frac{\partial^2 \varphi}{\partial z^2} = 0, \qquad \varphi = E_0 z \tag{6.1}$$

$$\frac{1}{r_{\mathrm{c}}}\frac{\partial}{\partial r_{\mathrm{c}}}\left(r_{\mathrm{c}}\frac{\partial F_\phi}{\partial r_{\mathrm{c}}}\right) + \frac{\partial^2 F_\phi}{\partial z^2} = 0, \qquad \hat{\boldsymbol{n}}\cdot\left[\hat{\boldsymbol{r}}_{\mathrm{c}}\frac{1}{r_{\mathrm{c}}}\frac{\partial(r_{\mathrm{c}}F_\phi)}{\partial r_{\mathrm{c}}} + \hat{\boldsymbol{z}}\frac{\partial F_\phi}{\partial z}\right] = -D_0\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{z}} \tag{6.2}$$

$$\frac{1}{r_{\mathrm{c}}}\frac{\partial}{\partial r_{\mathrm{c}}}\left(r_{\mathrm{c}}\frac{\partial \psi}{\partial r_{\mathrm{c}}}\right) + \frac{\partial^2 \psi}{\partial z^2} = 0, \qquad \hat{\boldsymbol{n}}\cdot\left[\hat{\boldsymbol{r}}_{\mathrm{c}}\frac{\partial \psi}{\partial r_{\mathrm{c}}} + \hat{\boldsymbol{z}}\frac{\partial \psi}{\partial z}\right] = H_0\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{z}} \tag{6.3}$$

$$\frac{1}{r_{\mathrm{c}}}\frac{\partial}{\partial r_{\mathrm{c}}}\left(r_{\mathrm{c}}\frac{\partial A_\phi}{\partial r_{\mathrm{c}}}\right) + \frac{\partial^2 A_\phi}{\partial z^2} = 0, \qquad A_\phi = -\frac{1}{2}B_0 r_{\mathrm{c}} \tag{6.4}$$

These equations are easily solved using software like Comsol Multiphysics, and we can compute the functionals J_{e}, K_{e}, J_{m}, and K_{m} using different discretizations. Each of these computations provides a new bound for the polarizabilities, and in Figure 2 we show how the bounds become progressively narrower as the discretization is made finer. The bounds are leveling out after just a few refinements of the grid. With the simple procedure of only refining the discretization, we conclude that we cannot expect more than about three digits accuracy using this program.

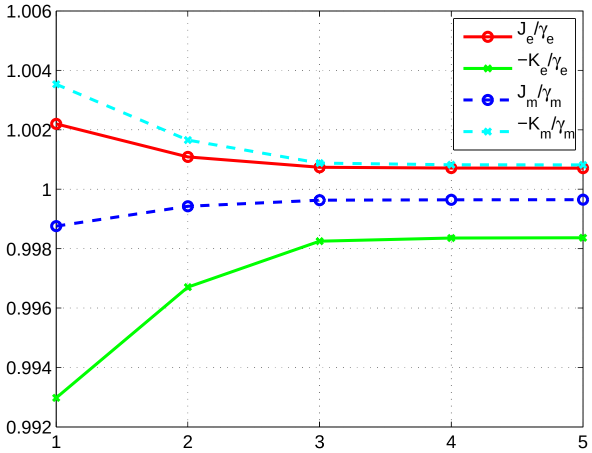

Figure 2: Demonstration of how numerical computations of the functionals provide bounds for γ_{e} and γ_{m}. Solid lines are for the electric case (normalized to γ_{e}), and dashed lines are for the magnetic case (normalized to γ_{m}). The x scale corresponds to the discretization used. The data points are for a PEC sphere descretized with 294, 665, 2660, 10640, and 42560 elements in a 2D axial symmetric geometry. The calculations are made with the commercial software Comsol Multiphysics 3.4 (http://www.comsol.com).

7 Conclusions

We have derived four variational principles from which the electric and magnetic polarizabilities can be computed or estimated. The polarizabilities are characterized as minimas and maximas of these functionals, providing strict error bounds when applying numerical methods to compute the polarizabilities. Similar functionals have been presented before, but this paper seems to be the first to give a unified presentation of anisotropic permittivity and permeability in combination with PEC inclusions.

The variational principles display important similarities and differences between electric and magnetic fields. If there are no PEC bodies present, there is a direct analogy between the functionals for the electric and magnetic case, making them formally identical to each other. However, when a PEC body is introduced, the fields satisfy different boundary conditions on the PEC surface, which leads to different variational principles for the electric and magnetic case, respectively. Specifically, the magnetic polarizability of a PEC body is negative, whereas the electric polarizability is positive. It is observed that in the electric case the boundary conditions are most easily expressed using a scalar potential, whereas the vector potential is most convenient in the magnetic case.

Appendix A Derivation of variational formulations for the polarizabilities

In this Appendix we derive the variational formulations for the polarizabilities. The electric and magnetic dipole moments are defined as

$$\boldsymbol{p} = \int (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0)\boldsymbol{E}\,\mathrm{d}V + \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}}\cdot\boldsymbol{D}\,\mathrm{d}S = \boldsymbol{p}_{\mathrm{mtrl}} + \boldsymbol{p}_{\mathrm{pec}} \tag{A.1}$$

$$\boldsymbol{m} = \int (\boldsymbol{\mu}_0^{-1} - \boldsymbol{\mu}^{-1})\boldsymbol{B}\,\mathrm{d}V + \frac{1}{2}\oint_{\partial\Omega} \boldsymbol{x}\times(\hat{\boldsymbol{n}}\times\boldsymbol{H})\,\mathrm{d}S = \boldsymbol{m}_{\mathrm{mtrl}} + \boldsymbol{m}_{\mathrm{pec}} \tag{A.2}$$

where $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ are the solutions to Maxwell's equations. We make use of the following integral identities, where Ω is a simply connected volume, $\hat{\boldsymbol{n}}$ is the normal vector pointing out of Ω, and ψ and $\boldsymbol{F}$ are arbitrary functions.

$$\oint_{\partial\Omega} \hat{\boldsymbol{n}}\boldsymbol{x}\,\mathrm{d}S = V_\Omega\mathbf{I} \tag{A.3}$$

$$\oint_{\partial\Omega} \hat{\boldsymbol{n}}\cdot\boldsymbol{x}\,\mathrm{d}S = 3V_\Omega \tag{A.4}$$

$$\oint_{\partial\Omega} \hat{\boldsymbol{n}}\psi\,\mathrm{d}S = -\frac{1}{2}\oint_{\partial\Omega} \boldsymbol{x}\times(\hat{\boldsymbol{n}}\times\nabla\psi)\,\mathrm{d}S \tag{A.5}$$

$$\oint_{\partial\Omega} \hat{\boldsymbol{n}}\times\boldsymbol{F}\,\mathrm{d}S = \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{F})\,\mathrm{d}S \tag{A.6}$$

The two first are easily proven.

$$\oint_{\partial\Omega} \hat{\boldsymbol{n}}\boldsymbol{x}\,\mathrm{d}S = \int_\Omega \nabla\boldsymbol{x}\,\mathrm{d}V = \int_\Omega \mathbf{I}\,\mathrm{d}V = V_\Omega\mathbf{I} \tag{A.7}$$

$$\oint_{\partial\Omega} \hat{\boldsymbol{n}}\cdot\boldsymbol{x}\,\mathrm{d}S = \int_\Omega \nabla\cdot\boldsymbol{x}\,\mathrm{d}V = \int_\Omega 3\,\mathrm{d}V = 3V_\Omega \tag{A.8}$$

The third is proven in [14], and using a similar trick to that paper we show the identity

$$\begin{aligned}(\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{F}))_i &= x_i\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{F}) = \hat{\boldsymbol{n}}\cdot(x_i\nabla\times\boldsymbol{F}) = \hat{\boldsymbol{n}}\cdot(\nabla\times(x_i\boldsymbol{F}) - \nabla x_i\times\boldsymbol{F})\\ &= \hat{\boldsymbol{n}}\cdot(\nabla\times(x_i\boldsymbol{F}) - \hat{\boldsymbol{n}}\cdot(\hat{\boldsymbol{x}}_i\times\boldsymbol{F}) = \hat{\boldsymbol{n}}\cdot(\nabla\times(x_i\boldsymbol{F}) + \hat{\boldsymbol{x}}_i\cdot(\hat{\boldsymbol{n}}\times\boldsymbol{F})\end{aligned} \tag{A.9}$$

Since $\oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot(\nabla\times(x_i\boldsymbol{F}))\,\mathrm{d}S = -\int\nabla\cdot(\nabla\times(x_i\boldsymbol{F}))\,\mathrm{d}V = 0$, the fourth integral identity follows.

A.1 Electric case, scalar potential

We start by looking at the field equations for the electric polarizability. The field equation $\nabla\times\boldsymbol{E} = \mathbf{0}$ implies $\boldsymbol{E} = \boldsymbol{E}_0 - \nabla\varphi$, and the equation $\nabla\cdot\boldsymbol{D} = 0$ is then

$$\nabla\cdot[\boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi)] = 0 \tag{A.10}$$

Assume there is one metallic inclusion with domain Ω. The potential then satisfies

$$\varphi = \boldsymbol{E}_0 \cdot \boldsymbol{x} + a \quad \text{on} \quad \partial\Omega \tag{A.11}$$

where the constant a is such that the potential φ is continuous and the total charge on the inclusion is zero, *i.e.*,

$$\oint_{\partial\Omega} \hat{\boldsymbol{n}} \cdot \boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi)\,\mathrm{d}S = 0 \tag{A.12}$$

This can easily be generalized to several inclusions. Multiply the field equation by φ and integrate to find

$$0 = \int \varphi\nabla\cdot[\boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi)]\,\mathrm{d}V = \int \nabla\varphi\cdot\boldsymbol{\epsilon}(-\boldsymbol{E}_0 + \nabla\varphi)\,\mathrm{d}V + \oint_{\partial\Omega} \hat{\boldsymbol{n}}\cdot\varphi\boldsymbol{\epsilon}(-\boldsymbol{E}_0 + \nabla\varphi)\,\mathrm{d}S \tag{A.13}$$

The last integral can be rewritten using the boundary condition

$$\begin{aligned}\oint_{\partial\Omega} \hat{\boldsymbol{n}}\cdot\varphi\boldsymbol{\epsilon}(-\boldsymbol{E}_0 + \nabla\varphi)\,\mathrm{d}S &= \oint_{\partial\Omega} (\boldsymbol{E}_0\cdot\boldsymbol{x} + a)\hat{\boldsymbol{n}}\cdot\boldsymbol{\epsilon}(-\boldsymbol{E}_0 + \nabla\varphi)\,\mathrm{d}S \\ &= \boldsymbol{E}_0\cdot\oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}}\cdot\boldsymbol{\epsilon}(-\boldsymbol{E}_0 + \nabla\varphi)\,\mathrm{d}S\end{aligned} \tag{A.14}$$

To include the effect of the polarization from the material, we rewrite the first integral in A.13 as

$$\begin{aligned}\int \nabla\varphi\cdot\boldsymbol{\epsilon}(-\boldsymbol{E}_0+\nabla\varphi)\,\mathrm{d}V &= \int \nabla\varphi\cdot\boldsymbol{\epsilon}\nabla\varphi\,\mathrm{d}V - \int \nabla\varphi\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V - \int \nabla\varphi\cdot\boldsymbol{\epsilon}_0\boldsymbol{E}_0\,\mathrm{d}V \\ &= \int \nabla\varphi\cdot\boldsymbol{\epsilon}\nabla\varphi\,\mathrm{d}V - 2\int \nabla\varphi\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V + \int (-\boldsymbol{E}_0 + \nabla\varphi)\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V \\ &\quad + \int \boldsymbol{E}_0\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V - \int \nabla\varphi\cdot\boldsymbol{\epsilon}_0\boldsymbol{E}_0\,\mathrm{d}V\end{aligned} \tag{A.15}$$

The last integral can be written

$$-\int \nabla\varphi\,\mathrm{d}V = \oint_{\partial\Omega} \hat{\boldsymbol{n}}\varphi\,\mathrm{d}S = \oint_{\partial\Omega} \hat{\boldsymbol{n}}\boldsymbol{x}\,\mathrm{d}S\cdot\boldsymbol{E}_0 = V_\Omega\boldsymbol{E}_0 \tag{A.16}$$

where we used (A.3). Collecting our results, we can write (A.13) as

$$\begin{aligned}0 &= \int \nabla\varphi\cdot\boldsymbol{\epsilon}\nabla\varphi\,\mathrm{d}V - 2\int \nabla\varphi\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V + \boldsymbol{E}_0\cdot\left[\int (\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\,\mathrm{d}V + \boldsymbol{\epsilon}_0 V_\Omega\right]\cdot\boldsymbol{E}_0 \\ &\quad - \boldsymbol{E}_0\cdot\left[\int (\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)(\boldsymbol{E}_0 - \nabla\varphi)\,\mathrm{d}V + \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}}\cdot\boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi)\,\mathrm{d}S\right]\end{aligned} \tag{A.17}$$

The expression in the last row in square brackets is identified as the total dipole moment $\boldsymbol{p}$. Define the following functional

$$\begin{aligned}J_\mathrm{e}(\varphi, \boldsymbol{E}_0) &= \int \nabla\varphi\cdot\boldsymbol{\epsilon}\nabla\varphi\,\mathrm{d}V - 2\int \nabla\varphi\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V \\ &\quad + \boldsymbol{E}_0\cdot\left[\int (\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\,\mathrm{d}V + V_\Omega\boldsymbol{\epsilon}_0\right]\cdot\boldsymbol{E}_0\end{aligned} \tag{A.18}$$

Note that the expression in square brackets does not depend on φ. For a fixed $\boldsymbol{E}_0$ this functional is minimized by a function φ_0 (in the space of potentials satisfying the proper boundary conditions), such that its minimal value is

$$\min_{\varphi} J_{\mathrm{e}}(\varphi, \boldsymbol{E}_0) = J_{\mathrm{e}}(\varphi_0, \boldsymbol{E}_0) = \boldsymbol{E}_0 \cdot \boldsymbol{p} \tag{A.19}$$

Consider now the variation of the functional at the minimum φ_0:

$$\begin{aligned}\frac{1}{2}\delta J_{\mathrm{e,s}} &= \frac{J_{\mathrm{e,s}}(\varphi_0+\delta\varphi, \boldsymbol{E}_0) - J_{\mathrm{e,s}}(\varphi_0, \boldsymbol{E}_0)}{2} = \int \nabla\delta\varphi\cdot\boldsymbol{\epsilon}\nabla\varphi_0\,\mathrm{d}V - \int \nabla\delta\varphi\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V \\ &= \int \delta\varphi\nabla\cdot[\boldsymbol{\epsilon}(\boldsymbol{E}_0-\nabla\varphi_0)]\,\mathrm{d}V + \oint_{\partial\Omega} \hat{\boldsymbol{n}}\cdot\delta\varphi\boldsymbol{\epsilon}(\boldsymbol{E}_0-\nabla\varphi_0)\,\mathrm{d}S - \oint_{\partial\Omega} \delta\varphi\hat{\boldsymbol{n}}\cdot(\boldsymbol{\epsilon}_0\boldsymbol{E}_0)\,\mathrm{d}S\end{aligned} \tag{A.20}$$

The last two integrals are identically zero since the variation $\delta\varphi$ must be zero on the PEC surface in order to comply with the boundary condition. Since the first variation of the functional should vanish at the extremum for all $\delta\varphi$, the minimizing potential must satisfy $\nabla\cdot[\boldsymbol{\epsilon}(\boldsymbol{E}_0-\nabla\varphi_0)] = 0$, *i.e.*, the electrostatic equation.

We finally consider the difference between two functionals for different geometries, primed and unprimed. We start by rewriting the functional as

$$\begin{aligned}J_{\mathrm{e}}(\varphi, \boldsymbol{E}_0) &= \int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi\cdot\boldsymbol{\epsilon}\nabla\varphi\,\mathrm{d}V - 2\int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V \\ &\quad + \boldsymbol{E}_0\cdot\left[\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\,\mathrm{d}V + V_\Omega\boldsymbol{\epsilon}_0\right]\boldsymbol{E}_0 = \int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{E}_0-\nabla\varphi)\cdot\boldsymbol{\epsilon}(\boldsymbol{E}_0-\nabla\varphi)\,\mathrm{d}V \\ &\quad + 2\int_{\mathbb{R}^3\setminus\Omega}\nabla\varphi\cdot\boldsymbol{\epsilon}_0\boldsymbol{E}_0\,\mathrm{d}V + \epsilon_0|\boldsymbol{E}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega}\mathrm{d}V + V_\Omega\right] \\ &= \int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{E}_0-\nabla\varphi)\cdot\boldsymbol{\epsilon}(\boldsymbol{E}_0-\nabla\varphi)\,\mathrm{d}V + \epsilon_0|\boldsymbol{E}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega}\mathrm{d}V - V_\Omega\right]\end{aligned} \tag{A.21}$$

where we used that $\int_{\mathbb{R}^3\setminus\Omega}\nabla\varphi\,\mathrm{d}V = -V_\Omega\boldsymbol{E}_0$, as shown previously. Even though this expression involves infinite integrals, they are cancelled when looking at the difference between two functionals. Assume the PEC bodies are in regions Ω and Ω'. We then have

$$\begin{aligned}J_{\mathrm{e}}(\varphi, \boldsymbol{E}_0) - J'_{\mathrm{e}}(\varphi', \boldsymbol{E}_0) &= \int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{E}_0-\nabla\varphi)\cdot\boldsymbol{\epsilon}(\boldsymbol{E}_0-\nabla\varphi)\,\mathrm{d}V \\ &\quad - \int_{\mathbb{R}^3\setminus\Omega'}(\boldsymbol{E}_0-\nabla\varphi')\cdot\boldsymbol{\epsilon}'(\boldsymbol{E}_0-\nabla\varphi')\,\mathrm{d}V\end{aligned} \tag{A.22}$$

A.2 Magnetic case, vector potential

The magnetic case is dual to the electric case. On the PEC boundary the condition $\hat{\boldsymbol{n}}\cdot\boldsymbol{B} = 0$ applies, and the field equation $\nabla\cdot\boldsymbol{B} = 0$ implies $\boldsymbol{B} = \boldsymbol{B}_0 + \nabla\times\boldsymbol{A}$. The field equation $\nabla\times\boldsymbol{H} = \boldsymbol{0}$ then becomes

$$\nabla\times[\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A})] = \boldsymbol{0} \tag{A.23}$$

The boundary condition $\hat{\boldsymbol{n}} \cdot \boldsymbol{B} = \hat{\boldsymbol{n}} \cdot (\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}) = 0$ implies that the tangential part of the vector potential satisfies

$$\hat{\boldsymbol{n}} \times \boldsymbol{A} = \hat{\boldsymbol{n}} \times \left(\frac{1}{2}\boldsymbol{x} \times \boldsymbol{B}_0 + \boldsymbol{a}\right) \quad \text{on} \quad \partial\Omega \tag{A.24}$$

where $\boldsymbol{a}$ is a constant vector. It is readily verified that $\nabla \times (\frac{1}{2}\boldsymbol{x} \times \boldsymbol{B}_0) = -\boldsymbol{B}_0$. Note that only the tangential part of $\boldsymbol{A}$ is specified, making only the normal component of $\nabla \times \boldsymbol{A}$ being prescribed. The tangential part of $\nabla \times \boldsymbol{A}$ corresponds to the tangential magnetic field, which is proportional to the resulting surface current.

The condition corresponding to zero total charge in the electric case, is that the total surface current on the PEC surface must be zero, *i.e.*,

$$\oint_{\partial\Omega} \hat{\boldsymbol{n}} \times [\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})] \,\mathrm{d}S = \boldsymbol{0} \tag{A.25}$$

Following the recipe from the electric case, we multiply the field equation with $\boldsymbol{A}$ and integrate to find

$$0 = \int \boldsymbol{A} \cdot \nabla \times [\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})] \,\mathrm{d}V = \int (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{\mu}^{-1}(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}) \,\mathrm{d}V + \oint_{\partial\Omega} \hat{\boldsymbol{n}} \cdot [\boldsymbol{A} \times \boldsymbol{\mu}^{-1}(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})] \,\mathrm{d}S \tag{A.26}$$

where we used the identity (here, $\hat{\boldsymbol{n}}$ is the outward unit normal for the volume V, whereas the unit normal above is pointing *into* the domain of integration)

$$\oint_{\partial V} \hat{\boldsymbol{n}} \cdot (\boldsymbol{A} \times \boldsymbol{B}) \,\mathrm{d}S = \int_V \nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) \,\mathrm{d}V = \int_V [\boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})] \,\mathrm{d}V \tag{A.27}$$

for arbitrary vector fields $\boldsymbol{A}$ and $\boldsymbol{B}$. The surface integral in (A.26) can be written (due to the occurrence of the unit normal $\hat{\boldsymbol{n}}$ only the tangential components of $\boldsymbol{A}$ are needed in the evaluation of the integral, in compliance with the boundary condition)

$$\begin{aligned}\oint_{\partial\Omega} \hat{\boldsymbol{n}} \cdot [\boldsymbol{A} \times \boldsymbol{\mu}^{-1}(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})] \,\mathrm{d}S &= \oint_{\partial\Omega} \hat{\boldsymbol{n}} \cdot [(\frac{1}{2}\boldsymbol{x} \times \boldsymbol{B}_0 + \boldsymbol{a}) \times \boldsymbol{\mu}^{-1}(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})] \,\mathrm{d}S \\ &= -\frac{1}{2} \oint_{\partial\Omega} (\boldsymbol{x} \times \boldsymbol{B}_0) \cdot [\hat{\boldsymbol{n}} \times \boldsymbol{\mu}^{-1}(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})] \,\mathrm{d}S \\ &= \boldsymbol{B}_0 \cdot \frac{1}{2} \oint_{\partial\Omega} \boldsymbol{x} \times [\hat{\boldsymbol{n}} \times \boldsymbol{\mu}^{-1}(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})] \,\mathrm{d}S\end{aligned} \tag{A.28}$$

The integral containing the constant $\boldsymbol{a}$ is zero due to the zero total current requirement. The first integral in (A.26) is

$$\begin{aligned}0 = \int (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{\mu}^{-1}(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}) \,\mathrm{d}V &= \int (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{\mu}^{-1} \nabla \times \boldsymbol{A} \,\mathrm{d}V \\ &+ \int (\nabla \times \boldsymbol{A}) \cdot (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}_0^{-1}) \boldsymbol{B}_0 \,\mathrm{d}V + \int (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{\mu}_0^{-1} \boldsymbol{B}_0 \,\mathrm{d}V\end{aligned} \tag{A.29}$$

By adding and subtracting the second integral can be written

$$\int (\nabla \times \boldsymbol{A}) \cdot (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}_0^{-1}) \boldsymbol{B}_0 \, \mathrm{d}V = 2 \int (\nabla \times \boldsymbol{A}) \cdot (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}_0^{-1}) \boldsymbol{B}_0 \, \mathrm{d}V \\ - \int (\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}) \cdot (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}_0^{-1}) \boldsymbol{B}_0 \, \mathrm{d}V + \int \boldsymbol{B}_0 \cdot (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}_0^{-1}) \boldsymbol{B}_0 \, \mathrm{d}V \quad \text{(A.30)}$$

whereas the last integral can be written

$$\int (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{\mu}_0^{-1} \boldsymbol{B}_0 \, \mathrm{d}V = \int \nabla \times \boldsymbol{A} \, \mathrm{d}V \cdot \boldsymbol{\mu}_0^{-1} \boldsymbol{B}_0 = - \oint_{\partial\Omega} \hat{\boldsymbol{n}} \times \boldsymbol{A} \, \mathrm{d}S \cdot \boldsymbol{\mu}_0^{-1} \boldsymbol{B}_0 \\ = - \oint_{\partial\Omega} \hat{\boldsymbol{n}} \times (\frac{1}{2} \boldsymbol{x} \times \boldsymbol{B}_0 + \boldsymbol{a}) \, \mathrm{d}S \cdot \boldsymbol{\mu}_0^{-1} \boldsymbol{B}_0 = -\frac{1}{2} \oint_{\partial\Omega} [\boldsymbol{x}(\hat{\boldsymbol{n}} \cdot \boldsymbol{B}_0) - \boldsymbol{B}_0(\hat{\boldsymbol{n}} \cdot \boldsymbol{x})] \, \mathrm{d}S \cdot \boldsymbol{\mu}_0^{-1} \boldsymbol{B}_0 \\ = \boldsymbol{B}_0 \cdot \frac{1}{2} \oint_{\partial\Omega} [(\hat{\boldsymbol{n}} \cdot \boldsymbol{x}) \mathbf{I} - \hat{\boldsymbol{n}} \boldsymbol{x}] \, \mathrm{d}S \cdot \boldsymbol{\mu}_0^{-1} \boldsymbol{B}_0 = \boldsymbol{B}_0 \cdot \frac{1}{2} (3V_\Omega - V_\Omega) \boldsymbol{\mu}_0^{-1} \boldsymbol{B}_0 \\ = V_\Omega \boldsymbol{B}_0 \cdot \boldsymbol{\mu}_0^{-1} \boldsymbol{B}_0 \quad \text{(A.31)}$$

where we used (A.3) and (A.4) in the last but one line. The integral containing the constant vector $\boldsymbol{a}$ is zero since it is proportional to the integral $\oint_{\partial\Omega} \hat{\boldsymbol{n}} \, \mathrm{d}S = \boldsymbol{0}$. Collecting the results for the terms in (A.26) we have

$$0 = \int (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{\mu}^{-1} \nabla \times \boldsymbol{A} \, \mathrm{d}V + 2 \int (\nabla \times \boldsymbol{A}) \cdot (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}_0^{-1}) \boldsymbol{B}_0 \, \mathrm{d}V \\ + \int \boldsymbol{B}_0 \cdot (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}_0^{-1}) \boldsymbol{B}_0 \, \mathrm{d}V + \boldsymbol{B}_0 \cdot V_\Omega \boldsymbol{\mu}_0^{-1} \boldsymbol{B}_0 \\ + \boldsymbol{B}_0 \cdot \left[- \int (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}_0^{-1})(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}) \, \mathrm{d}V + \frac{1}{2} \oint_{\partial\Omega} \boldsymbol{x} \times [\hat{\boldsymbol{n}} \times \boldsymbol{\mu}^{-1} (\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})] \, \mathrm{d}S \right] \quad \text{(A.32)}$$

The expression in square brackets is the total magnetic dipole moment for the structure, $\boldsymbol{m}$. Define the following functional

$$K_\mathrm{m}(\boldsymbol{A}, \boldsymbol{B}_0) = \int (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{\mu}^{-1} \nabla \times \boldsymbol{A} \, \mathrm{d}V - 2 \int (\nabla \times \boldsymbol{A}) \cdot (\boldsymbol{\mu}_0^{-1} - \boldsymbol{\mu}^{-1}) \boldsymbol{B}_0 \, \mathrm{d}V \\ \boldsymbol{B}_0 \cdot \left[- \int (\boldsymbol{\mu}_0^{-1} - \boldsymbol{\mu}^{-1}) \, \mathrm{d}V + V_\Omega \boldsymbol{\mu}_0^{-1} \right] \boldsymbol{B}_0 \quad \text{(A.33)}$$

Note that the expression in square brackets does not depend on $\boldsymbol{A}$. For a fixed $\boldsymbol{B}_0$ this functional is minimized by a vector potential $\boldsymbol{A}_0$ (in the space of vector potentials satisfying the proper boundary conditions), such that its minimal value is

$$\min_{\boldsymbol{A}} K_\mathrm{m}(\boldsymbol{A}, \boldsymbol{B}_0) = K_\mathrm{m}(\boldsymbol{A}_0, \boldsymbol{B}_0) = -\boldsymbol{B}_0 \cdot \boldsymbol{m} \quad \text{(A.34)}$$

The symmetry with the electric case is interesting, but also the difference: in the electric case the minimum was attained as $\boldsymbol{E}_0 \cdot \boldsymbol{p}$, whereas in the magnetic case it is the negative, $-\boldsymbol{B}_0 \cdot \boldsymbol{m}$.

Consider now the variation of the functional at the minimum:

$$\frac{1}{2}\delta K_\mathrm{m} = \frac{K_\mathrm{m}(\boldsymbol{A}_0+\delta\boldsymbol{A},\boldsymbol{B}_0) - K_\mathrm{m}(\boldsymbol{A}_0,\boldsymbol{B}_0)}{2} = \int (\nabla\times\delta\boldsymbol{A})\cdot\boldsymbol{\mu}^{-1}\nabla\times\boldsymbol{A}_0\,\mathrm{d}V$$
$$- \int (\nabla\times\delta\boldsymbol{A})\cdot(\boldsymbol{\mu}_0^{-1}-\boldsymbol{\mu}^{1})\boldsymbol{B}_0\,\mathrm{d}V = \int \delta\boldsymbol{A}\cdot(\nabla\times[\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A}_0)]\,\mathrm{d}V$$
$$- \oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot(\delta\boldsymbol{A}\times\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A}_0)\,\mathrm{d}S + \oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot(\delta\boldsymbol{A}\times\boldsymbol{\mu}_0^{-1}\boldsymbol{B}_0)\,\mathrm{d}S \quad \text{(A.35)}$$

The last two integrals are identically zero since the variation of the tangential part of $\delta\boldsymbol{A}$ must be zero on the PEC surface in order to comply with the boundary condition. Since the first variation of the functional should vanish at the extremum for all $\delta\boldsymbol{A}$, the minimizing potential must satisfy $\nabla\times[\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A}_0)] = \boldsymbol{0}$, *i.e.*, the magnetostatic equation.

We finally consider the difference between two functionals. Rewriting the functional as

$$K_\mathrm{m}(\boldsymbol{A},\boldsymbol{B}_0) = \int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{A})\cdot\boldsymbol{\mu}^{-1}\nabla\times\boldsymbol{A}\,\mathrm{d}V - 2\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{A})\cdot(\boldsymbol{\mu}_0^{-1}-\boldsymbol{\mu}^{-1})\boldsymbol{B}_0\,\mathrm{d}V$$
$$\boldsymbol{B}_0\cdot\left[-\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{\mu}_0^{-1}-\boldsymbol{\mu}^{-1})\,\mathrm{d}V + V_\Omega\boldsymbol{\mu}_0^{-1}\right]\boldsymbol{B}_0$$
$$= \int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A})\cdot\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A})\,\mathrm{d}V - 2\int_{\mathbb{R}^3\setminus\Omega}\nabla\times\boldsymbol{A}\,\mathrm{d}V\boldsymbol{\mu}_0^{-1}\boldsymbol{B}_0$$
$$+\mu_0^{-1}|\boldsymbol{B}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega}\mathrm{d}V + V_\Omega\right] = \int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A})\cdot\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A})\,\mathrm{d}V$$
$$+\mu_0^{-1}|\boldsymbol{B}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega}\mathrm{d}V - V_\Omega\right] \quad \text{(A.36)}$$

where we used that $\int_{\mathbb{R}^3\setminus\Omega}\nabla\times\boldsymbol{A}\,\mathrm{d}V = V_\Omega\boldsymbol{B}_0$, as shown previously. The difference between two functionals can now be written

$$K_\mathrm{m}(\boldsymbol{A},\boldsymbol{B}_0) - K'_\mathrm{m}(\boldsymbol{A}',\boldsymbol{B}_0) = \int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A})\cdot\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A})\,\mathrm{d}V$$
$$- \int_{\mathbb{R}^3\setminus\Omega'}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A}')\cdot(\boldsymbol{\mu}')^{-1}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A}')\,\mathrm{d}V \quad \text{(A.37)}$$

A.3 Electric case, vector potential

In the electric case, the scalar potential fits nicely with the boundary condition on PEC surfaces. We now investigate what can be learned from treating the electric case with a vector potential. The equation $\nabla\cdot\boldsymbol{D} = 0$ implies that $\boldsymbol{D} = \boldsymbol{D}_0+\nabla\times\boldsymbol{F}$, and we have the field equation

$$\nabla\times[\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})] = \boldsymbol{0} \quad \text{(A.38)}$$

with boundary condition

$$\hat{\boldsymbol{n}} \times \boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}) = \boldsymbol{0} \tag{A.39}$$

The zero total charge condition is

$$\oint_{\partial\Omega} \hat{\boldsymbol{n}} \cdot (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F})\,\mathrm{d}S = 0 \tag{A.40}$$

Multiplying the field equation with $\boldsymbol{F}$ and integrating implies

$$\begin{aligned} 0 = \int \boldsymbol{F} \cdot \nabla \times [\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0 + \nabla \times \boldsymbol{F})]\,\mathrm{d}V = \int (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0 + \nabla \times \boldsymbol{F})\,\mathrm{d}V \\ + \oint_{\partial\Omega} \hat{\boldsymbol{n}} \cdot [\boldsymbol{F} \times \boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0 + \nabla \times \boldsymbol{F})]\,\mathrm{d}S \end{aligned} \tag{A.41}$$

The last integral is zero due to the boundary conditions. We then have

$$\begin{aligned} 0 = \int (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}^{-1} \nabla \times \boldsymbol{F})\,\mathrm{d}V + \int (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}^{-1} \boldsymbol{D}_0\,\mathrm{d}V = \int (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}^{-1} \nabla \times \boldsymbol{F})\,\mathrm{d}V \\ + \int (\nabla \times \boldsymbol{F}) \cdot (\boldsymbol{\epsilon}^{-1} - \boldsymbol{\epsilon}_0^{-1}) \boldsymbol{D}_0\,\mathrm{d}V + \int (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}_0^{-1} \boldsymbol{D}_0\,\mathrm{d}V \end{aligned} \tag{A.42}$$

The last but one integral can be written

$$\begin{aligned} \int (\nabla \times \boldsymbol{F}) \cdot (\boldsymbol{\epsilon}^{-1} - \boldsymbol{\epsilon}_0^{-1}) \boldsymbol{D}_0\,\mathrm{d}V = 2 \int (\nabla \times \boldsymbol{F}) \cdot (\boldsymbol{\epsilon}^{-1} - \boldsymbol{\epsilon}_0^{-1}) \boldsymbol{D}_0\,\mathrm{d}V \\ - \int (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}) \cdot (\boldsymbol{\epsilon}^{-1} - \boldsymbol{\epsilon}_0^{-1}) \boldsymbol{D}_0\,\mathrm{d}V + \boldsymbol{D}_0 \cdot \int (\boldsymbol{\epsilon}^{-1} - \boldsymbol{\epsilon}_0^{-1}) \boldsymbol{D}_0\,\mathrm{d}V \end{aligned} \tag{A.43}$$

whereas the last integral is

$$\int (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}_0^{-1} \boldsymbol{D}_0\,\mathrm{d}V = -\oint_{\partial\Omega} \hat{\boldsymbol{n}} \times \boldsymbol{F}\,\mathrm{d}S \cdot \boldsymbol{\epsilon}_0^{-1} \boldsymbol{D}_0 \tag{A.44}$$

Using (A.6), this integral is transformed to

$$\begin{aligned} -\oint_{\partial\Omega} \hat{\boldsymbol{n}} \times \boldsymbol{F}\,\mathrm{d}S = -\oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}} \cdot (\nabla \times \boldsymbol{F})\,\mathrm{d}S = \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}} \cdot (\boldsymbol{D}_0 - \boldsymbol{D}_0 - \nabla \times \boldsymbol{F})\,\mathrm{d}S \\ = \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}}\,\mathrm{d}S \cdot \boldsymbol{D}_0 - \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}} \cdot (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F})\,\mathrm{d}S = V_\Omega \boldsymbol{D}_0 - \boldsymbol{p}_{\mathrm{pec}} \end{aligned} \tag{A.45}$$

Collecting the results, we have

$$\begin{aligned} 0 = \int (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}^{-1} \nabla \times \boldsymbol{F})\,\mathrm{d}V + 2 \int (\nabla \times \boldsymbol{F}) \cdot (\boldsymbol{\epsilon}^{-1} - \boldsymbol{\epsilon}_0^{-1}) \boldsymbol{D}_0\,\mathrm{d}V \\ + \boldsymbol{D}_0 \cdot \int (\boldsymbol{\epsilon}^{-1} - \boldsymbol{\epsilon}_0^{-1}) \boldsymbol{D}_0\,\mathrm{d}V + V_\Omega \boldsymbol{D}_0 \cdot \boldsymbol{\epsilon}_0^{-1} \boldsymbol{D}_0 \\ - \int (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}) \cdot (\boldsymbol{\epsilon}^{-1} - \boldsymbol{\epsilon}_0^{-1}) \boldsymbol{D}_0\,\mathrm{d}V - \boldsymbol{D}_0 \cdot \boldsymbol{\epsilon}_0^{-1} \boldsymbol{p}_{\mathrm{pec}} \end{aligned} \tag{A.46}$$

By adding and subtracting $2\boldsymbol{p}_{\mathrm{pec}}$, we can identify the total electric dipole moment $\boldsymbol{p}_{\mathrm{mtrl}} + \boldsymbol{p}_{\mathrm{pec}}$. The functional

$$K_{\mathrm{e}}(\boldsymbol{F},\boldsymbol{D}_0) = \int (\nabla\times\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}\nabla\times\boldsymbol{F})\,\mathrm{d}V - 2\int(\nabla\times\boldsymbol{F})\cdot(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1})\boldsymbol{D}_0\,\mathrm{d}V \\ - 2\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{p}_{\mathrm{pec}} + \boldsymbol{D}_0\cdot\left[-\int(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1})\,\mathrm{d}V + V_\Omega\boldsymbol{\epsilon}_0^{-1}\right]\cdot\boldsymbol{D}_0 \quad \text{(A.47)}$$

then has minimum

$$\min_{\boldsymbol{F}} K_{\mathrm{e}}(\boldsymbol{F},\boldsymbol{D}_0) = K_{\mathrm{e}}(\boldsymbol{F}_0,\boldsymbol{D}_0) \\ = -\boldsymbol{D}_0\cdot\left[\int(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1})(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}V + \oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}S\right] \\ = -\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{p} \quad \text{(A.48)}$$

The necessity of adding and subtracting the term $\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{p}_{\mathrm{pec}}$ appears when considering the variation of the functional at the minimum:

$$\frac{1}{2}\delta K_{\mathrm{e}} = \frac{K_{\mathrm{e}}(\boldsymbol{F}_0+\delta\boldsymbol{F},\boldsymbol{D}_0) - K_{\mathrm{e}}(\boldsymbol{F}_0,\boldsymbol{D}_0)}{2} = \int(\nabla\times\delta\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}\nabla\times\boldsymbol{F}_0\,\mathrm{d}V \\ - \int(\nabla\times\delta\boldsymbol{F})\cdot(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1})\boldsymbol{D}_0\,\mathrm{d}V - \boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\nabla\times\delta\boldsymbol{F})\,\mathrm{d}S \\ = \int\delta\boldsymbol{F}\cdot(\nabla\times(\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)))\,\mathrm{d}V + \oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot(\delta\boldsymbol{F}\times\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0))\,\mathrm{d}S \\ - \int\delta\boldsymbol{F}\cdot(\nabla\times\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0)\,\mathrm{d}V + \oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot(\delta\boldsymbol{F}\times\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0)\,\mathrm{d}S - \oint_{\partial\Omega}\hat{\boldsymbol{n}}\times\delta\boldsymbol{F}\,\mathrm{d}S\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0 \\ \text{(A.49)}$$

The second integral is identically zero since $\hat{\boldsymbol{n}}\times\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}) = 0$ on the PEC boundary. The third is zero since $\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0$ is constant, and the last two integrals cancel each other. The only integral remaining is the first one, and since we should have $\delta K_{\mathrm{e}} = 0$ for any $\delta\boldsymbol{F}$ at the extremum, we see that the minimizing potential must satisfy the equation $\nabla\times(\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)) = \boldsymbol{0}$, *i.e.*, the electrostatic equation.

We finally consider the difference between two functionals. We first rewrite the functional as

$$K_{\rm e}(\boldsymbol{F},\boldsymbol{D}_0)=\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}\nabla\times\boldsymbol{F})\,\mathrm{d}V-2\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F})\cdot(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1})\boldsymbol{D}_0\,\mathrm{d}V$$
$$-2\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{p}_{\rm pec}+\boldsymbol{D}_0\cdot\left[-\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1})\,\mathrm{d}V+V_\Omega\boldsymbol{\epsilon}_0^{-1}\right]\cdot\boldsymbol{D}_0$$
$$=\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})\,\mathrm{d}V-2\int_{\mathbb{R}^3\setminus\Omega}\nabla\times\boldsymbol{F}\,\mathrm{d}V\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0-2\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{p}_{\rm pec}$$
$$+\epsilon_0^{-1}|\boldsymbol{D}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega}\mathrm{d}V+V_\Omega\right]=\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})\,\mathrm{d}V$$
$$+\epsilon_0^{-1}|\boldsymbol{D}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega}\mathrm{d}V-V_\Omega\right]\quad\text{(A.50)}$$

where we used that $\int_{\mathbb{R}^3\setminus\Omega}\nabla\times\boldsymbol{F}\,\mathrm{d}V=V_\Omega\boldsymbol{D}_0-\boldsymbol{p}_{\rm pec}$, which was shown previously. The difference between two functionals can then be written

$$K_{\rm e}(\boldsymbol{F},\boldsymbol{D}_0)-K_{\rm e}'(\boldsymbol{F}',\boldsymbol{D}_0)=\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})\,\mathrm{d}V$$
$$-\int_{\mathbb{R}^3\setminus\Omega'}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}')\cdot(\boldsymbol{\epsilon}')^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}')\,\mathrm{d}V\quad\text{(A.51)}$$

A.4 Magnetic case, scalar potential

We now use the scalar potential for the magnetic case. To start with, $\nabla\times\boldsymbol{H}=\boldsymbol{0}$ implies $\boldsymbol{H}=\boldsymbol{H}_0-\nabla\psi$. The remaining field equation $\nabla\cdot\boldsymbol{B}=0$ is then

$$\nabla\cdot[\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi)]=0\quad\text{(A.52)}$$

with the boundary condition $\hat{\boldsymbol{n}}\cdot\boldsymbol{B}=0$ on the PEC boundary being

$$\hat{\boldsymbol{n}}\cdot[\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi)]=0\quad\text{(A.53)}$$

Multiplying the field equation with ψ and integrating implies

$$0=\int\psi\nabla\cdot[\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi)]=\int\nabla\psi\cdot\boldsymbol{\mu}(-\boldsymbol{H}_0+\nabla\psi)]\,\mathrm{d}V+\oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot\psi\boldsymbol{\mu}(-\boldsymbol{H}_0+\nabla\psi)\,\mathrm{d}S$$
$$=\int\nabla\psi\cdot\boldsymbol{\mu}\nabla\psi\,\mathrm{d}V-\int\nabla\psi\cdot(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\boldsymbol{H}_0\,\mathrm{d}V-\int\nabla\psi\cdot\boldsymbol{\mu}_0\boldsymbol{H}_0\,\mathrm{d}V\quad\text{(A.54)}$$

Proceeding along now familiar lines, we write

$$-\int\nabla\psi\cdot(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\boldsymbol{H}_0\,\mathrm{d}V=-2\int\nabla\psi\cdot(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\boldsymbol{H}_0\,\mathrm{d}V$$
$$+\int(-\boldsymbol{H}_0+\nabla\psi)\cdot(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\boldsymbol{H}_0\,\mathrm{d}V+\boldsymbol{H}_0\cdot\int(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\,\mathrm{d}V\cdot\boldsymbol{H}_0\quad\text{(A.55)}$$

and

$$-\int \nabla\psi \cdot \boldsymbol{\mu}_0 \boldsymbol{H}_0 \,\mathrm{d}V = \oint_{\partial\Omega} \hat{\boldsymbol{n}}\psi \,\mathrm{d}S \cdot \boldsymbol{\mu}_0 \boldsymbol{H}_0 \tag{A.56}$$

Using (A.5), we can write

$$\begin{aligned}\oint_{\partial\Omega} \hat{\boldsymbol{n}}\psi \,\mathrm{d}S &= \oint_{\partial\Omega} \hat{\boldsymbol{n}}(\psi - \boldsymbol{H}_0 \cdot \boldsymbol{x} + \boldsymbol{H}_0 \cdot \boldsymbol{x}) \,\mathrm{d}S \\ &= \frac{1}{2}\oint_{\partial\Omega} \boldsymbol{x} \times (\hat{\boldsymbol{n}} \times (\boldsymbol{H}_0 - \nabla\psi)) \,\mathrm{d}S + \oint_{\partial\Omega} \hat{\boldsymbol{n}}\boldsymbol{x} \,\mathrm{d}S \cdot \boldsymbol{H}_0 = \boldsymbol{m}_{\mathrm{pec}} + V_\Omega \boldsymbol{H}_0 \end{aligned} \tag{A.57}$$

so that we have

$$\begin{aligned} 0 = &\int \nabla\psi \cdot \boldsymbol{\mu}\nabla\psi \,\mathrm{d}V - 2\int \nabla\psi \cdot (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\boldsymbol{H}_0 \,\mathrm{d}V + \boldsymbol{H}_0 \cdot \int (\boldsymbol{\mu} - \boldsymbol{\mu}_0) \,\mathrm{d}V \cdot \boldsymbol{H}_0 \\ &- \boldsymbol{H}_0 \cdot \int (\boldsymbol{\mu} - \boldsymbol{\mu}_0)(\boldsymbol{H}_0 - \nabla\psi) \,\mathrm{d}V + \boldsymbol{m}_{\mathrm{pec}} \cdot \boldsymbol{\mu}_0 \boldsymbol{H}_0 + V_\Omega \boldsymbol{H}_0 \cdot \boldsymbol{\mu}_0 \boldsymbol{H}_0 \end{aligned} \tag{A.58}$$

As with the vector potential in the electric case, we add and subtract $2\boldsymbol{H}_0 \cdot \boldsymbol{\mu}_0 \boldsymbol{m}_{\mathrm{pec}}$. This suggests that the minimum of the functional

$$\begin{aligned} J_{\mathrm{m}}(\psi, \boldsymbol{H}_0) = &\int \nabla\psi \cdot \boldsymbol{\mu}\nabla\psi \,\mathrm{d}V - 2\int \nabla\psi \cdot (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\boldsymbol{H}_0 \,\mathrm{d}V + 2\boldsymbol{H}_0 \cdot \boldsymbol{\mu}_0 \boldsymbol{m}_{\mathrm{pec}} \\ &+ \boldsymbol{H}_0 \cdot \left[\int (\boldsymbol{\mu} - \boldsymbol{\mu}_0) \,\mathrm{d}V + V_\Omega \boldsymbol{\mu}_0\right] \boldsymbol{H}_0 \end{aligned} \tag{A.59}$$

is given by

$$\begin{aligned} \min_\psi J_{\mathrm{m}}(\psi, \boldsymbol{H}_0) &= J_{\mathrm{m}}(\psi_0, \boldsymbol{H}_0) \\ &= \boldsymbol{H}_0 \cdot \left[\int (\boldsymbol{\mu} - \boldsymbol{\mu}_0)(\boldsymbol{H}_0 - \nabla\psi_0) \,\mathrm{d}V + \boldsymbol{\mu}_0 \frac{1}{2}\oint_{\partial\Omega} \boldsymbol{x} \times (\hat{\boldsymbol{n}} \times (\boldsymbol{H}_0 - \nabla\psi_0)) \,\mathrm{d}S\right] \\ &= \boldsymbol{H}_0 \cdot \boldsymbol{\mu}_0 \boldsymbol{m} \end{aligned} \tag{A.60}$$

The necessity of adding and subtracting $2\boldsymbol{H}_0 \cdot \boldsymbol{\mu}_0 \boldsymbol{m}_{\mathrm{pec}}$ is shown by considering the variation of the functional at the minimum:

$$\begin{aligned} \frac{1}{2}\delta J_{\mathrm{m}} = &\frac{J_{\mathrm{m}}(\psi_0 + \delta\psi, \boldsymbol{H}_0) - J_{\mathrm{m}}(\psi_0, \boldsymbol{H}_0)}{2} = \int \nabla\delta\psi \cdot \boldsymbol{\mu}\nabla\psi_0 \,\mathrm{d}V \\ &- \int \nabla\delta\psi \cdot (\boldsymbol{\mu} - \boldsymbol{\mu}_0)\boldsymbol{H}_0 \,\mathrm{d}V + \boldsymbol{H}_0 \cdot \boldsymbol{\mu}_0 \frac{1}{2}\oint_{\partial\Omega} \boldsymbol{x} \times (\hat{\boldsymbol{n}} \times (-\nabla\delta\psi)) \,\mathrm{d}S \\ &= \int \delta\psi\nabla \cdot (\boldsymbol{\mu}(\boldsymbol{H}_0 - \nabla\psi_0)) \,\mathrm{d}V - \oint_{\partial\Omega} \hat{\boldsymbol{n}} \cdot (\psi\boldsymbol{\mu}(\boldsymbol{H}_0 - \nabla\psi_0) \,\mathrm{d}S \\ &- \int \delta\psi\nabla \cdot (\boldsymbol{\mu}_0 \boldsymbol{H}_0) \,\mathrm{d}V - \oint_{\partial\Omega} \delta\psi\hat{\boldsymbol{n}} \cdot \boldsymbol{\mu}_0 \boldsymbol{H}_0 \,\mathrm{d}S + \oint_{\partial\Omega} \hat{\boldsymbol{n}}\delta\psi \,\mathrm{d}S \cdot \boldsymbol{\mu}_0 \boldsymbol{H}_0 \end{aligned} \tag{A.61}$$

The second integral is zero since $\hat{\boldsymbol{n}} \cdot \boldsymbol{\mu}(\boldsymbol{H}_0 - \nabla\psi_0) = 0$ on the PEC boundary. The third integral is zero since $\boldsymbol{\mu}_0 \boldsymbol{H}_0$ is constant, and the two last integrals cancel each other. This leaves onle the first integral, and since the first variation of the functional should be zero at the extremum, the potential must satisfy $\nabla\cdot(\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi_0))=0$, *i.e.*, the magnetostatic equation.

Finally, we study the difference between two functionals. We start by rewriting the functional as

$$\begin{aligned}
J_\mathrm{m}(\psi,\boldsymbol{H}_0) &= \int_{\mathbb{R}^3\setminus\Omega}\nabla\psi\cdot\boldsymbol{\mu}\nabla\psi\,\mathrm{d}V - 2\int_{\mathbb{R}^3\setminus\Omega}\nabla\psi\cdot(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\boldsymbol{H}_0\,\mathrm{d}V + 2\boldsymbol{H}_0\cdot\boldsymbol{\mu}_0\boldsymbol{m}_\mathrm{pec} \\
&+ \boldsymbol{H}_0\cdot\left[\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\,\mathrm{d}V + V_\Omega\boldsymbol{\mu}_0\right]\boldsymbol{H}_0 = \int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{H}_0-\nabla\psi)\cdot\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi)\,\mathrm{d}V \\
&+ 2\int_{\mathbb{R}^3\setminus\Omega}\nabla\psi\,\mathrm{d}V\boldsymbol{\mu}_0\boldsymbol{H}_0 + 2\boldsymbol{H}_0\cdot\boldsymbol{m}_\mathrm{pec} + \mu_0|\boldsymbol{H}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega}\mathrm{d}V + V_\Omega\right] \\
&= \int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{H}_0-\nabla\psi)\cdot\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi)\,\mathrm{d}V + \mu_0|\boldsymbol{H}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega}\mathrm{d}V - V_\Omega\right] \quad \text{(A.62)}
\end{aligned}$$

where we used that $\int_{\mathbb{R}^3\setminus\Omega}\nabla\psi\,\mathrm{d}V = -\boldsymbol{m}_\mathrm{pec} - V_\Omega\boldsymbol{H}_0$, as was shown previously. The difference between two functionals can then be written

$$\begin{aligned}
J_\mathrm{m}(\psi,\boldsymbol{H}_0) - J'_\mathrm{m}(\psi',\boldsymbol{H}_0) &= \int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{H}_0-\nabla\psi)\cdot\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi)\,\mathrm{d}V \\
&- \int_{\mathbb{R}^3\setminus\Omega'}(\boldsymbol{H}_0-\nabla\psi')\cdot\boldsymbol{\mu}'(\boldsymbol{H}_0-\nabla\psi')\,\mathrm{d}V \quad \text{(A.63)}
\end{aligned}$$

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