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# HYSTERESIS ANALYSIS BASED ON INTEGRAL QUADRATIC CONSTRAINTS

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## Abstract.

*It is shown how the framework of Integral Quadratic Constraints can be applied to analyze systems with hysteresis, in spite of the fact that hysteresis operators are unbounded and that not all system variable can be expected to approach zero.*

## 1. Introduction

Backlash and other hysteresis phenomena are common in mechanical and hydraulic systems. They may severely limit the overall system performance, but the effects are often hard to analyse. The purpose of this paper is to demonstrate how recently developed tools for nonlinear system analysis can be applied to systems with hysteresis and in particular backlash.

The most common approach to analysis of hysteresis is maybe passivity theory. Such analysis is based on the physical insight that passive components can only extract energy from the system, not generate new energy. Mathematically, the passivity property characterizes a nonlinearity in terms of positivity of a certain quadratic integral. However, the passivity property alone carries very little information about the nonlinearity. In order to better predict the system behaviour, it is strongly desirable to exploit more quantitative information.

Stability criteria based on Lyapunov functions, passivity and absolute stability have been developed over several decades. Many results of this kind were recently unified and generalized using the notion integral quadratic constraint (IQC) [6]. In general, the more information about the nonlinearity that can be captured in the form of IQC's, the better analysis one can make. The computational treatment of the inequalities leads to a convex optimization problem with linear matrix in-

equality constraints. Such problems can be solved with great efficiency using interior point methods [2]. Furthermore, it was shown that technical problems in the earlier treatments of anti-causal multipliers [3] can be avoided using a so called homotopy argument.

Many of the earlier treatments of hysteresis, such as [7, 1] can be interpreted and generalized in terms of IQC's, see [5]. The contribution of this paper is to prove a new set of integral quadratic constraints for hysteresis and to demonstrate how these constraints can be used in analysis of stability and performance.

## Notation

$\mathbf{L}^n$  is the linear space of all functions  $f : (0, \infty) \rightarrow \mathbf{R}^n$  which are square integrable on any finite interval.  $\mathbf{L}_2^n$  is the subspace of signals  $f \in \mathbf{L}^n$  with  $\|f\| < \infty$ , where

$$\|f\| = \langle f, f \rangle^{1/2}, \quad \langle f, g \rangle = \int_0^\infty f(t)^* g(t) dt.$$

The set of proper rational transfer matrices  $G = G(s)$  of size  $k$  by  $m$  is denoted by  $\mathbf{RL}_\infty^{k \times m}$ . The subset of stable  $G \in \mathbf{RL}_\infty^{k \times m}$  is denoted by  $\mathbf{RH}_\infty^{k \times m}$ . Each element  $G \in \mathbf{RL}_\infty^{k \times m}$  is associated with a corresponding causal LTI operator  $G : \mathbf{L}^m \rightarrow \mathbf{L}^k$ . An element  $G \in \mathbf{RL}_\infty^{k \times m}$  is called *proper* if  $G(\infty) = 0$ . For  $a, b \geq 0$  and  $f \in \mathbf{L}^k$ , the projection  $P_b^a f \in \mathbf{L}^k$  is defined by

$$(P_b^a f)(t) = \begin{cases} f(t), & b < t \leq a \\ 0, & \text{otherwise} \end{cases}$$

The shorthand  $P_T f$  is used for  $P_T^0 f$ , and  $P^T f$  means  $P_0^T f$ .

## 2. Multivalued Nonlinear Operators

The word *operator* will be used to denote an input/output system. Mathematically, it simply means

any function (possibly multi-valued) from one signal space  $\mathbf{L}^k$  to another: an operator  $\Delta : \mathbf{L}^l \rightarrow \mathbf{L}^m$  is defined by a subset  $\mathcal{S}_\Delta \subset \mathbf{L}^l \times \mathbf{L}^m$  such that for every  $v \in \mathbf{L}^l$  there exists  $w \in \mathbf{L}^m$  with  $(v, w) \in \mathcal{S}_\Delta$ . The notation

$$w = \Delta(v) \quad (1)$$

simply means that  $(v, w) \in \mathcal{S}_\Delta$ .

The notion of causality is introduced to represent existence and continuability of solutions of such equations forward in time: an operator  $\Delta$  is said to be *causal* if the set of past projections  $P_T w$  of possible outputs  $w = \Delta(v)$  corresponding to a particular input  $v$  does not depend on the future  $P^T v$  of the input, i.e.  $P_T \Delta = P_T \Delta P_T$  for all  $T \geq 0$ .

The operator  $\Delta$  is *bounded* if there exists  $C$  such that

$$\|P_T w\| \leq C \|P_T v\| \quad \forall T > 0, w = \Delta(v), v \in \mathbf{L}^l \quad (2)$$

The *gain*  $\|\Delta\|$  of  $\Delta$  is defined as the infimum of all such  $C$ . The operator  $\Delta$  is *exponentially bounded* if there exist  $a > 0$  and  $C > 0$  such that

$$\|e^{at} P_T w\| \leq C \|e^{at} P_T v\| \quad \forall T > 0, w = \Delta(v), v \in \mathbf{L}^l \quad (3)$$

For proofs of exponential stability of feedback systems, the following concept will be used: the operator  $\Delta$  is said to have *fading memory* if there exist  $C > 0$  and  $a > 0$  such that for every  $h = (v, \Delta(v)) \in \mathbf{L}^{l+m}$  and for every  $\tau \geq 0$  there exists  $h_\tau = (v_\tau, \Delta(v_\tau))$  such that  $P_\tau h_\tau = 0$  and

$$\|e^{at} P^\tau (h - h_\tau)\| \leq C_f \|e^{at} P_\tau h\| \quad (4)$$

The fading memory condition is somehow related to controllability and observability, since apparently only the “unexposed” memory needs to be fading. The next lemmata state some important facts about the concept fading memory.

LEMMA 1

Every linear time-invariant operator of finite order has fading memory.  $\square$

LEMMA 2

Every bounded operator with fading memory is exponentially bounded.  $\square$

Proofs of Lemma 1 and Lemma 2 are given in section 6.

For example, a pure integrator is unbounded but has fading memory by Lemma 1. In contrast, the composition of a pure integrator and saturation does not have fading memory. (To see that this is indeed true, apply non-zero constant input at the saturation level.) The example also shows that a composition of

two operators with fading memory does not necessarily have fading memory itself.

We also need a notion of distance between two operators. For this, we define the *gap* between  $G$  and  $H$  as  $\delta(G, H)$ , where

$$\begin{aligned} \tilde{\delta}(G, H) &:= \sup_{g \in \mathcal{S}_G} \inf_{h \in \mathcal{S}_H} \sup_{T > 0} \frac{\|P_T g - P_T h\|}{\|P_T g\|} \\ \delta(G, H) &:= \max(\tilde{\delta}(G, H), \tilde{\delta}(H, G)) \end{aligned}$$

and supremum is taken over all nonzero  $g \in G$  and all  $T > 0$  with  $\|P_T g\| \neq 0$ . The operator  $G_\tau$  is said to depend *continuously* on  $\tau$  if  $\delta(G_{\tau_1}, G_{\tau_2}) \rightarrow 0$  as  $|\tau_1 - \tau_2| \rightarrow 0$ . This definition of gap is very close to the one suggested by Georgiou and Smith in [4]. The notion of gap can be used to verify boundedness in the following way:

LEMMA 3

Let the operator  $\Delta_0$  be causal and bounded and let  $\Delta$  be causal. If

$$\delta(\Delta_0, \Delta) < (2 + \|\Delta_0\|)^{-1}$$

then  $\Delta$  is bounded.  $\square$

### 3. Interconnections

It is standard to analyse systems with nonlinearities by writing them as a feedback interconnection of a linear time-invariant operator  $G$  and a possibly nonlinear and uncertain operator  $\Delta$ , described by integral quadratic constraints. The interconnection is a relation of the form

$$\begin{cases} v = G(w) + f \\ w = \Delta(v) + e \end{cases} \quad (5)$$

and it is said to be *well posed* if the set of all solutions to (5) defines a causal operator  $[G, \Delta] : (f, e) \mapsto (v, w)$ . The interconnection is called *stable* or *exponentially stable* if in addition  $[G, \Delta]$  is bounded or exponentially bounded respectively. Noting that  $[G, \Delta]$  has fading memory whenever  $G$  and  $\Delta$  do so, Lemma 2 can be reformulated in terms of stability as follows.

COROLLARY 1

If  $G$  and  $\Delta$  have fading memory and their interconnection is stable, then the interconnection is exponentially stable.  $\square$

To derive well-posedness of interconnections with hysteresis, one has to work with operators that are not open-loop bounded. We then use the notions of continuity and boundedness in a local context.

We say that operator  $F$  is *locally incremental* ( $F \in F_{li}$ ) if for any  $T > 0$  there exist  $C_0, C_1, \tau > 0$ , and  $\theta < 1$  such that

$$\|P_{t+\tau}F(w)\| \leq \theta\|P_{t+\tau}^t w\| + C_0 + C_1\|P_t w\| \quad (6)$$

for all  $t \in [0, T]$ ,  $w \in \mathbf{L}$ .

We write  $w_i \rightarrow w \in \mathbf{L}^n$  when  $\|w_i - w\| \rightarrow 0$ , and  $w_i \rightarrow^* w$  if  $\sup\|w_i - w\| < \infty$  and  $\langle g, w_i - w \rangle \rightarrow 0$  for every  $g \in \mathbf{L}_2^n$ . An operator  $F$  is said to be *locally \*-continuous* if for every  $t > 0$  there exists  $d > 0$  such that from every input-output sequence  $w_i = F(y_i)$  with  $P_{t-d}(w_i - w_0) = 0$ ,  $P_{t-d}(y_i - y_0) = 0$ , and  $P_{t+d}y_i \rightarrow^* P_{t+d}y_0$ , one can extract a subsequence  $w_{i(j)}$  such that  $P_{t+d}w_{i(j)} \rightarrow^* w$  and  $P_{t+d}w = P_{t+d}F(y)$ . In this case we write  $F \in F_{ls}$ .

#### PROPOSITION 1

Let  $F : \mathbf{L}^n \rightarrow \mathbf{L}^n$  be a causal operator which is both locally \*-continuous and locally incremental. Then equation  $w = F(w + v)$  has a solution for every  $v \in \mathbf{L}^n$ , and the corresponding operator  $v \mapsto w$  is causal, locally \*-continuous, and locally incremental.  $\square$

Theorem 1 is a general result which helps to establish well-posedness of various interconnections. The following statement describes an important special case when the proposition can be applied.

#### PROPOSITION 2

Let  $f$  be a function that maps  $\mathbf{R}^n$  into the set of convex compact subsets of  $\mathbf{R}^n$ . Assume  $f$  is linearly bounded, i.e.

$$|z| \leq c_0 + c_1|x| \quad \forall z \in f(x), x \in \mathbf{R}^n,$$

and continuous, i.e.

$$z_i \in f(x_i), x_i \rightarrow x, z_i \rightarrow z \Rightarrow z \in f(x).$$

Define the operator  $\Delta_f : \mathbf{L}^n \rightarrow \mathbf{L}^n$  by

$$\Delta_f(v) = w \Leftrightarrow w(t) \in f(v(t)) \forall t.$$

Let  $L$  be the causal LTI operator with an impulse response  $h(t)$  which is integrable over an interval  $t \in (0, d)$ . Then the composition  $F = \Delta_f \circ L$  is locally \*-continuous and locally incremental.  $\square$

## 4. Stability via Integral Quadratic Constraints

A functional  $\sigma : \mathbf{L}_2^n \rightarrow \mathbf{R}$  is called *quadratically continuous* if for every  $\varepsilon > 0$  there exists  $C > 0$  such that

$$\sigma(h) \leq \sigma(g) + \varepsilon\|g\|^2 + C\|h - g\|^2 \quad \forall g, h \in \mathbf{L}_2^n \quad (7)$$

The operator  $\Delta : \mathbf{L}^l \rightarrow \mathbf{L}^m$  is said to satisfy the *integral quadratic constraint* (IQC) defined by  $\sigma$  if

$$\sigma(h) \geq 0 \quad \forall h = (v, \Delta(v)) \in \mathbf{L}_2^{l+m}$$

Integral quadratic constraints are most often used on the form

$$\int_{-\infty}^{\infty} \left[ \begin{array}{c} \hat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{array} \right]^* \Pi(j\omega) \left[ \begin{array}{c} \hat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{array} \right] d\omega \geq 0$$

where  $\Pi(j\omega)$  is a bounded Hermitean  $k + m$  by  $k + m$  matrix-valued function. The corresponding functional will be denoted  $\sigma_\Pi$ . Moreover, the operator  $G$  will be replaced by a transfer matrix  $G(s)$  and the stability criterion can be written on the following form, recognized from Theorem 2 in [6].

#### PROPOSITION 3

Consider a  $G \in \mathbf{RH}_\infty^{k \times m}$  and a causal, locally continuous and incremental operator  $\Delta : \mathbf{L}^l \rightarrow \mathbf{L}^m$  such that  $G(\infty)\Delta$  is lower triangular. Suppose that  $\Pi \in \mathbf{RH}_\infty^{(l+m) \times (l+m)}$ ,  $\varepsilon > 0$  and  $\sigma_\Pi(0, w) + \varepsilon\|w\|^2 \leq 0 \leq \sigma_\Pi(v, 0)$  for all  $v, w$ . If

$$\left[ \begin{array}{c} G(j\omega) \\ I \end{array} \right]^* \Pi(j\omega) \left[ \begin{array}{c} G(j\omega) \\ I \end{array} \right] \leq -\varepsilon I \quad (8)$$

for all  $\omega \in \mathbf{R}$  and

$$\sigma_\Pi(v, \Delta(v)) \geq 0$$

for all  $v \in \mathbf{L}_2^l[0, \infty)$ , then the interconnection of  $G$  and  $\Delta$  is stable. Moreover, if  $\Delta$  has fading memory, then the interconnection is exponentially stable.  $\square$

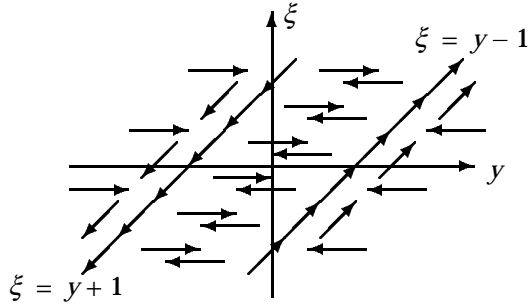
An alternative to the condition  $\sigma_\Pi(0, w) + \varepsilon\|w\|^2 \leq 0$  is to assume existence of some  $G_0 \in \mathbf{RH}_\infty^{k \times m}$  such that (8) holds with  $G$  replaced by  $G_0$  and the interconnection of  $G_0$  and  $\Delta$  is stable. The conditions above are recovered with  $G_0 = 0$ .

## 5. Application to Hysteresis

Let  $f_+$  and  $f_-$  be bounded continuous functions, mapping vectors in  $\mathbf{R}^n$  to convex subsets of  $\mathbf{R}^n$ . Let  $h : \mathbf{R}^n \rightarrow \mathbf{R}$  be a function, and let  $U \subset \mathbf{R}^n$  be a set. We say that two scalar signals  $y, \xi$  satisfy the *hysteresis relation* defined by  $f$  and  $h$  (notation  $\xi = \text{hys}(y) = \text{hys}_{f,h}(y)$ ), if  $\dot{y} \in \mathbf{L}$  and there exists a locally Lipschitz function  $x : [0, \infty) \rightarrow \mathbf{R}^n$  such that

$$\dot{x} \in \begin{cases} f_+(x)\dot{y}, & \text{if } \dot{y} \geq 0 \\ f_-(x)\dot{y}, & \text{if } \dot{y} \leq 0 \end{cases} \quad x(0) \in U \quad (9)$$

$$\xi(t) = h(x(t))$$



**Figure 1.** Backlash hysteresis

In (9),  $f_+$  and  $f_-$  define the admissible trajectories for the phase vector  $x$  for the cases when  $y$  is respectively increasing or decreasing, while  $h$  defines how the possibly higher-dimensional dynamics of  $x$  results in variations of the scalar output  $\xi$ .

For example, the ideal backlash relation, defined by the condition

$$y - \xi = \text{sgn} \dot{\xi},$$

is a special case of (9) with  $n = 2$  and

$$\begin{aligned} f_{\pm}(x_1, x_2) &= (1, 0.5[1 - \text{sgn}(1 \mp x_1 \pm x_2)]) \\ h(x_1, x_2) &= x_2 \\ U &= \{(0, \xi) : |\xi| \leq 1\} \end{aligned}$$

See Figure 1 showing the motion of the  $x$ -state as  $y$  varies.

Interconnections with hysteresis nonlinearities typically have many possible equilibria for  $y, \xi$ , so they are unstable in the sense that  $y$  does not tend to zero. Nevertheless, it is possible to prove exponential decay of  $\dot{y}$  and  $\dot{\xi}$ .

**THEOREM 1**

Assume that  $U$  is compact,  $h$  is a globally Lipschitz function,

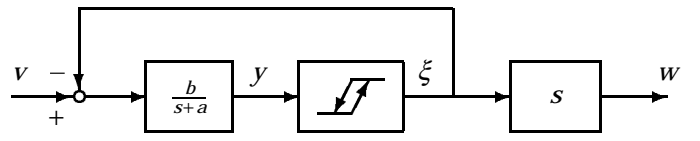
$$|h(x_1) - h(x_2)| \leq L|x_1 - x_2| \quad \forall x_1, x_2 \in \mathbf{R}^n,$$

and  $|r| \leq R$  for any  $r \in f_{\pm}(x)$ . Then the set of all pairs  $(\dot{y}, \dot{\xi})$ , where  $y$  and  $\xi$  satisfy the hysteresis relation (9), defines a causal, locally \*-continuous, and locally incremental operator  $\dot{y} \rightarrow \dot{\xi}$ . Moreover, the operator  $\dot{y} \rightarrow \dot{\xi}$  is causal and bounded (with the norm not exceeding  $RL$ ).  $\square$

**COROLLARY 2**

Let the assumptions of Theorem 1 be satisfied and suppose that  $\text{hys}_{f,h}$  has fading memory. Let  $G(s) = C(sI - A)^{-1}B$ . If

$$RL|G(i\omega)| < 1 \quad \forall \omega$$



**Figure 2.** Feedback "encapsulation" of hysteresis

then  $\dot{y}$  and  $\dot{\xi}$  tend to zero exponentially for all solutions to

$$\dot{x} = Ax - B\xi, \quad \xi = \text{hys}_{f,h}(y), \quad y = Cx \quad (10)$$

$\square$

The corollary can be viewed as a generalization of the circle criterion. Our next step will be a more specific result for the ideal backlash relation described above.

For  $a > 0$ ,  $b > -a$  define the operator  $w = \Delta_{\text{bkl}}(v)$  by the relations

$$\begin{cases} \dot{y} = -ay + b(v - \xi), & y(0) = 0 \\ \xi = \text{hys}_{f,h,U}(y) \\ w = \dot{\xi} \end{cases} \quad (11)$$

See Figure 2. The operator has properties as follows.

**THEOREM 2**

The operator  $(1/s) \circ \Delta_{\text{bkl}}$  is causal, locally \*-continuous, and locally incremental. The operator  $\Delta_{\text{bkl}}$  is causal, bounded, has fading memory, and satisfies the IQC's

$$0 \leq \left\langle w, \frac{b[1 + H(s)]}{s + a}(v + w/a) \right\rangle - (1 + b/a) \left\langle w, \frac{H(s) - H(0)}{s} w \right\rangle \quad (12)$$

$$0 = \left\langle w, \frac{bs}{s + a}v - \frac{b}{s + a}w - w \right\rangle \quad (13)$$

for every  $H \in \mathbf{RL}_{\infty}$  with  $\|H\|_{L_1} \leq 1$ .  $\square$

Combination with Proposition 3 gives the following stability criterion for systems with backlash.

**COROLLARY 3**

Let  $G(s) = C(sI - A)^{-1}B$ . If there exist  $\varepsilon > 0$ ,  $\eta \in \mathbf{R}$  and  $H \in \mathbf{RL}_{\infty}$  with  $\|H\|_{L_1} \leq 1$  such that

$$G(0) > -1 \quad (14)$$

$$\text{Re} \left[ (G(i\omega) + 1) \left( \eta + \frac{1 + H(i\omega)}{i\omega} \right) \right] > \varepsilon \quad \forall \omega \neq 0 \quad (15)$$

then for the solutions of (10) with  $f, h, U$  corresponding to backlash, the derivatives  $\dot{\xi}$  and  $\dot{y}$  tend to zero exponentially.  $\square$

## 6. Selected Proofs

*Proof of Lemma 1.* Let the linear time-invariant operator  $v = G(w)$  have the minimal realization

$$\begin{aligned}\dot{x} &= Ax + Bw \\ v &= Cx + Dw\end{aligned}$$

Given some  $g = (G(w), w)$  and  $\tau > 0$ , define  $g_\tau = (G(w_\tau), w_\tau)$  as follows. Choose  $g_\tau$  to minimize  $\int_\tau^{\tau+1} |g_\tau(t)|^2 dt$  under the constraints

$$0 = P_\tau g_\tau = P^{\tau+1}(g_\tau - g)$$

By controllability of  $(A, B)$  there exists a  $c_1 > 0$ , independent of  $g$  and  $\tau$ , such that

$$\|P_{\tau+1}^r(g_\tau - g)\| \leq c_1 |x(\tau)|$$

and by observability of  $(C, A)$  there exists a  $c_2 > 0$ , independent of  $g$  and  $\tau$ , such that

$$|x(\tau)| \leq c_2 \|P_\tau^{r-1} g\|$$

Hence

$$\begin{aligned}\|e^t P^r(g_\tau - g)\| &= \|e^t P_{\tau+1}^r(g_\tau - g)\| \\ &\leq e^{\tau+1} c_1 \|x(\tau)\| \\ &\leq e^{\tau+1} c_1 c_2 \|P_\tau^{r-1} g\| \\ &\leq e^2 c_1 c_2 \|e^t P_\tau g\|\end{aligned}$$

*Proof of Lemma 2.* By the definition of fading memory, there exists  $a > 0$  and  $C_a > 0$  such that for every  $z = (v, \Delta(v))$  and for every  $\tau \geq 0$  there exists  $z_\tau = (v_\tau, \Delta(v_\tau))$  such that  $P_\tau z_\tau = 0$  and

$$\|e^{at} P^r(z - z_\tau)\| \leq C_a \|e^{at} P_\tau z\|$$

Boundedness of  $\Delta$  implies existence of  $C$  such that for all  $T > \tau > 0$

$$\begin{aligned}\|P_T^r z\| &\leq \|P_T z_\tau\| + \|P_T^r(z - z_\tau)\| \\ &\leq C \|P_T v_\tau\| + \|P_T^r(z - z_\tau)\| \\ &= C \|P_T^r[v + (v_\tau - v)]\| + \|P_T^r(z - z_\tau)\| \\ &\leq C \|P_T^r v\| + (2C + 1) \|P_T^r(z - z_\tau)\| \\ &\leq C \|P_T^r v\| + (2C + 1) e^{-a\tau} C_a \|e^{at} P_\tau z\|\end{aligned}$$

In particular, there exist constants  $C_0, C_1$ , independent of  $\tau$  and  $T$ , such that

$$\int_\tau^T |z|^2 dt \leq C_0 e^{-2a\tau} \int_0^\tau e^{2at} |z|^2 dt + C_1 \int_\tau^T |v|^2 dt \quad (16)$$

for any  $\tau \in [0, T]$ . Multiplying (16) by  $2\varepsilon e^{2\varepsilon\tau}$ , where  $\varepsilon \in [0, a)$ , integrating the products from  $\tau = 0$  to  $\tau = T$ , and adding (16) with  $\tau = 0$  to the result yields

$$\int_0^T e^{2\varepsilon t} |z|^2 dt \leq \frac{C_0 \varepsilon}{a - \varepsilon} \int_0^T e^{2\varepsilon t} |z|^2 dt + C_1 \int_0^T e^{2\varepsilon t} |v|^2 dt$$

When  $C_0 \varepsilon < a - \varepsilon$ , the exponential bound is proved.  $\square$

*Proof of the IQC's in Theorem 2.* First note that

$$0 \leq \dot{\xi}(t)(y - \xi)(t) \quad \forall t$$

In addition,  $\dot{\xi}(t)$  can be nonzero only if  $|y - \xi| = 1$ . It is therefore possible to add perturbations to the factor  $y - \xi$  without violating the inequality. More precisely,

$$0 \leq \dot{\xi}(t)[y - \xi + h * (y - \xi)](t) \quad \forall t$$

for every convolution kernel  $h$  with  $\int_{-\infty}^{\infty} |h| dt \leq 1$ , because the magnitude of the term  $h * (y - \xi)$  is then at most one. Noting that  $\langle \dot{\xi}, \xi \rangle = 0$  for  $\xi, \dot{\xi} \in \mathbf{L}_2$  gives

$$\begin{aligned}0 &\leq \langle \dot{\xi}, y - \xi + h * (y - \xi) \rangle \\ &= \langle w, [1 + H(s)]y - \xi \rangle \\ &= \langle w, [1 + H(s)](y - \xi) + [1 + H(0)](1 + b/a)\xi \rangle \\ &= \langle w, [1 + H](y + \xi b/a) \rangle - \langle w, [H - H(0)](1 + b/a)\xi \rangle \\ &= \left\langle w, \frac{b[1 + H(s)]}{s + a}(v + w/a) \right\rangle \\ &\quad - (1 + b/a) \left\langle w, \frac{H(s) - H(0)}{s} w \right\rangle\end{aligned}$$

$\square$  The equality (13) is just that

$$\begin{aligned}0 &= \langle \dot{\xi}, y - \dot{\xi} \rangle \\ &= \left\langle w, \frac{bs}{s + a} v - \frac{b}{s + a} w - w \right\rangle\end{aligned}$$

$\square$

*Proof of Corollary 3.* Let  $b = aG(0)$ . Then  $b > -a$  so  $\Delta_{\text{bkl}}$  is properly defined and

$$G_1(s) := \left[ \frac{s+a}{b} G(s) + 1 \right] / s$$

is stable because the  $b$  was chosen to cancel the unstable pole. From (10) and (11) follows that

$$\begin{cases} v = G_1(w) + b^{-1}(aC + CA)e^{At}x(0) \\ w = \Delta_{\text{bkl}}(v) \end{cases}$$

so to conclude exponential decay of  $\dot{\xi}$  and  $\dot{y}$ , it is sufficient to verify (8) where  $\Pi$  corresponds to a convex combination of (12) and (13). This condition becomes the inequality (15).  $\square$

## 7. Acknowledgments

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## 8. References

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