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# Small antenna analysis using convex optimization

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**Abstract**— The performance of small antennas is constrained by the electrical size of the antenna structure. Here, convex optimization is used to analyze small antennas. It is shown that stored electric and magnetic energies, radiated power, and radiated field expressed in the current density can be combined to form convex optimization problems for many interesting antenna cases. The solution of the optimization problem determines physical bounds on the antenna performance.

Fundamental limitations on small antennas were first studied more than 60 years ago by Wheeler [14] and Chu [2], see also [9, 13]. The approach by Chu is based on mode expansions of the fields outside the smallest circumscribing sphere [2] and explicit calculations of the stored electric and magnetic energies and the radiated power to determine the Q-factor. This approach has dominated the research but is unfortunately restricted to canonical geometries such as spheres. The analysis was generalized to arbitrarily shaped antennas by a reformulation of the antenna problem into a scattering problem and utilizing the forward scattering sum rule [4, 5], see also [6, 10, 12, 15] for alternative approaches.

In this presentation, the analysis is generalized to many new and important antenna problems using convex optimization [1, 8]. The results are based on formulations of the antenna problems as convex optimization problems, where we use that the stored electric and magnetic energies, and radiated power are positive semi-definite quadratic forms in the current density. We use the explicit expressions by Vanderbosch [11] for the stored energies, see also [7] for an alternative derivation and interpretation. Moreover, the presented results are for current densities in free space.

Following the notation in [8] and expand the current density,  $\mathbf{J}$ , in local basis functions as in the method of moments (MoM), and let

$$\begin{cases} W_e \approx \mathbf{J}^H \mathbf{X}_e \mathbf{J} & \text{stored electric energy} \\ W_m \approx \mathbf{J}^H \mathbf{X}_m \mathbf{J} & \text{stored magnetic energy} \\ P_r \approx \mathbf{J}^H \mathbf{R}_r \mathbf{J} & \text{radiated power} \\ \hat{\mathbf{e}}^* \cdot \mathbf{F}(\hat{\mathbf{k}}) \approx \mathbf{F}^H \mathbf{J} & \text{far field in the } \hat{\mathbf{k}} \text{ direction and the } \hat{\mathbf{e}} \text{ polarization,} \end{cases} \quad (1)$$

where  $\mathbf{J}$  is a column matrix with the expansion coefficients for  $\mathbf{J}$  and the superscripts  $*$  and  $^H$  denote the complex conjugate and Hermitian transpose, respectively. The quality factor is defined as a weighted quotient between the stored energy and the radiated power [2, 9, 13]

$$Q = \frac{2\omega \max\{W_e, W_m\}}{P_r} \approx \frac{2\omega \max\{\mathbf{J}^H \mathbf{X}_e \mathbf{J}, \mathbf{J}^H \mathbf{X}_m \mathbf{J}\}}{\mathbf{J}^H \mathbf{R}_r \mathbf{J}}. \quad (2)$$

Similarly, the partial directivity is a weighted quotient between the radiation intensity in the direction  $\hat{\mathbf{k}}$  and polarization  $\hat{\mathbf{e}}$  and the radiated power, *i.e.*,

$$D(\hat{\mathbf{k}}, \hat{\mathbf{e}}) = \frac{\beta |\hat{\mathbf{e}}^* \cdot \mathbf{F}(\hat{\mathbf{k}})|^2}{P_r} \approx \frac{\beta |\mathbf{F}^H \mathbf{J}|^2}{\mathbf{J}^H \mathbf{R}_r \mathbf{J}}, \quad (3)$$

where  $\beta$  is a normalization constant. To form convex optimization problems, we first consider the partial directivity Q-factor quotient

$$\frac{D(\hat{\mathbf{k}}, \hat{\mathbf{e}})}{Q} \leq \max_{\mathbf{J}} \frac{D(\hat{\mathbf{k}}, \hat{\mathbf{e}})}{Q} = \max_{\mathbf{J}} \frac{\beta |\hat{\mathbf{e}}^* \cdot \mathbf{F}(\hat{\mathbf{k}})|^2}{2\omega \max\{W_e, W_m\}} \approx \max_{\mathbf{J}} \frac{\beta |\mathbf{F}^H \mathbf{J}|^2}{2\omega \max\{\mathbf{J}^H \mathbf{X}_e \mathbf{J}, \mathbf{J}^H \mathbf{X}_m \mathbf{J}\}}. \quad (4)$$

The quotient is invariant for a multiplicative scaling  $\mathbf{J} \rightarrow \alpha\mathbf{J}$  showing that it is sufficient to consider  $|\mathbf{F}^H\mathbf{J}|^2 = 1$  or  $\text{Re}\{\mathbf{F}^H\mathbf{J}\} = 1$ , see [6]. This gives the convex optimization problem

$$\begin{aligned} & \text{minimize} && \max\{\mathbf{J}^H\mathbf{X}_e\mathbf{J}, \mathbf{J}^H\mathbf{X}_m\mathbf{J}\} \\ & \text{subject to} && \text{Re}\{\mathbf{F}^H\mathbf{J}\} = 1 \end{aligned} \quad (5)$$

where the constant 1 is used for simplicity. Convex optimization problems can be solved with *e.g.*, the matlab toolbox CVX [3], see [8]. The main advantage with the formulation as a convex optimization problem is that we can easily extend the  $D/Q$  problem (5) to other problems. By *e.g.*, adding a constraint on the radiated power  $P_r$ , we get the convex optimization problem

$$\begin{aligned} & \text{minimize} && \max\{\mathbf{J}^H\mathbf{X}_e\mathbf{J}, \mathbf{J}^H\mathbf{X}_m\mathbf{J}\} \\ & \text{subject to} && \text{Re}\{\mathbf{F}^H\mathbf{J}\} = 1 \\ & && \mathbf{J}^H\mathbf{R}_r\mathbf{J} \leq \beta D_0^{-1}, \end{aligned} \quad (6)$$

that can be interpreted as a constraint on the partial directivity  $D(\hat{\mathbf{k}}, \hat{\mathbf{e}}) \geq D_0$ . The formulation (6) can be used to analyze the minimum  $Q$  for superdirective antennas. We can also add other constraints that determines bounds on antennas that have a specified radiated field or are integrated in passive (metallic and/or dielectric) structures [8].

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