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The Theory of the Propagation of TEM-Pulses in Dispersive Bi-Isotropic Slabs

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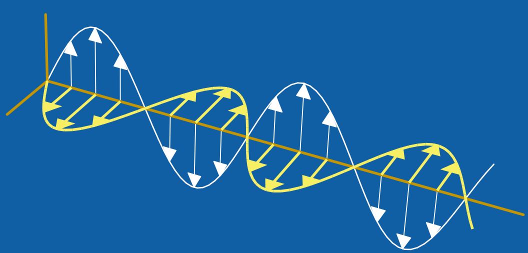

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Abstract

A survey of the theory of propagation of transient transverse electromagnetic waves in temporally dispersive, bi-isotropic slabs is given, and a novel wave splitting, which completely separates right-going and left-going waves in the dispersive medium, is proposed. The new approach leads to a simple scattering relation in terms of wave propagators and single-interface scattering operators only. These temporal integral operators are related to the four time-dependent susceptibility kernels of the medium through non-linear Volterra equations of the second kind. In a subsequent article, the corresponding inverse scattering problem is addressed on the basis of the new results.

1 Introduction

The study of the electromagnetic properties of chiral media or, more generally, bi-an-isotropic media, and the potential use for these materials in various electronic components, is an active research field. An extensive amount of articles have been published during the latter part of the 1980's and during the 1990's, and several new books with the stress on microwave applications are available as well, see, e.g., Lakhtakia *et al.* [13] and Lindell *et al.* [14]. The present paper concerns the bi-isotropic medium, which is both chiral and isotropic. For a review, see the article by Engheta and Jaggard [3] or the one by Lakhtakia [12].

The characteristic property of the isotropic chiral medium is the chirality, which continuously twists and distorts the plane of polarization of an initially linearly polarized electromagnetic wave. At fixed frequency, these phenomena are interpreted as the result of the superposition of one left-circularly polarized (LCP) wave and one right-circularly polarized (RCP) wave, traveling through the complex medium with different phase velocities and subject to unequal absorption. If, standing at the receiver and looking towards the source, the plane of polarization is rotated clockwise, the medium is said to be dextro-rotatory; otherwise, it is levo-rotatory. The transmitted wave is elliptically polarized except for lossless (non-absorbing) media, which merely rotate the plane of polarization. Although intimately tied together — both effects are explained by the presence of the chirality parameter in the constitutive relations — the rotation and absorption phenomena have been given special names: optical rotatory power (ORP) and circular dichroism (CD), respectively. Media that exhibit these effects in the optical regime are said to be optically active, and the epithet electromagnetic activity has been suggested in the general electromagnetic case [13, 14].

The physical origin of electromagnetic activity is resonance phenomena in the handed (chiral) structure of the medium; therefore, chirality is presumed to be a highly dispersive property. In particular, the rotatory dispersion is anomalous in the sense that the angle of rotation may change sign in a narrow frequency-band. Thus, the chiral medium is both dextro-rotatory and levo-rotatory depending on frequency. In natural isotropic chiral media, e.g., maple syrup, ORP occurs in the optical regime, due to the presence of randomly oriented chiral molecules. In order to observe electromagnetic activity at lower frequencies, the dimensions of the chiral scatterers must be considerably larger, see, e.g., Lindman's pioneer works [15, 16]. For applications in the microwave regime, man-made chiral media are required.

In recent years, the interaction between electromagnetic pulses and bi-isotropic media has attracted certain attention [4, 5, 9–11, 18–21]. One interesting line of research within this field is the study of the early time behavior of the signal in these media. Knowledge of these transients may be of significance in technical applications such as pulsed radars and ultrafast lasers. Both traditional methods [5, 21] and time-domain techniques [19] have been employed. As far as the Sommerfeld forerunners are concerned, the time-domain approach has proved to be very efficient; results are available even for general stratified bi-isotropic slabs subject to arbitrary normal incidence [19].

The study of the inverse problem of reconstructing the time-dependent material parameters of the bi-isotropic slab is another active time-domain research field. The accessibility of reliable inverse algorithms is, naturally, of importance for many applications. Specifically, an inverse algorithm using transient TEM scattering data has been developed, and examples of excellent reconstructions with synthetic scattering data have been presented [10, 18]. However, unique solubility of this inverse problem has not been proven yet. Neither has transient experimental scattering data been presented.

The propagation of transient TEM-waves in dispersive bi-isotropic slabs has been studied extensively using time-domain methods based on the wave splitting technique, see, e.g., Refs 9, 10, 18, 19. In the present article, these studies are developed further. As a result, a scattering relation in terms of wave propagators and single-interface reflection operators at the boundary is obtained. This is accomplished with the aid of a new wave splitting, which completely separates right-going and left-going waves in the medium. The wave propagators and the single-interface scattering operators are temporal integral operators, which are related to the four time-dependent susceptibility kernels of the bi-isotropic medium through non-linear Volterra equations of the second kind.

The obtained scattering relation has a simple structure, which admits derivation of all the well-known features of TEM-wave propagation in bi-isotropic media both in the time domain and at fixed frequency. In a second article, the corresponding inverse problem is addressed on the basis of this scattering relation. Specifically, it is shown that the generic inverse scattering problem is, indeed, uniquely soluble. Furthermore, additional examples of successful reconstructions are presented.

In Section 2 of the present article, a brief outline of the time-domain methods employed in the Refs 9–11, 18–20 mentioned above is given. In Section 3, the wave propagation problem for the (optically) impedance matched bi-isotropic slab is formulated. In Section 4, the new wave splitting for the dispersive bi-isotropic medium is presented. The wave propagators of the bi-isotropic medium are defined in Section 5 and the scattering relation is derived in Section 6. Explicit integral representations of the Green functions and the imbedding kernels used in Refs 10, 18 are given in Section 7. In Section 8, the special results for the semi-infinite medium are emphasized. Finally, in an appendix, the scattering relation obtained in Section 6 is generalized to the high-frequency mismatch case.

2 Survey of the time-domain analysis

In this section, the time-domain methods employed in the above references are reviewed.

The time-domain analysis is based on the wave splitting technique, see Ref. 2. A wave splitting is a change of the dependent variables — in this context, preferably the electric and magnetic fields — such that, in the surrounding non-dispersive, isotropic media, the two new, so called, split vector field variables represent the right-going and left-going waves, respectively. A wave splitting can be performed in several ways, and at least two different wave splittings for bi-isotropic media have been presented so far. The split vector fields satisfy partial integro-differential equations obtained from the Maxwell equations and from the constitutive relations of the complex medium. The latter involve temporal convolutions modeling dispersion and chirality effects. The wave splittings presented so far are based on the principal part of the intrinsic impedance of the medium only. In other words, the fact that the medium is dispersive is not considered (at this step). As a result, the right-going and left-going waves are still coupled inside the dispersive medium.

The next step in the analysis is to define the scattering operators by applying Duhamel's principle to the linear, time-invariant, causal scattering problem. The kernels of these integral operators are generic in the sense that they are independent of the excitation of the slab, and depend on the time-dependent susceptibility kernels of the medium only. Similarly, the Green functions and the imbedding kernels are defined. These functions are the generic quantities of the internal (split) vector fields. Finally, the Green functions equations and the imbedding equations are derived. These equations are the partial integro-differential equations for the corresponding integral kernels.

In the propagation problem (the direct scattering problem), either of these equations is solved numerically in space-time for given material parameters (susceptibility kernels). The method of integration along the characteristics is employed in both cases. In particular, the scattering kernels, which are seen to be appropriate restrictions of the Green functions and imbedding kernels to the boundaries, are obtained. By convolution of the scattering kernels and the incident electric field, the scattered fields, i.e., the reflected and transmitted fields, are computed. The accuracy of the results is confirmed, e.g., by comparing the results of the two employed methods [9]. (It ought to be remarked, that, in spite of being of quite different physical origin, the Green functions and the imbedding kernels are intimately related mathematically; specifically, they are obtained from one another by solving Volterra equations of the second kind, see Ref. 18.) The internal fields are obtained by convolution of the Green functions and the incident electric field.

In Ref. 20 it is shown that the propagation problem referred to above is uniquely soluble, and, as a consequence of this, it is proved that the Green functions equations are uniquely soluble too, given the specific properties of the medium. This fact is used in Ref. 18 to prove that the imbedding equations are also uniquely soluble, and exact solutions to these equations in terms of infinite series were given in the homogeneous case.

The inverse scattering problem for the impedance matched, but otherwise arbitrary, homogeneous bi-isotropic slab subject to normal incidence is discussed in Ref. 10 using generic reflection and transmission data, and several examples of good reconstructions of the four susceptibility kernels that characterize the medium are presented. The inverse algorithm rests heavily on the direct problem. The discussion of this inverse problem is continued in Ref. 18, where the more realistic case, when data consists of the physical reflected and transmitted field quantities instead of the corresponding scattering kernels, is considered. Still, good reconstructions are obtained. The effect of the ill-posed nature of deconvolution can be minimized by choosing sharp pulses as incident fields.

In the next section, the wave propagation problem to be discussed is formulated. The theory presented in, e.g., Ref. 10 is the appropriate starting point. At the convenience of the reader, the basic equations of this investigation are repeated.

3 Basic equations

A dispersive bi-isotropic slab is located between the surfaces $z = 0$ and $z = d$ in a right-handed rectangular coordinate system $\mathcal{O}(x, y, z)$, where the three basis vectors are denoted by $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$, see Figure 1. For the sake of simplicity, the slab is surrounded by vacuum with permittivity ϵ_0, permeability μ_0, and with the speed of light and intrinsic impedance given by

$$\begin{cases} c = 1/\sqrt{\epsilon_0\mu_0}, \\ \eta = \sqrt{\mu_0/\epsilon_0}, \end{cases} \tag{3.1}$$

respectively. Furthermore, it is assumed that the slab is (optically) impedance matched, that is, the optical intrinsic impedance of the medium equals the vacuum value (3.1). Since so called hard reflectors at the front and rear walls can be removed by appropriate techniques (time-delayed Volterra equations of the second kind involving various scattering kernels), this assumption is not really a restriction [8, 19]. Nevertheless, for completeness, a generalization to the general mismatch case based on the results obtained in this article is presented in Appendix A.

The constitutive relations of the bi-isotropic medium at the point $\boldsymbol{r} \equiv (x, y, z)$ and at the time t are

$$\begin{cases} c\eta\boldsymbol{D}(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{r}, t) + \big((G + F) * \boldsymbol{E}\big)(\boldsymbol{r}, t) + \eta\big((K + L) * \boldsymbol{H}\big)(\boldsymbol{r}, t), \\ c\boldsymbol{B}(\boldsymbol{r}, t) = \big((-K + L) * \boldsymbol{E}\big)(\boldsymbol{r}, t) + \eta\boldsymbol{H}(\boldsymbol{r}, t) + \eta\big((G - F) * \boldsymbol{H}\big)(\boldsymbol{r}, t), \end{cases} \tag{3.2}$$

where the dispersive effects as well as the characteristic properties of the complex medium are introduced by temporal convolutions, e.g.,

$$(G * \boldsymbol{E})(\boldsymbol{r}, t) = \int_{-\infty}^{t} G(t - t')\boldsymbol{E}(\boldsymbol{r}, t')\, dt'.$$

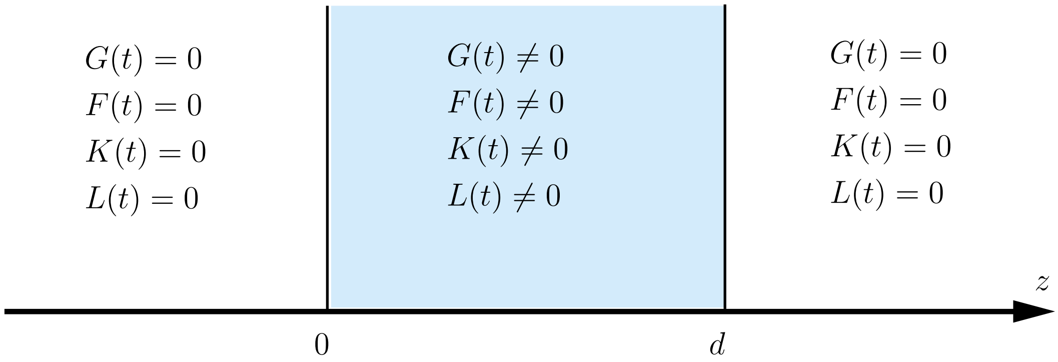

Figure 1: The infinite bi-isotropic scatterer.

Thus, both the polarization

$$\boldsymbol{P}(\boldsymbol{r},t) = c^{-1}\eta^{-1}\big((G+F)*\boldsymbol{E}\big)(\boldsymbol{r},t) + c^{-1}\big((K+L)*\boldsymbol{H}\big)(\boldsymbol{r},t)$$

and the magnetization

$$\boldsymbol{M}(\boldsymbol{r},t) = \eta^{-1}\big((-K+L)*\boldsymbol{E}\big)(\boldsymbol{r},t) + \big((G-F)*\boldsymbol{H}\big)(\boldsymbol{r},t)$$

of the medium lack direct terms. All electromagnetic fields in the slab are assumed to be quiescent before the time $t = 0$.

The susceptibility kernels $G(t)$ and $F(t)$ model the ordinary dispersive effects of the slab, whereas the chirality $K(t)$ and the non-reciprocity $L(t)$ are the characteristic properties of the bi-isotropic medium. If $F = L = 0$, then the constitutive relations (3.2) describe the non-reflective chiral medium, which plays an important role in the present investigation. The analysis in Section 7 suggests the alternative names *co-reflectivity* for $F(t)$ and *cross-reflectivity* for $L(t)$: if $F = 0$, then the co-component of the reflected electric field is zero, and vice versa. The influence of L on the cross-component of the reflected electric field is analogous.

At non-negative times, the susceptibility kernels $G(t)$, $F(t)$, $K(t)$, and $L(t)$ are assumed to be twice continuously differentiable functions of time with bounded derivatives. At negative times, they all equal zero due to causality arguments [6]. By assuming these susceptibility kernels to be integrable, the slab vanishes in the high-frequency limit (i.e., at optical frequencies) in electromagnetic sense. This is an immediate consequence of the Riemann-Lebesgue lemma:

$$\lim_{\omega\to\infty} \int_0^\infty e^{-i\omega t} G(t)\, dt = 0. \tag{3.3}$$

The slab is excited by transient transverse plane waves, one right-going and one left-going, see Figure 2. The reason for the presence of two incident pulses is that the medium is not entirely symmetric. The electric field of the right-going wave at the front wall $z = 0$ at time t is denoted by $\boldsymbol{E}^i_{\text{left}}(t)$, whereas the electric field of the left-going wave at the rear wall $z = d$ at time t is $\boldsymbol{E}^i_{\text{right}}(t)$. These functions

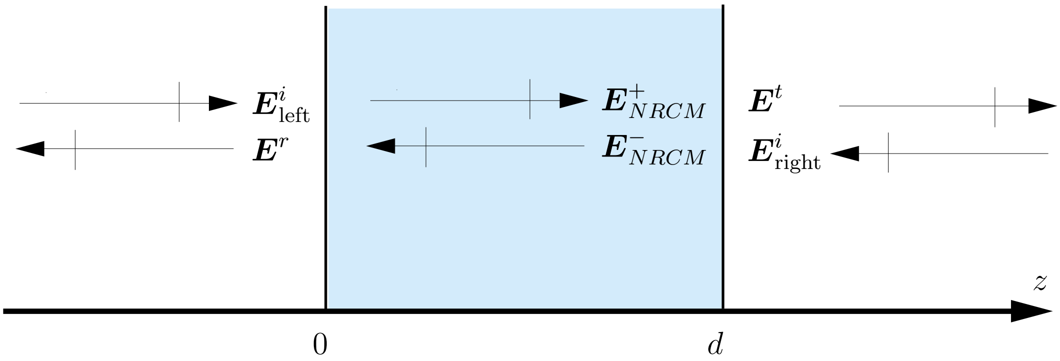

Figure 2: Incident, scattered, and internal electric fields.

are continuously differentiable with bounded derivative, except possibly at a finite number of times. Moreover, they are causal, i.e., identically zero at negative times.

In each bounded time interval, there exists a transverse solution to the source-free Maxwell equations

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \qquad \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}.$$

This solution is independent of the transverse variables (x, y), that is,

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{e}_x E_x(z,t) + \boldsymbol{e}_y E_y(z,t) \equiv \begin{pmatrix} E_x(z,t) \\ E_y(z,t) \end{pmatrix}, \tag{3.4}$$

and similarly for the magnetic field $\boldsymbol{H}(\boldsymbol{r}, t)$, and for the flux densities $\boldsymbol{D}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$. The solution (3.4) to the propagation problem is unique, and the internal and scattered electromagnetic fields inherit the regularity of the incident electric field. For future reference, the scattered electric fields at the front and rear walls are introduced. At the time t, they are denoted by $\boldsymbol{E}^r(t)$ and $\boldsymbol{E}^t(t)$, respectively, in agreement with the notation in, e.g., Ref. 10, where $\boldsymbol{E}^i_{\text{right}}(t) \equiv \boldsymbol{0}$. The scattering geometry is depicted in Figure 2.

With the assumption (3.4), the Maxwell equations for the dispersive bi-isotropic medium read

$$c\frac{\partial}{\partial z}\begin{pmatrix} \boldsymbol{E} \\ \eta \mathbf{J}\boldsymbol{H} \end{pmatrix} = \frac{\partial}{\partial t}\left\{ \begin{pmatrix} (-\mathbf{K}+\mathbf{L})* & \mathbf{I}+(\mathbf{G}-\mathbf{F})* \\ \mathbf{I}+(\mathbf{G}+\mathbf{F})* & -(\mathbf{K}+\mathbf{L})* \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \eta \mathbf{J}\boldsymbol{H} \end{pmatrix} \right\}, \tag{3.5}$$

where the 2×2-matrices $\mathbf{I}$ and $\mathbf{J}$ are defined by

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{J} = \boldsymbol{e}_z \times \mathbf{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and where the susceptibility matrices are

$$\mathbf{G} = G\mathbf{I}, \qquad \mathbf{K} = K\mathbf{J}, \qquad \mathbf{F} = F\mathbf{I}, \qquad \mathbf{L} = L\mathbf{J}.$$

All matrices that appear in this paper are typed in Roman boldface. Vectors are typed in italic boldface, while calliographic letters are reserved for matrix-valued operators. The scalar identity operator is denoted by 1.

A wave splitting with respect to the principal part of the intrinsic impedance of the medium is now applied:

$$\begin{pmatrix} \boldsymbol{E}^+(z,t) \\ \boldsymbol{E}^-(z,t) \end{pmatrix} = \mathbf{W}_{\text{opt}} \begin{pmatrix} \boldsymbol{E}(z,t) \\ \eta \mathbf{J} \boldsymbol{H}(z,t) \end{pmatrix}, \quad \mathbf{W}_{\text{opt}} = \frac{1}{2} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix}, \tag{3.6}$$

$$\begin{pmatrix} \boldsymbol{E}(z,t) \\ \eta \mathbf{J} \boldsymbol{H}(z,t) \end{pmatrix} = \mathbf{W}_{\text{opt}}^{-1} \begin{pmatrix} \boldsymbol{E}^+(z,t) \\ \boldsymbol{E}^-(z,t) \end{pmatrix}, \quad \mathbf{W}_{\text{opt}}^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix}. \tag{3.7}$$

This simple change of basis is local both in the space-variable and the time-variable, and can be referred to as an *optical wave splitting*. The transformation (3.7) shows that, in free space, the split vector fields $\boldsymbol{E}^{\pm}(z,t)$ are the electric fields of the right-going and left-going waves, respectively. With exception for the non-reflective medium, this interpretation does not hold inside the dispersive slab, see the dynamical equation (3.8) below. This is hardly surprising, since the dispersive contribution to the intrinsic impedance of the medium is not considered at optical wave splitting. Nevertheless, since the only non-vanishing component of the Poynting vector $\boldsymbol{S}$ is

$$\boldsymbol{e}_z \cdot \boldsymbol{S} = \boldsymbol{e}_z \cdot (\boldsymbol{E} \times \boldsymbol{H}) = -\boldsymbol{E} \cdot \mathbf{J} \boldsymbol{H} = \frac{1}{\eta}(\boldsymbol{E}^+ \cdot \boldsymbol{E}^+ - \boldsymbol{E}^- \cdot \boldsymbol{E}^-),$$

the power flow in the $+\boldsymbol{e}_z$-direction at each point (z,t) in the space-time is fully determined by the vector field $\boldsymbol{E}^+(z,t)$. The analogous result holds for $\boldsymbol{E}^-(z,t)$.

Combination of the Maxwell equations (3.5) and the wave splitting (3.6)–(3.7) yields the dynamical equation for the split vector fields $\boldsymbol{E}^{\pm}(z,t)$ in the bi-isotropic medium:

$$\begin{pmatrix} (c\partial_z + \partial_t)\boldsymbol{E}^+ \\ (c\partial_z - \partial_t)\boldsymbol{E}^- \end{pmatrix} = \partial_t \left\{ \begin{pmatrix} -\mathbf{G} - \mathbf{K} & -\mathbf{F} + \mathbf{L} \\ \mathbf{F} + \mathbf{L} & \mathbf{G} - \mathbf{K} \end{pmatrix} * \begin{pmatrix} \boldsymbol{E}^+ \\ \boldsymbol{E}^- \end{pmatrix} \right\}. \tag{3.8}$$

The free space contribution is recognized as the left term, whereas the right term represents the medium effects. The 4×4-matrix to the right is referred to as the medium matrix. Note that, since the (tangential components) of the electric and magnetic fields are continuous in the spatial variable z, the split vector fields are continuous in this variable too; hence, the boundary values are

$$\begin{cases} \boldsymbol{E}^+(0,t) = \boldsymbol{E}^i_{\text{left}}(t), \\ \boldsymbol{E}^-(0,t) = \boldsymbol{E}^r(t), \end{cases} \qquad \begin{cases} \boldsymbol{E}^+(d,t) = \boldsymbol{E}^t(t), \\ \boldsymbol{E}^-(d,t) = \boldsymbol{E}^i_{\text{right}}(t). \end{cases} \tag{3.9}$$

A generalization of these conditions to various mismatch cases, e.g., the metal-backed bi-isotropic slab is given in the appendix.

In the next section, a new wave splitting for homogeneous isotropic and bi-isotropic media, which pays due attention to dispersion effects, is proposed.

4 Dispersive wave splitting

In this section, new dependent vector field variables are introduced with intent to reduce the dynamical equation in the general bi-isotropic case (3.8) to the dynamics of the Non-Reflective Chiral Medium:

$$\begin{pmatrix} (c\partial_z + \partial_t)\boldsymbol{E}^+_{NRCM} \\ (c\partial_z - \partial_t)\boldsymbol{E}^-_{NRCM} \end{pmatrix} = \partial_t \left\{ \begin{pmatrix} -\mathbf{N} - \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} - \mathbf{K} \end{pmatrix} * \begin{pmatrix} \boldsymbol{E}^+_{NRCM} \\ \boldsymbol{E}^-_{NRCM} \end{pmatrix} \right\}. \tag{4.1}$$

In the $NRCM$, which is an isotropic chiral medium, the split vector fields do not couple. Wave propagation in the $NRCM$ is easy to analyze in terms of *wave propagators*, see Section 5 below. These operators correspond to the (exponential) propagation factors, which arise at the steady-state analysis of the problem. By equation (4.1), the new vector field variables $\boldsymbol{E}^\pm_{NRCM}(z,t)$ do not couple in the general bi-isotropic medium either, except through boundary conditions. In the achiral case, the $NRCM$ is reduced to the non-reflective isotropic medium, which is the simplest linear medium.

The matrix-valued integral kernel $\mathbf{N}$ introduced in the dynamics (4.1) is uniquely determined by the susceptibility matrices $\mathbf{G}$, $\mathbf{F}$, and $\mathbf{L}$ of the general bi-isotropic medium. Thus, it depends on time only. Implicitly, $\mathbf{N}$ is defined by the non-linear Volterra equation of the second kind

$$2\mathbf{N} + \mathbf{N} * \mathbf{N} = 2\mathbf{G} + \mathbf{G} * \mathbf{G} - \mathbf{F} * \mathbf{F} + \mathbf{L} * \mathbf{L}, \tag{4.2}$$

which is stable numerically. Use of the matrix identity $\mathbf{JJ} = -\mathbf{I}$ shows that $\mathbf{N} = N\,\mathbf{I}$, where the scalar integral kernel $N(t)$ satisfies the integral equation

$$2N + N * N = 2G + G * G - F * F - L * L. \tag{4.3}$$

Observe that $N(t)$ can be expanded in a power series of temporal convolutions, which converges for all finite t, provided the susceptibility kernels are bounded:

$$N = \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (M*)^{k-1} M, \qquad M = 2G + G * G - F * F - L * L.$$

In the special case of an isotropic medium, and in the sense of operators, equation (4.3) can be factored as

$$(1 + N*)^2 = \big(1 + (G + F) *\big)\big(1 + (G - F) *\big).$$

A fixed frequency analysis now reveals that the operator $c(1 + N*)^{-1}$ corresponds to the spectral density (the inverse Fourier transform) of the complex phase velocity of the medium as a function of angular frequency. By $(1 + N*)^{-1}$ is meant the temporal integral operator $1 + N_{\mathrm{res}}*$, where the resolvent kernel $N_{\mathrm{res}}(t)$ is uniquely determined by the kernel $N(t)$ through the Volterra equation of the second kind

$$N_{\mathrm{res}}(t) + N(t) + (N_{\mathrm{res}} * N)(t) = 0.$$

In other words, the temporal integral operator $1 + N*$ is the time-domain equivalent of the complex index of refraction of the medium as a function of angular frequency. This notion is well known from the analysis of absorbing isotropic media, see, e.g., Brillouin [1, p. 43].

The Volterra equation (4.2) is now derived in the general bi-isotropic case. Let the time-dependent matrix-valued functions $\mathbf{W}_{ij}$, $1 \le i,j \le 2$, be arbitrary linear combinations of the matrices $\mathbf{I}$ and $\mathbf{J}$, and consider the change of variables

$$\begin{pmatrix} \boldsymbol{E}^+_{NRCM} \\ \boldsymbol{E}^-_{NRCM} \end{pmatrix} = \begin{pmatrix} \mathbf{I} + \mathbf{W}_{11}* & \mathbf{W}_{12}* \\ \mathbf{W}_{21}* & \mathbf{I} + \mathbf{W}_{22}* \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^+ \\ \boldsymbol{E}^- \end{pmatrix} \equiv \mathcal{W}_{\text{disp}} \begin{pmatrix} \boldsymbol{E}^+ \\ \boldsymbol{E}^- \end{pmatrix}. \tag{4.4}$$

This transformation is non-local in time. Outside the slab, the entries $\mathbf{W}_{ij}$ are defined to be zero, reducing $\mathcal{W}_{\text{disp}}$ to the identity operator in this region.

By equations (3.8) and (4.1), the condition on the integral kernel $\mathbf{N}$ becomes

$$\mathcal{W}_{\text{disp}}^{-1} \begin{pmatrix} -(\mathbf{I} + \mathbf{N}*) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} + \mathbf{N}* \end{pmatrix} \mathcal{W}_{\text{disp}} = \begin{pmatrix} -(\mathbf{I} + \mathbf{G}*) & (-\mathbf{F} + \mathbf{L})* \\ (\mathbf{F} + \mathbf{L})* & \mathbf{I} + \mathbf{G}* \end{pmatrix}.$$

Assuming $\mathbf{N}$ to be a linear combination of $\mathbf{I}$ and $\mathbf{J}$, the square of this operator identity is precisely equation (4.2), and the derivation is finished. The temporal integral operators

$$\mathcal{N}^{\pm}(t) := \mathbf{I} + (\mathbf{N} \pm \mathbf{K})*,$$

which appear in the dynamical equation (4.1), are referred to as *the generalized indices of refraction of the bi-isotropic medium for right-going and left-going waves*, respectively. $\mathbf{N} \pm \mathbf{K}$ are the matrix-valued kernels of these operators.

The vector fields $\boldsymbol{E}^{\pm}(z,t)$ can be expressed in the new vector field variables $\boldsymbol{E}^{\pm}_{NRCM}(z,t)$ in several ways in order to bring the dynamical equation (3.8) into the non-reflective diagonal form (4.1)–(4.2). The wave splitting presented in this paper aims at identifying $\boldsymbol{E}^{+}_{NRCM}(z,t)$ as the right-going electric field and $\boldsymbol{E}^{-}_{NRCM}(z,t)$ as the left-going electric field. This admits meaningful definitions of transmission operators and reflection operators at the boundaries, see below.

As a result of this identification, the total electric field $\boldsymbol{E}(z,t)$ becomes

$$\boldsymbol{E}(z,t) = \boldsymbol{E}^{+}_{NRCM}(z,t) + \boldsymbol{E}^{-}_{NRCM}(z,t). \tag{4.5}$$

Moreover, in heuristic agreement with plane-wave propagation in free space, the total magnetic field $\boldsymbol{H}(z,t)$ is expected to be able to be written in the form

$$\mathbf{J}\boldsymbol{H}(z,t) = -\mathcal{Z}^{+}(t)^{-1}\boldsymbol{E}^{+}_{NRCM}(z,t) + \mathcal{Z}^{-}(t)^{-1}\boldsymbol{E}^{-}_{NRCM}(z,t). \tag{4.6}$$

The introduced temporal integral operators $\mathcal{Z}^{\pm}$ are referred to as *the generalized intrinsic impedances of the bi-isotropic medium associated with right-going and left-going waves*, respectively. The principal form of these operators is

$$\mathcal{Z}^{\pm}(t) \equiv \eta(\mathbf{I} + \mathbf{Z}^{\pm}*),$$

where $\mathbf{Z}^{\pm}$ are the matrix-valued kernels of the generalized intrinsic impedances. By spatial reflection ($z \to -z$) of the axially symmetric space, one deduces that

$$\mathcal{Z}^{-} = (\mathcal{Z}^{+})^{t}, \tag{4.7}$$

where $(\mathcal{Z}^{+})^{t}$ denotes the transpose of $\mathcal{Z}^{+}$, i.e., $\mathbf{Z}^{-} = (\mathbf{Z}^{+})^{t}$. Axial symmetry also implies that the kernels $\mathbf{Z}^{\pm}$ are linear combinations of the matrices $\mathbf{I}$ and $\mathbf{J}$.

In the isotropic case, the definition of the generalized intrinsic impedances is

$$\mathcal{Z}^- = \mathcal{Z}^+ = (1 + Z*)\mathcal{I},$$

where $\mathcal{I}$ is the identity operator and the scalar kernel $Z(t)$ satisfies the operator identity

$$(1 + Z*)^2 = \big(1 + (G + F) *\big)^{-1}\big(1 + (G - F) *\big). \tag{4.8}$$

In microwave engineering, the Fourier transform of the operator $\eta(1 + Z*)$ is the relevant property. The kernel $Z(t)$ can be expanded in the power series

$$Z = \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (A*)^{k-1} A, \qquad A = G - F + (G + F)_{\mathrm{res}} + (G - F) * (G + F)_{\mathrm{res}},$$

where $(G + F)_{\mathrm{res}}(t)$ is the resolvent kernel of $G(t) + F(t)$. The definition of the generalized intrinsic impedances of the general bi-isotropic medium is postponed.

As a consequence of the imposed conditions (4.5) and (4.6), the principal form of the *temporally dispersive wave splitting* in the bi-isotropic medium becomes

$$\begin{pmatrix} \boldsymbol{E}^+_{NRCM} \\ \boldsymbol{E}^-_{NRCM} \end{pmatrix} = \mathcal{W} \begin{pmatrix} \boldsymbol{E} \\ \eta \mathbf{J}\boldsymbol{H} \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{E} \\ \eta \mathbf{J}\boldsymbol{H} \end{pmatrix} = \mathcal{W}^{-1} \begin{pmatrix} \boldsymbol{E}^+_{NRCM} \\ \boldsymbol{E}^-_{NRCM} \end{pmatrix}, \tag{4.9}$$

where

$$\mathcal{W}(t) = \frac{1}{2}\left(\mathbf{I} + \frac{\mathbf{Z}^+ + \mathbf{Z}^-}{2} *\right)^{-1} \otimes \begin{pmatrix} (\mathbf{I} + \mathbf{Z}^+ *) & -(\mathbf{I} + \mathbf{Z}^+ *)(\mathbf{I} + \mathbf{Z}^- *) \\ (\mathbf{I} + \mathbf{Z}^- *) & (\mathbf{I} + \mathbf{Z}^+ *)(\mathbf{I} + \mathbf{Z}^- *) \end{pmatrix}$$

and

$$\mathcal{W}^{-1}(t) = \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -(\mathbf{I} + \mathbf{Z}^+ *)^{-1} & (\mathbf{I} + \mathbf{Z}^- *)^{-1} \end{pmatrix}.$$

The introduced notation means, that each of the four 2×2 matrix operator entries of the 4×4 matrix operator to the right of symbol $\otimes$ is to be multiplied by the 2×2 matrix operator to the left. The change of variables (4.4) is specified by the composition

$$\mathcal{W} = \mathcal{W}_{\mathrm{disp}} \mathcal{W}_{\mathrm{opt}},$$

where $\mathcal{W}_{\mathrm{opt}}$ is given by equation (3.6). Explicitly,

$$\mathcal{W}_{\mathrm{disp}}(t) = \left(\mathbf{I} + \frac{\mathbf{Z}^+ + \mathbf{Z}^-}{2} *\right)^{-1} \otimes \begin{pmatrix} (\mathbf{I} + \mathbf{Z}^+ *)(\mathbf{I} + \frac{\mathbf{Z}^-}{2} *) & -(\mathbf{I} + \mathbf{Z}^+ *)\frac{\mathbf{Z}^-}{2} * \\ -(\mathbf{I} + \mathbf{Z}^- *)\frac{\mathbf{Z}^+}{2} * & (\mathbf{I} + \mathbf{Z}^+ *)(\mathbf{I} + \frac{\mathbf{Z}^+}{2} *) \end{pmatrix}.$$

The inverse of the transformation (4.4) is

$$\begin{pmatrix} \boldsymbol{E}^+ \\ \boldsymbol{E}^- \end{pmatrix} = \mathcal{W}^{-1}_{\mathrm{disp}} \begin{pmatrix} \boldsymbol{E}^+_{NRCM} \\ \boldsymbol{E}^-_{NRCM} \end{pmatrix}, \tag{4.10}$$

where

$$\mathcal{W}_{\mathrm{disp}}^{-1}(t) = \begin{pmatrix} (\mathbf{I}+\mathbf{Z}^{+}*)^{-1}(\mathbf{I}+\frac{\mathbf{Z}^{+}}{2}*) & (\mathbf{I}+\mathbf{Z}^{-}*)^{-1}\frac{\mathbf{Z}^{-}}{2}* \\ (\mathbf{I}+\mathbf{Z}^{+}*)^{-1}\frac{\mathbf{Z}^{+}}{2}* & (\mathbf{I}+\mathbf{Z}^{-}*)^{-1}(\mathbf{I}+\frac{\mathbf{Z}^{-}}{2}*) \end{pmatrix}.$$

By introducing single-interface scattering operators at the boundaries, these transformations are simplified considerably.

In the general bi-isotropic case, the intrinsic impedances are defined in terms of the temporal integral operators

$$\begin{cases} \mathcal{R}_0(t) = \mathbf{R}_0 * = \left(\mathcal{Z}^{+}+\eta\mathcal{I}\right)^{-1}\left(\mathcal{Z}^{+}-\eta\mathcal{I}\right) = \left(\mathbf{I}+\dfrac{\mathbf{Z}^{+}}{2}*\right)^{-1}\dfrac{\mathbf{Z}^{+}}{2}* \\ \mathcal{R}_1(t) = \mathbf{R}_1 * = \left(\mathcal{Z}^{-}+\eta\mathcal{I}\right)^{-1}\left(\mathcal{Z}^{-}-\eta\mathcal{I}\right) = \left(\mathbf{I}+\dfrac{\mathbf{Z}^{-}}{2}*\right)^{-1}\dfrac{\mathbf{Z}^{-}}{2}* \end{cases}$$

Explicitly,

$$\begin{cases} \mathbf{I}+\mathbf{Z}^{+}* = \mathcal{Z}^{+}/\eta = (\mathcal{I}-\mathcal{R}_0)^{-1}(\mathcal{I}+\mathcal{R}_0), \\ \mathbf{I}+\mathbf{Z}^{-}* = \mathcal{Z}^{-}/\eta = (\mathcal{I}-\mathcal{R}_1)^{-1}(\mathcal{I}+\mathcal{R}_1). \end{cases} \tag{4.11}$$

The operators $\mathcal{R}_0$ and $\mathcal{R}_1$ are the reflection operators at the front and rear walls, $z=0$ and $z=d$, respectively, viewed from the surrounding non-dispersive media. In terms of these operators, transformation (4.10) reads

$$\begin{pmatrix} \boldsymbol{E}^{+} \\ \boldsymbol{E}^{-} \end{pmatrix} = \begin{pmatrix} (\mathcal{I}+\mathcal{R}_0)^{-1} & (\mathcal{I}+\mathcal{R}_1)^{-1}\mathcal{R}_1 \\ (\mathcal{I}+\mathcal{R}_0)^{-1}\mathcal{R}_0 & (\mathcal{I}+\mathcal{R}_1)^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^{+}_{NRCM} \\ \boldsymbol{E}^{-}_{NRCM} \end{pmatrix}, \tag{4.12}$$

with inverse

$$\begin{pmatrix} \boldsymbol{E}^{+}_{NRCM} \\ \boldsymbol{E}^{-}_{NRCM} \end{pmatrix} = (\mathcal{I}-\mathcal{R}_0\mathcal{R}_1)^{-1} \otimes \begin{pmatrix} (\mathcal{I}+\mathcal{R}_0) & -(\mathcal{I}+\mathcal{R}_0)\mathcal{R}_1 \\ -(\mathcal{I}+\mathcal{R}_1)\mathcal{R}_0 & (\mathcal{I}+\mathcal{R}_1) \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^{+} \\ \boldsymbol{E}^{-} \end{pmatrix}. \tag{4.13}$$

Recall the property (4.7) of the generalized intrinsic impedances of the axially symmetric medium. Then, by definition,

$$\mathcal{R}_1 = \mathcal{R}_0^{t}.$$

Furthermore, explicit evaluation of the cross-components — the co-components have already been exploited resulting in the integral equation (4.2) — of the product

$$\begin{pmatrix} -(\mathbf{I}+\mathbf{N}*) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}+\mathbf{N}* \end{pmatrix} = \mathcal{W}_{\mathrm{disp}} \begin{pmatrix} -(\mathbf{I}+\mathbf{G}*) & (-\mathbf{F}+\mathbf{L})* \\ (\mathbf{F}+\mathbf{L})* & \mathbf{I}+\mathbf{G}* \end{pmatrix} \mathcal{W}_{\mathrm{disp}}^{-1}$$

yields the imbedding equation of the identical semi-infinite bi-isotropic medium $z \in (0,\infty)$, see Ref. 18, equation (5.11):

$$2(\mathbf{I}+\mathbf{G}*)\mathbf{R}_0 + \mathbf{F} + \mathbf{L} + (\mathbf{F}-\mathbf{L}) * \mathbf{R}_0 * \mathbf{R}_0 = \mathbf{0}. \tag{4.14}$$

Hence, $\mathbf{R}_0(t)$ is the imbedding kernel of the right optically impedance-matched bi-isotropic half-space $z \in (0,\infty)$. Analogously, $\mathbf{R}_1(t)$ is the imbedding kernel of the left optically impedance-matched bi-isotropic half-space $z \in (-\infty, d)$. In terms of integral kernels introduced in this paper, the solution of the imbedding equation (4.14) is simple to obtain:

$$\begin{cases} \mathbf{R}_0 = -\left(\mathbf{I} + \dfrac{\mathbf{N}+\mathbf{G}}{2}*\right)^{-1} \dfrac{\mathbf{F}+\mathbf{L}}{2} = -\left(1 + \dfrac{N+G}{2}*\right)^{-1} \dfrac{\mathbf{F}+\mathbf{L}}{2}, \\ \mathbf{R}_1 = -\left(\mathbf{I} + \dfrac{\mathbf{N}+\mathbf{G}}{2}*\right)^{-1} \dfrac{\mathbf{F}-\mathbf{L}}{2} = -\left(1 + \dfrac{N+G}{2}*\right)^{-1} \dfrac{\mathbf{F}-\mathbf{L}}{2}. \end{cases} \tag{4.15}$$

Recall that the imbedding equation is uniquely soluble, see the fifth paragraph of Section 2 for references.

The generalized intrinsic impedances are obtained by substituting the reflection operators $\mathcal{R}_0$ and $\mathcal{R}_1$ into definition (4.11). The result, which by equation (4.2) is reduced to equation (4.8) in the reciprocal case, is

$$\mathscr{Z}^{\pm} = \eta(\mathbf{I} + \mathbf{Z}^{\pm}*) = \left(\mathbf{I} + \frac{\mathbf{N}+\mathbf{G}+\mathbf{F}\pm\mathbf{L}}{2}*\right)^{-1} \left(\mathbf{I} + \frac{\mathbf{N}+\mathbf{G}-\mathbf{F}\mp\mathbf{L}}{2}*\right).$$

In other words, the kernels $\mathbf{Z}^{\pm}(t)$ are obtained by solving linear matrix-valued Volterra equations of the second kind.

The change of variables (4.12)–(4.13) and its inverse are now interpreted in terms of single-interface scattering operators. The result is

$$\begin{aligned} \begin{pmatrix} \boldsymbol{E}^{+} \\ \boldsymbol{E}^{-} \end{pmatrix} &= \begin{pmatrix} (\mathcal{T}_0^{+})^{-1} & -(\mathcal{T}_0^{+})^{-1}\mathcal{R}_0^{+} \\ -(\mathcal{T}_1^{-})^{-1}\mathcal{R}_1^{-} & (\mathcal{T}_1^{-})^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^{+}_{NRCM} \\ \boldsymbol{E}^{-}_{NRCM} \end{pmatrix}, \\ \begin{pmatrix} \boldsymbol{E}^{+}_{NRCM} \\ \boldsymbol{E}^{-}_{NRCM} \end{pmatrix} &= \begin{pmatrix} (\mathcal{T}_1^{+})^{-1} & -(\mathcal{T}_1^{+})^{-1}\mathcal{R}_1^{+} \\ -(\mathcal{T}_0^{-})^{-1}\mathcal{R}_0^{-} & (\mathcal{T}_0^{-})^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^{+} \\ \boldsymbol{E}^{-} \end{pmatrix}. \end{aligned} \tag{4.16}$$

At both walls, the superscripts $\pm$ denote contribution to waves propagating in the $\pm z$-directions, respectively. The scattering operators at the wall $z = 0$ are endowed with the subscript 0:

$$\begin{cases} \mathcal{R}_0^{-} = \text{the reflection operator viewed from the non-dispersive medium,} \\ \mathcal{T}_0^{+} = \text{the transmission operator for the transition,} \\ \qquad\quad \text{from the non-dispersive medium to the dispersive slab,} \\ \mathcal{R}_0^{+} = \text{the reflection operator viewed from the dispersive slab,} \\ \mathcal{T}_0^{-} = \text{the transmission operator for the transition,} \\ \qquad\quad \text{from the dispersive slab to the non-dispersive medium.} \end{cases}$$

Similarly, the scattering operators at the wall $z = d$ are

$$\begin{cases} \mathcal{R}_1^{-} = \text{the reflection operator viewed from the dispersive slab,} \\ \mathcal{T}_1^{+} = \text{the transmission operator for the transition,} \\ \qquad\quad \text{from the dispersive slab to the non-dispersive medium,} \\ \mathcal{R}_1^{+} = \text{the reflection operator viewed from the non-dispersive medium,} \\ \mathcal{T}_1^{-} = \text{the transmission operator for the transition,} \\ \qquad\quad \text{from the non-dispersive medium to the dispersive slab.} \end{cases}$$

In terms of the reflection operators $\mathcal{R}_0$ and $\mathcal{R}_1$, one obtains

$$\begin{cases} \mathcal{R}_0^- = \mathcal{R}_0, \\ \mathcal{T}_0^+ = \mathcal{I} + \mathcal{R}_0, \\ \mathcal{R}_0^+ = -(\mathcal{I} + \mathcal{R}_1)^{-1}(\mathcal{I} + \mathcal{R}_0)\mathcal{R}_1, \\ \mathcal{T}_0^- = (\mathcal{I} + \mathcal{R}_1)^{-1}(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1). \end{cases} \tag{4.17}$$

Similarly, the scattering operators at the wall $z = d$ are

$$\begin{cases} \mathcal{R}_1^- = -(\mathcal{I} + \mathcal{R}_0)^{-1}(\mathcal{I} + \mathcal{R}_1)\mathcal{R}_0, \\ \mathcal{T}_1^+ = (\mathcal{I} + \mathcal{R}_0)^{-1}(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1), \\ \mathcal{R}_1^+ = \mathcal{R}_1, \\ \mathcal{T}_1^- = \mathcal{I} + \mathcal{R}_1. \end{cases} \tag{4.18}$$

Note the special relations between these operators:

$$\mathcal{T}_0^+ - \mathcal{R}_0^- = \mathcal{T}_0^- - \mathcal{R}_0^+ = \mathcal{T}_1^+ - \mathcal{R}_1^- = \mathcal{T}_1^- - \mathcal{R}_1^+ = \mathcal{I}. \tag{4.19}$$

5 Wave propagators

The solution to the transformed dynamical equation (4.1) is easy to obtain. In appropriate operator notation, it takes the form

$$\boldsymbol{E}_{NRCM}^{\pm}\left(z, t \pm \frac{z}{c}\right) = \mathcal{P}^{\pm}(z, 0; t)\boldsymbol{E}_{NRCM}^{\pm}(0, t),$$

where

$$\begin{aligned} \mathcal{P}^{\pm}(z, 0; t) &= \exp\left(\frac{z}{c}\frac{d}{dt}(\mp\mathbf{N} - \mathbf{K}) *\right) = \\ &= \exp\big(\mp \frac{z}{c}(\mathbf{N}(0) + \mathbf{N}'*)\big) \exp\big(- \frac{z}{c}(\mathbf{K}(0) + \mathbf{K}'*)\big) \end{aligned}$$

are the wave propagators for the subslab $(0, z)$. More generally, one has

$$\boldsymbol{E}_{NRCM}^{\pm}(z_2, t \pm (z_2 - z_1)/c) = \mathcal{P}^{\pm}(z_2, z_1; t)\boldsymbol{E}_{NRCM}^{\pm}(z_1, t), \tag{5.1}$$

for all space points $z_1, z_2 \in (0, d)$, where

$$\mathcal{P}^{\pm}(z_2, z_1; t) = \mathcal{P}^{\pm}(z_2 - z_1, 0; t),$$

are the *wave propagators for the subslab* (z_1, z_2). The invariance of the wave propagators under translations in the spatial variable z is due to homogeneity. Wave propagators for dielectrics were introduced by Karlsson and Stewart [7].

The wave propagators $\mathcal{P}^{\pm}$ fully determine the propagating fields along the characteristics and satisfy the relations

$$\begin{cases} \mathcal{P}^{\pm}(z_3, z_2; t)\mathcal{P}^{\pm}(z_2, z_1; t) = \mathcal{P}^{\pm}(z_3, z_1; t), \\ \mathcal{P}^{\pm}(z, z; t) = \mathcal{I}, \\ \mathcal{P}^{\pm}(z_2, z_1; t)^{-1} = \mathcal{P}^{\pm}(z_1, z_2; t). \end{cases}$$

An additional property of the propagators of the axially symmetric medium is

$$\mathcal{P}^{\pm}(z_2, z_1; t)^t = \mathcal{P}^{\mp}(z_1, z_2; t). \tag{5.2}$$

The temporal integral operators

$$\mathcal{G}^{\pm}(t) = \frac{1}{c}\frac{d}{dt}\big(\mp \mathbf{N} - \mathbf{K}\big)* = \frac{1}{c}\big(\mp \mathbf{N}(0) - \mathbf{K}(0)\big) + \frac{1}{c}\big(\mp \mathbf{N}'(t) - \mathbf{K}'(t)\big)*$$

are the *generators* of the wave propagators, and the functions

$$\mathbf{G}^{\pm}(t) := \frac{1}{c}\big(\mp \mathbf{N}'(t) - \mathbf{K}'(t)\big)$$

are referred to as the *generator kernels*. Thus, in terms of their generators, the wave propagators are

$$\mathcal{P}^{\pm}(z_2, z_1; t) = \exp\left(\int_{z_1}^{z_2} \mathcal{G}^{\pm}(t)\, dz\right) = \exp\big((z_2 - z_1)\mathcal{G}^{\pm}(t)\big).$$

Moreover, they satisfy the operator equations

$$\partial_{z_2}\mathcal{P}^{\pm}(z_2, z_1; t) = \mathcal{G}^{\pm}(t)\mathcal{P}^{\pm}(z_2, z_1; t).$$

The generators of the stratified medium depend on the spatial variable z.

The wave propagators can be factored as

$$\mathcal{P}^{\pm}(z_2, z_1; t) = \mathbf{Q}^{\pm}(z_2, z_1)\left(\mathbf{I} + \mathbf{P}^{\pm}(z_2, z_1; t)*\right),$$

where the matrices $\mathbf{Q}^{\pm}(z_2, z_1)$ determine the attenuation and rotation of the wave front [18]. Specifically,

$$\mathbf{Q}^{\pm}(z_2, z_1) = e^{\frac{1}{c}\int_{z_1}^{z_2}(\mp\mathbf{N}(0)-\mathbf{K}(0))dz} = e^{\pm a(z_1,z_2)}\begin{pmatrix}\cos\phi(z_2, z_1) & -\sin\phi(z_2, z_1)\\ \sin\phi(z_2, z_1) & \cos\phi(z_2, z_1)\end{pmatrix},$$

where the attenuation factor is

$$a(z_2, z_1) = -N(0)(z_2 - z_1)/c$$

and the angle of rotation is

$$\phi(z_2, z_1) = -K(0)(z_2 - z_1)/c.$$

These formulae agree with the heuristic picture of electromagnetic activity.

The operators $\mathbf{I} + \mathbf{P}^{\pm}(z_2, z_1; t)*$ involve the generator kernels $\mathbf{G}^{\pm}(t)$:

$$\begin{aligned}\mathbf{I} + \mathbf{P}^{\pm}(z_2, z_1; t)* &= \exp\big((z_2 - z_1)\mathbf{G}^{\pm}*\big) = \\ &= \exp\Big(\mp \frac{(z_2 - z_1)}{c}N'*\Big)\exp\Big(-\frac{(z_2 - z_1)}{c}\mathbf{K}'*\Big),\end{aligned}$$

where the matrix-valued factor is

$$\exp\big(-\frac{(z_2-z_1)}{c}\mathbf{K}'*\big)=\begin{pmatrix}\cos\big(-\frac{(z_2-z_1)}{c}K'*\big) & -\sin\big(-\frac{(z_2-z_1)}{c}K'*\big)\\ \sin\big(-\frac{(z_2-z_1)}{c}K'*\big) & \cos\big(-\frac{(z_2-z_1)}{c}K'*\big)\end{pmatrix}.$$

The functions $\mathbf{P}^{\pm}(z_2,z_1;t)$ are the *propagator kernels for the subslab* (z_1,z_2). These functions are generalizations of the Green functions of the $NRCM$, see Section 7. Observe that the propagator kernels $\mathbf{P}^{\pm}(z_2,z_1;t)$ are linear combinations of the matrices $\mathbf{I}$ and $\mathbf{J}$ in axially symmetric cases.

The formulae presented in this section must be interpreted appropriately as products of integral operators. The exponentials are expanded in their power series

$$\exp\big(\mp\frac{z}{c}N'*\big)=1\mp\frac{z}{c}(N'*)+\frac{z^2}{c^2}\frac{(N'*N'*)}{2!}\mp\frac{z^3}{c^3}\frac{(N'*N'*N'*)}{3!}+\dots$$

Similarly, it is understood that the trigonometrical functions are

$$\cos\big(-\frac{z}{c}K'*\big)=1-\frac{z^2}{c^2}\frac{(K'*K'*)}{2!}+\frac{z^4}{c^4}\frac{(K'*K'*K'*K'*)}{4!}+\dots$$

and

$$\sin\big(-\frac{z}{c}K'*\big)=-\frac{z}{c}(K'*)+\frac{z^3}{c^3}\frac{(K'*K'*K'*)}{3!}-\dots$$

respectively.

6 Complete solution

The solution to the general wave propagation problem is now derived. It is given in terms of wave propagators and the single-interface scattering operators. For convenience, the delta measure $\delta(t)$ is employed, and, furthermore, the general time dependence is suppressed. The notation for time-delay now becomes

$$\delta_a*\boldsymbol{E}^+(z):=\boldsymbol{E}^+(z,t-a).$$

All 2×2-operators that appear commute.

Straightforward use of equations (4.12), (4.13), and (5.1) yields the expansion

$$\begin{aligned}\begin{pmatrix}\boldsymbol{E}^+(z)\\ \boldsymbol{E}^-(z)\end{pmatrix}&=\delta_{\frac{z}{c}}*\mathcal{P}^+(z,0)(\mathcal{I}-\mathcal{R}_0\mathcal{R}_1)^{-1}\otimes\begin{pmatrix}\mathcal{I} & -\mathcal{R}_1\\ \mathcal{R}_0 & -\mathcal{R}_0\mathcal{R}_1\end{pmatrix}\begin{pmatrix}\boldsymbol{E}^+(0)\\ \boldsymbol{E}^-(0)\end{pmatrix}+\\ &+\delta_{\frac{d-z}{c}}*\mathcal{P}^-(z,d)(\mathcal{I}-\mathcal{R}_0\mathcal{R}_1)^{-1}\otimes\begin{pmatrix}-\mathcal{R}_0\mathcal{R}_1 & \mathcal{R}_1\\ -\mathcal{R}_0 & \mathcal{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{E}^+(d)\\ \boldsymbol{E}^-(d)\end{pmatrix}\end{aligned}\tag{6.1}$$

of the internal split vector fields in terms of their values at the edges. The first relation in equation (6.1) evaluated at $z=d$ and the second relation evaluated at $z=0$ form the system of equations

$$\begin{aligned}&\begin{pmatrix}\mathcal{I} & -\mathcal{R}_1\\ -\mathcal{R}_0\delta_{\frac{d}{c}}*\mathcal{P}^-(0,d) & \delta_{\frac{d}{c}}*\mathcal{P}^-(0,d)\end{pmatrix}\begin{pmatrix}\boldsymbol{E}^+(d)\\ \boldsymbol{E}^-(d)\end{pmatrix}=\\ &\qquad\begin{pmatrix}\delta_{\frac{d}{c}}*\mathcal{P}^+(d,0) & -\mathcal{R}_1\delta_{\frac{d}{c}}*\mathcal{P}^+(d,0)\\ -\mathcal{R}_0 & \mathcal{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{E}^+(0)\\ \boldsymbol{E}^-(0)\end{pmatrix},\end{aligned}\tag{6.2}$$

which is equivalent to

$$\begin{pmatrix} \mathcal{I} & \mathcal{R}_1\delta_{\frac{d}{c}} * \mathcal{P}^+(d,0) \\ \mathcal{R}_0\delta_{\frac{d}{c}} * \mathcal{P}^-(0,d) & \mathcal{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^+(d) \\ \boldsymbol{E}^-(0) \end{pmatrix} = \begin{pmatrix} \delta_{\frac{d}{c}} * \mathcal{P}^+(d,0) & \mathcal{R}_1 \\ \mathcal{R}_0 & \delta_{\frac{d}{c}} * \mathcal{P}^-(0,d) \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^+(0) \\ \boldsymbol{E}^-(d) \end{pmatrix}.$$

The scattering relation is now easy to obtain. It reads

$$\begin{pmatrix} \boldsymbol{E}^+(d) \\ \boldsymbol{E}^-(0) \end{pmatrix} = \mathcal{M}(0,d) \otimes \mathcal{C} \begin{pmatrix} \boldsymbol{E}^+(0) \\ \boldsymbol{E}^-(d) \end{pmatrix}, \tag{6.3}$$

where

$$\mathcal{M}(0,d;t) = \big(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d)\big)^{-1}$$

and

$$\mathcal{C}(t) = \begin{pmatrix} (\mathcal{I} - \mathcal{R}_0\mathcal{R}_1)\delta_{\frac{d}{c}} * \mathcal{P}^+(d,0) & \mathcal{R}_1\big(\mathcal{I} - \delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d)\big) \\ \mathcal{R}_0\big(\mathcal{I} - \delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d)\big) & (\mathcal{I} - \mathcal{R}_0\mathcal{R}_1)\delta_{\frac{d}{c}} * \mathcal{P}^-(0,d) \end{pmatrix}.$$

An equivalent form is

$$\begin{pmatrix} \boldsymbol{E}^t \\ \boldsymbol{E}^r \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathcal{R}_1 \\ \mathcal{R}_0 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}}(t) \\ \boldsymbol{E}^i_{\text{right}}(t) \end{pmatrix} + (\mathcal{I} - \mathcal{R}_0\mathcal{R}_1)\mathcal{M}(0,d) \otimes$$
$$\otimes \begin{pmatrix} \delta_{\frac{d}{c}} * \mathcal{P}^+(d,0) & -\mathcal{R}_1\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d) \\ -\mathcal{R}_0\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d) & \delta_{\frac{d}{c}} * \mathcal{P}^-(0,d) \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}} \\ \boldsymbol{E}^i_{\text{right}} \end{pmatrix},$$

where the boundary values (3.9) also have been employed. Note that the latter result can be interpreted heuristically in terms of the scattering operators (4.17)-(4.18):

$$\begin{pmatrix} \boldsymbol{E}^t \\ \boldsymbol{E}^r \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathcal{R}_1^+ \\ \mathcal{R}_0^- & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}} \\ \boldsymbol{E}^i_{\text{right}} \end{pmatrix} + \mathcal{M}(0,d) \otimes \tag{6.4}$$
$$\otimes \begin{pmatrix} \mathcal{T}_0^+\mathcal{T}_1^+\delta_{\frac{d}{c}} * \mathcal{P}^+(d,0) & \mathcal{T}_1^+\mathcal{T}_1^-\mathcal{R}_0^+\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d) \\ \mathcal{T}_0^+\mathcal{T}_0^-\mathcal{R}_1^-\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d) & \mathcal{T}_0^-\mathcal{T}_1^-\delta_{\frac{d}{c}} * \mathcal{P}^-(0,d) \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}} \\ \boldsymbol{E}^i_{\text{right}} \end{pmatrix}.$$

Recalling the geometric series, one concludes that the operator

$$\mathcal{M}(0,d) = \big(\mathcal{I} - \mathcal{R}_0^+\mathcal{R}_1^-\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d)\big)^{-1} \tag{6.5}$$

represents multiple propagation through the slab.

With the aid of the relation (5.2), the general scattering relation (6.3) can be written in the principal form

$$\begin{pmatrix} \boldsymbol{E}^t \\ \boldsymbol{E}^r \end{pmatrix} = \begin{pmatrix} \mathcal{T}\delta_{\frac{d}{c}} * & \mathcal{R}^t \\ \mathcal{R} & \mathcal{T}^t\delta_{\frac{d}{c}} * \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}} \\ \boldsymbol{E}^i_{\text{right}} \end{pmatrix},$$

which coincides with results obtained elsewhere, see, e.g., Ref. 18. Thus, it is not necessary to refer to Duhamel's principle when the physical reflection and transmission operators

$$\begin{cases} \mathcal{R} := \mathbf{R}*, \\ \mathcal{T} := \mathbf{Q}^+(d,0)(\mathbf{I} + \mathbf{T}*) \end{cases}$$

are defined. Explicitly, the physical scattering operators are given by

$$\begin{cases} \mathcal{R} = \mathcal{R}_0 - \mathcal{M}(0,d)(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1)\mathcal{R}_0\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d), \\ \mathcal{T} = \mathcal{M}(0,d)(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1)\mathcal{P}^+(d,0). \end{cases}$$

These equations are the appropriate starting point for the inverse scattering problem.

Attention is now focused on the internal fields. If equations (4.13) and (5.1) are combined, one obtains

$$\begin{pmatrix} \boldsymbol{E}^+_{NRCM}(z) \\ \boldsymbol{E}^-_{NRCM}(z) \end{pmatrix} = \delta_{\frac{z}{c}} * \mathcal{P}^+(z,0)(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1)^{-1}(\mathcal{I} + \mathcal{R}_0) \otimes \begin{pmatrix} \mathcal{I} & -\mathcal{R}_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^+(0) \\ \boldsymbol{E}^-(0) \end{pmatrix} + \\ + \delta_{\frac{d-z}{c}} * \mathcal{P}^-(z,d)(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1)^{-1}(\mathcal{I} + \mathcal{R}_1) \otimes \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -\mathcal{R}_0 & \mathcal{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^+(d) \\ \boldsymbol{E}^-(d) \end{pmatrix}. \tag{6.6}$$

Use of the scattering relation (6.3) then yields the relation between the internal electric fields and the excitation in terms of wave propagators and scattering operators at the boundary:

$$\begin{pmatrix} \boldsymbol{E}^+_{NRCM}(z) \\ \boldsymbol{E}^-_{NRCM}(z) \end{pmatrix} = \left(\mathcal{I} - \mathcal{R}_0^+\mathcal{R}_1^-\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d)\right)^{-1} \otimes \tag{6.7}$$

$$\otimes \begin{pmatrix} \mathcal{T}_0^+\delta_{\frac{z}{c}} * \mathcal{P}^+(z,0) & \mathcal{T}_1^-\mathcal{R}_0^+\delta_{\frac{d+z}{c}} * \mathcal{P}^+(z,0)\mathcal{P}^-(0,d) \\ \mathcal{T}_0^+\mathcal{R}_1^-\delta_{\frac{2d-z}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(z,d) & \mathcal{T}_1^-\delta_{\frac{d-z}{c}} * \mathcal{P}^-(z,d) \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}} \\ \boldsymbol{E}^i_{\text{right}} \end{pmatrix}.$$

This final formula confirms the interpretation of the operators (4.17)–(4.18).

7 Green functions and imbedding kernels

In this section, the explicit expressions for the Green functions and reflection imbedding kernels defined in Refs 11, 18 are given. It is assumed that $\boldsymbol{E}^i_{\text{right}}(t) \equiv \mathbf{0}$.

Substitution of equation (6.7) into equation (4.16) using definitions (4.17)–(4.18) immediately yields the following formulae for the split vector fields:

$$\begin{cases} \boldsymbol{E}^+(z, t+z/c) = \mathcal{P}^+(z,0)\left(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d)\right)^{-1} \\ \qquad\qquad \left(\mathcal{I} - \mathcal{R}_1\mathcal{R}_0\delta_{2\frac{d-z}{c}} * \mathcal{P}^+(d,z)\mathcal{P}^-(z,d)\right)\boldsymbol{E}^i_{\text{left}}(t), \\ \boldsymbol{E}^-(z, t+z/c) = \mathcal{R}_0\mathcal{P}^+(z,0)\left(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d)\right)^{-1} \\ \qquad\qquad \left(\mathcal{I} - \delta_{2\frac{d-z}{c}} * \mathcal{P}^+(d,z)\mathcal{P}^-(z,d)\right)\boldsymbol{E}^i_{\text{left}}(t). \end{cases} \tag{7.1}$$

The Green functions $\mathbf{G}^{\pm}_{\text{Green}}(z,t)$, defined as

$$\begin{cases} \boldsymbol{E}^{+}(z,t+z/c) = \mathbf{Q}^{+}(z,0)\left\{\boldsymbol{E}^{+}(0,t) + \left(\mathbf{G}^{+}_{\text{Green}}(z,\cdot) * \boldsymbol{E}^{+}(0,\cdot)\right)(t)\right\}, \\ \boldsymbol{E}^{-}(z,t+z/c) = \mathbf{Q}^{+}(z,0)\left(\mathbf{G}^{-}_{\text{Green}}(z,\cdot) * \boldsymbol{E}^{+}(0,\cdot)\right)(t), \end{cases}$$

are then easily identified. The result is

$$\begin{cases} \mathbf{Q}^{+}(z,0)\left(\mathbf{I} + \mathbf{G}^{+}_{\text{Green}}(z,\cdot)*\right) = \mathcal{P}^{+}(z,0)\left(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1\delta_{2\frac{d}{c}} * \mathcal{P}^{+}(d,0)\mathcal{P}^{-}(0,d)\right)^{-1} \\ \qquad\qquad \left(\mathcal{I} - \mathcal{R}_1\mathcal{R}_0\delta_{2\frac{d-z}{c}} * \mathcal{P}^{+}(d,z)\mathcal{P}^{-}(z,d)\right), \\ \mathbf{Q}^{+}(z,0)\mathbf{G}^{-}_{\text{Green}}(z,\cdot)* = \mathcal{R}_0\mathcal{P}^{+}(0,z)\left(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1\delta_{2\frac{d}{c}} * \mathcal{P}^{+}(d,0)\mathcal{P}^{-}(0,d)\right)^{-1} \\ \qquad\qquad \left(\mathcal{I} - \delta_{2\frac{d-z}{c}} * \mathcal{P}^{+}(d,z)\mathcal{P}^{-}(z,d)\right). \end{cases}$$

The explicit expression for the reflection imbedding kernel $\mathbf{R}(z,t)$ for the subsection (z,d) of the physical medium $(0,d)$ is readily obtained from equation (7.1). Since the imbedding kernels are defined as

$$\boldsymbol{E}^{-}(z,t) = \left(\mathbf{R}(z,\cdot) * \boldsymbol{E}^{+}(z,\cdot)\right)(t),$$

the result is

$$\begin{aligned} \mathbf{R}(z,\cdot)* = \mathcal{R}_0 &\left(\mathcal{I} - \mathcal{R}_1\mathcal{R}_0\delta_{2\frac{d-z}{c}} * \mathcal{P}^{+}(d,z)\mathcal{P}^{-}(z,d)\right)^{-1} \\ &\left(\mathcal{I} - \delta_{2\frac{d-z}{c}} * \mathcal{P}^{+}(d,z)\mathcal{P}^{-}(z,d)\right). \end{aligned} \tag{7.2}$$

Note that $\mathbf{R}(0,t)$ equals the physical reflection kernel $\mathbf{R}(t)$. The imbedding kernel $\mathbf{R}(z,t)$ can be represented as

$$\mathbf{R}(z,t) = \begin{pmatrix} R_{\text{co}}(z,t) & -R_{\text{cross}}(z,t) \\ R_{\text{cross}}(z,t) & R_{\text{co}}(z,t) \end{pmatrix}.$$

By equations (4.15) and (7.2), the susceptibility kernels $F(t)$ and $L(t)$ can be characterized as follows: $F(t) \equiv 0$ iff $R_{\text{co}}(z,t) \equiv 0$ for some $z \in (0,d)$ and $L(t) \equiv 0$ iff $R_{\text{cross}}(z,t) \equiv 0$ for some $z \in (0,d)$. The result for the non-reciprocity is well-known.

8 The semi-infinite case

If the slab $z \in (0,d)$ extends to infinity ($d \to +\infty$), the solution (6.7) to the propagation problem becomes very simple:

$$\begin{cases} \boldsymbol{E}^{+}_{NRCM}\left(z, t + \dfrac{z}{c}\right) = \mathcal{T}_0^{+}\mathcal{P}^{+}(z,0)\boldsymbol{E}^{i}_{\text{left}}(t) = (\mathcal{I} + \mathcal{R}_0)\mathcal{P}^{+}(z,0)\boldsymbol{E}^{i}_{\text{left}}(t), \\ \boldsymbol{E}^{-}_{NRCM}(z,t) = \mathbf{0}. \end{cases}$$

By equation (4.12), one obtains

$$\begin{cases} \boldsymbol{E}^{+}\left(z, t+\dfrac{z}{c}\right) = \mathcal{P}^{+}(z,0)\boldsymbol{E}^{i}_{\text{left}}(t), \\ \boldsymbol{E}^{-}\left(z, t+\dfrac{z}{c}\right) = \mathcal{R}_0\mathcal{P}^{+}(z,0)\boldsymbol{E}^{i}_{\text{left}}(t) \end{cases}$$

for the split vector fields. This equation shows explicitly that the reflection imbedding operator $\mathcal{R}_\infty = \mathcal{R}_0$ for the subsection (z,∞) of the physical medium $(0,\infty)$ is independent of the spatial variable z. This result is often proved in imbedding theory with invariance arguments. Restricted to the first roundtrip, $0 < t < 2\frac{d}{c}$, this result holds in the finite slab case $z \in (0,d)$ as well, see equation (6.7).

Finally, by the optical wave splitting (3.7), the electric and magnetic fields are

$$\begin{cases} \boldsymbol{E}\left(z, t+\dfrac{z}{c}\right) = (\mathcal{I}+\mathcal{R}_0)\mathcal{P}^{+}(z,0)\boldsymbol{E}^{i}_{\text{left}}(t), \\ \eta\boldsymbol{H}\left(z, t+\dfrac{z}{c}\right) = (\mathcal{I}-\mathcal{R}_0)\mathcal{P}^{+}(z,0)\mathbf{J}\boldsymbol{E}^{i}_{\text{left}}(t). \end{cases}$$

Appendix A The mismatch case

In Section 3, the optical intrinsic impedance η was required to be constant throughout space. As a consequence of this limitation, several interesting technical cases are excluded. Among these cases, the problem with metal (PEC)-backed dispersive slab particularly deserves to be mentioned. This problem is of importance in radar applications.

The purpose with this appendix is to extend the analysis to various optical impedance mismatch cases. This is done directly without employing so called Redheffer star products [17] or other related means for decomposition of the actual problem into easier ones [8, 19]. In this sense, the approach seems to be new.

The first step in the analysis of the general mismatch problem is to introduce optical reflection coefficients at the walls. Non-zero reflection coefficients generate (periodically repeated) non-classical contributions to the impulse responses of the medium. In the case with non-dispersive, homogeneous, and isotropic embedding of the bi-isotropic slab, the reflection coefficients at the front and rear walls are

$$r_0 = \frac{\eta(+0)-\eta(-0)}{\eta(+0)+\eta(-0)} \quad \text{and} \quad r_1 = \frac{\eta(d-0)-\eta(d+0)}{\eta(d-0)+\eta(d+0)}, \tag{A.1}$$

respectively. The intrinsic impedance of the medium to the left of the dispersive slab is here denoted by $\eta(-0)$, whereas the notation for the corresponding property of the medium to the right is $\eta(d+0)$. Moreover,

$$\eta = \eta(+0) = \eta(d-0), \tag{A.2}$$

since the slab is homogeneous, see the constitutive relations (3.2). All these numbers may be chosen arbitrarily (positive). Notice that the reflection coefficients (A.1) have been defined with respect to the surrounding non-dispersive media. The reflection coefficients viewed from the dispersive slab are $-r_0$ and $-r_1$, respectively. Similarly, the transmission coefficients for transition from the dispersive slab to the non-dispersive media are $1 - r_0$ and $1 - r_1$, whereas, considering the transition from non-dispersive media to the dispersive slab, they become $1 + r_0$ and $1 + r_1$, respectively. The reflection coefficient at the rear wall of the PEC-backed slab is $r_1 = 1$.

Secondly, the optical wave splitting (3.6) is introduced. This wave splitting is now interpreted local in space, that is, the optical intrinsic impedance is given by equation (A.2) inside the dispersive slab, and by $\eta(-0)$ and $\eta(d + 0)$ in the surrounding non-dispersive media. This implies that the split vector fields $\boldsymbol{E}^{\pm}(z, t)$ still equal the right-going and left-going electric fields outside the slab. In terms of the optical reflection coefficients, the boundary conditions for the split vector fields at the front and rear walls are

$$\begin{pmatrix} \boldsymbol{E}^{+}(0,t) \\ \boldsymbol{E}^{-}(0,t) \end{pmatrix} = \frac{1}{1-r_0} \begin{pmatrix} \mathbf{I} & -r_0\mathbf{I} \\ -r_0\mathbf{I} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^{i}_{\text{left}}(t) \\ \boldsymbol{E}^{r}(t) \end{pmatrix} \tag{A.3}$$

and

$$\begin{pmatrix} \boldsymbol{E}^{t}(t) \\ \boldsymbol{E}^{i}_{\text{right}}(t) \end{pmatrix} = \frac{1}{1+r_1} \begin{pmatrix} \mathbf{I} & r_1\mathbf{I} \\ r_1\mathbf{I} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^{+}(d,t) \\ \boldsymbol{E}^{-}(d,t) \end{pmatrix}, \tag{A.4}$$

respectively. These equations, which are reduced to equation (3.9) in the optically impedance matched case, are immediate consequences of the optical wave splitting (3.6) and the continuity in the space-variable of the electric and magnetic fields. Observe that the functions $\boldsymbol{E}^{\pm}(0, t)$ and $\boldsymbol{E}^{\pm}(d, t)$ are the limit values viewed from the dispersive slab. This notation is used throughout this appendix.

Due to the boundary conditions

$$\boldsymbol{E}(d,t) = \boldsymbol{E}^{+}(d,t) + \boldsymbol{E}^{-}(d,t) = \boldsymbol{E}^{t}(t) = \boldsymbol{E}^{i}_{\text{right}}(t) = \mathbf{0},$$

equation (A.4) holds for the PEC-backed slab. Hence, this problem does not require special consideration. The derivation of the general scattering relation is, however, simplified if equation (A.4) is invertible. Therefore, assume that $r_1 \neq 1$, and pass in the limit $r_1 \to 1$ to obtain the results for the PEC-backed slab.

The theory presented in the previous sections can now be applied to the split vector fields $\boldsymbol{E}^{\pm}(d, t)$. The solution to the general scattering problem is obtained by substituting the boundary conditions (A.3) and (A.4) into equation (6.2):

$$\begin{aligned} &\frac{1}{1-r_1} \begin{pmatrix} \mathcal{I} + r_1\mathcal{R}_1 & -(r_1\mathcal{I} + \mathcal{R}_1) \\ -(r_1\mathcal{I} + \mathcal{R}_0)\delta_{\frac{d}{c}} * \mathcal{P}^{-}(0,d) & (\mathcal{I} + r_1\mathcal{R}_0)\delta_{\frac{d}{c}} * \mathcal{P}^{-}(0,d) \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^{t} \\ \boldsymbol{E}^{i}_{\text{right}} \end{pmatrix} = \\ &= \frac{1}{1-r_0} \begin{pmatrix} (\mathcal{I} + r_0\mathcal{R}_1)\delta_{\frac{d}{c}} * \mathcal{P}^{+}(d,0) & -(r_0\mathcal{I} + \mathcal{R}_1)\delta_{\frac{d}{c}} * \mathcal{P}^{+}(d,0) \\ -(r_0\mathcal{I} + \mathcal{R}_0) & \mathcal{I} + r_0\mathcal{R}_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^{i}_{\text{left}} \\ \boldsymbol{E}^{r} \end{pmatrix}. \end{aligned} \tag{A.5}$$

The final result can be written in the form (6.4)–(6.5), where the single-interface scattering operators at the walls $z = 0$ and $z = d$ are given by

$$\begin{cases} \mathcal{R}_0^- = (\mathcal{I} + r_0\mathcal{R}_0)^{-1}(r_0\mathcal{I} + \mathcal{R}_0), \\ \mathcal{T}_0^+ = (1 + r_0)(\mathcal{I} + r_0\mathcal{R}_0)^{-1}(\mathcal{I} + \mathcal{R}_0), \\ \mathcal{R}_0^+ = -(\mathcal{I} + \mathcal{R}_1)^{-1}(\mathcal{I} + \mathcal{R}_0)(\mathcal{I} + r_0\mathcal{R}_0)^{-1}(r_0\mathcal{I} + \mathcal{R}_1), \\ \mathcal{T}_0^- = (1 - r_0)(\mathcal{I} + \mathcal{R}_1)^{-1}(\mathcal{I} + r_0\mathcal{R}_0)^{-1}(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1) \end{cases} \tag{A.6}$$

and

$$\begin{cases} \mathcal{R}_1^- = -(\mathcal{I} + \mathcal{R}_0)^{-1}(\mathcal{I} + \mathcal{R}_1)(\mathcal{I} + r_1\mathcal{R}_1)^{-1}(r_1\mathcal{I} + \mathcal{R}_0), \\ \mathcal{T}_1^+ = (1 - r_1)(\mathcal{I} + \mathcal{R}_0)^{-1}(\mathcal{I} + r_1\mathcal{R}_1)^{-1}(\mathcal{I} - \mathcal{R}_0\mathcal{R}_1), \\ \mathcal{R}_1^+ = (\mathcal{I} + r_1\mathcal{R}_1)^{-1}(r_1\mathcal{I} + \mathcal{R}_1), \\ \mathcal{T}_1^- = (1 + r_1)(\mathcal{I} + r_1\mathcal{R}_1)^{-1}(\mathcal{I} + \mathcal{R}_1), \end{cases} \tag{A.7}$$

respectively. These definitions are natural: consider the transformations (4.12)–(4.13) subject to the boundary conditions (A.3) and (A.4), respectively. Notice that the relations (4.19) hold and that

$$\begin{cases} \mathcal{R}_0^+ = \big(\mathcal{Z}^+ + \eta(-0)\mathcal{I}\big)^{-1}\big(\mathcal{Z}^+ - \eta(-0)\mathcal{I}\big), \\ \mathcal{R}_1^- = \big(\mathcal{Z}^- + \eta(d+0)\mathcal{I}\big)^{-1}\big(\mathcal{Z}^- - \eta(d+0)\mathcal{I}\big) \end{cases}$$

in the case with isotropic embedding.

The solution (6.4)–(6.5), (A.6)–(A.7) of the general scattering problem is now derived. An equivalent scattering relation is

$$\begin{pmatrix} \boldsymbol{E}^t \\ \boldsymbol{E}^r \end{pmatrix} = \mathcal{M}(0,d) \otimes \mathcal{C} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}} \\ \boldsymbol{E}^i_{\text{right}} \end{pmatrix}, \tag{A.8}$$

where the temporal operator $\mathcal{C}(t)$ is

$$\mathcal{C} = \begin{pmatrix} \mathcal{T}_0^+\mathcal{T}_1^+\delta_{\frac{d}{c}} * \mathcal{P}^+(d,0) & \mathcal{R}_1^+ - \mathcal{R}_0^+\mathcal{F}_1\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d) \\ \mathcal{R}_0^- - \mathcal{R}_1^-\mathcal{F}_0\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d) & \mathcal{T}_0^-\mathcal{T}_1^-\delta_{\frac{d}{c}} * \mathcal{P}^-(0,d) \end{pmatrix}$$

and

$$\begin{cases} \mathcal{F}_0 = \mathcal{R}_0^+\mathcal{R}_0^- - \mathcal{T}_0^+\mathcal{T}_0^-, \\ \mathcal{F}_1 = \mathcal{R}_1^+\mathcal{R}_1^- - \mathcal{T}_1^+\mathcal{T}_1^-. \end{cases}$$

By rearranging the terms of equation (A.5), one obtains

$$\begin{pmatrix} \frac{\mathcal{I}+r_1\mathcal{R}_1}{1-r_1} & \frac{r_0\mathcal{I}+\mathcal{R}_1}{1-r_0}\delta_{\frac{d}{c}} * \mathcal{P}^+(d,0) \\ \frac{r_1\mathcal{I}+\mathcal{R}_0}{1-r_1}\delta_{\frac{d}{c}} * \mathcal{P}^-(0,d) & \frac{\mathcal{I}+r_0\mathcal{R}_0}{1-r_0} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^t \\ \boldsymbol{E}^r \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\mathcal{I}+r_0\mathcal{R}_1}{1-r_0}\delta_{\frac{d}{c}} * \mathcal{P}^+(d,0) & \frac{r_1\mathcal{I}+\mathcal{R}_1}{1-r_1} \\ \frac{r_0\mathcal{I}+\mathcal{R}_0}{1-r_0} & \frac{\mathcal{I}+r_1\mathcal{R}_0}{1-r_1}\delta_{\frac{d}{c}} * \mathcal{P}^-(0,d) \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}} \\ \boldsymbol{E}^i_{\text{right}} \end{pmatrix}.$$

Straightforward algebraic computations using definitions (A.6)–(A.7) show that equation (A.8) holds, and the derivation is finished.

Attention is now paid to some special cases. In the reciprocal case, the reflection operators are

$$\begin{cases} \mathcal{R}_0^- = \big(\eta(1+Z*)+\eta(-0)\big)^{-1}\big(\eta(1+Z*)-\eta(-0)\big)\mathcal{I}, \\ \mathcal{R}_1^+ = \big(\eta(1+Z*)+\eta(d+0)\big)^{-1}\big(\eta(1+Z*)-\eta(d+0)\big)\mathcal{I}, \qquad \text{(isotropic backing)} \\ \mathcal{R}_1^- = -\mathcal{I} \qquad \text{(PEC-backing)}, \end{cases}$$

where η is the optical intrinsic impedance of the dispersive medium defined by equation (A.2). The integral operator $1+Z*$ is given by equation (4.8). Moreover, if the surrounding isotropic media are the same, i.e., if

$$\eta_{\text{out}} := \eta(-0) = \eta(d+0),$$

then $\mathcal{R}_0^+ = \mathcal{R}_1^- = R\mathcal{I}$, where

$$R = \big(\eta(1+Z*)+\eta_{\text{out}}\big)^{-1}\big(\eta(1+Z*)-\eta_{\text{out}}\big)$$

and equation (6.4) is reduced to

$$\begin{pmatrix} \boldsymbol{E}^t \\ \boldsymbol{E}^r \end{pmatrix} = \begin{pmatrix} \mathbf{0} & R\mathcal{I} \\ R\mathcal{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}} \\ \boldsymbol{E}^i_{\text{right}} \end{pmatrix} + (1-R^2)\left(\mathcal{I} - R^2\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d)\right)^{-1} \otimes$$
$$\otimes \begin{pmatrix} \delta_{\frac{d}{c}} * \mathcal{P}^+(d,0) & -R\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d) \\ -R\delta_{2\frac{d}{c}} * \mathcal{P}^+(d,0)\mathcal{P}^-(0,d) & \delta_{\frac{d}{c}} * \mathcal{P}^-(0,d) \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}} \\ \boldsymbol{E}^i_{\text{right}} \end{pmatrix}.$$

In the case $\boldsymbol{E}^i_{\text{right}} \equiv \mathbf{0}$, the expansion of the multiple-propagation operator $\mathcal{M}(0,d)$ with respect to its principal part leads to the representation

$$\boldsymbol{E}^r(t) = r_0\boldsymbol{E}^i_{\text{left}}(t) + (1-r_0)(1+r_0)\left(\mathbf{R}_{\text{ph}} * \boldsymbol{E}^i_{\text{left}}\right)(t) +$$
$$-(1-r_0)r_1(1+r_0)\sum_{k=1}^{\infty}(r_1r_0)^{k-1}\big(\mathbf{Q}^-(0,d)\mathbf{Q}^+(d,0)\big)^k\boldsymbol{E}^i_{\text{left}}(t-kt_r)$$

of the reflected electric field, where $t_r := 2\frac{d}{c}$ is one roundtrip. For the transmitted electric field, the analogous result is

$$\boldsymbol{E}^t(t+t_r/2) = (1-r_1)\mathbf{Q}^+(d,0)(1+r_0)\left(\mathbf{T}_{\text{ph}}(\cdot) * \boldsymbol{E}^i_{\text{left}}(\cdot)\right)(t) +$$
$$+(1-r_1)\mathbf{Q}^+(d,0)(1+r_0)\sum_{k=0}^{\infty}\left(r_0\mathbf{Q}^-(0,d)r_1\mathbf{Q}^+(d,0)\right)^k\boldsymbol{E}^i_{\text{left}}(t-kt_r).$$

Both these formulae are in agreement with the results in Ref. 19. In the inverse scattering problem, finite traces of the physical scattering kernels $\mathbf{R}_{\text{ph}}(t)$ and $\mathbf{T}_{\text{ph}}(t)$ are the input data. The matrix $\mathbf{Q}^+(d,0)$ and the coefficients r_0, r_1 can also be obtained from scattering data.

The dynamics of the internal electric fields can be written in the form (6.7), where the single-interface scattering operators are given by equations (A.6)–(A.7).

This is now proved. If the boundary conditions (A.3) and (A.4) are substituted into equation (6.6), one readily obtains the relation

$$\begin{pmatrix} \boldsymbol{E}^+_{NRCM}(z) \\ \boldsymbol{E}^-_{NRCM}(z) \end{pmatrix} =$$
$$= \frac{(\mathcal{I}-\mathcal{R}_0\mathcal{R}_1)^{-1}(\mathcal{I}+\mathcal{R}_0)}{1-r_0}\delta_{\frac{z}{c}} * \mathcal{P}^+(z,0) \otimes \begin{pmatrix} \mathcal{I}+r_0\mathcal{R}_1 & -(r_0+\mathcal{R}_1) \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^i_{\text{left}} \\ \boldsymbol{E}^r \end{pmatrix} +$$
$$+ \frac{(\mathcal{I}-\mathcal{R}_0\mathcal{R}_1)^{-1}(\mathcal{I}+\mathcal{R}_1)}{1-r_1}\delta_{\frac{d-z}{c}} * \mathcal{P}^-(z,d) \otimes \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -(r_1+\mathcal{R}_0) & \mathcal{I}+r_1\mathcal{R}_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{E}^t \\ \boldsymbol{E}^i_{\text{right}} \end{pmatrix}.$$

The scattering relation (A.8) then admits elimination of the scattered fields. The definitions (A.6) and (A.7) finally yield the desired result.

If the slab $z \in (0,d)$ extends to infinity ($d \to +\infty$), one readily obtains

$$\boldsymbol{E}^+_{NRCM}\left(z, t+\frac{z}{c}\right) = \mathcal{T}_0^+\mathcal{P}^+(z,0)\boldsymbol{E}^i_{\text{left}}(t), \qquad \boldsymbol{E}^-_{NRCM}(z,t) = \mathbf{0}.$$

By equations (4.9)–(4.11), the electric and the magnetic fields are

$$\begin{cases} \boldsymbol{E}\left(z, t+\frac{z}{c}\right) = \mathcal{T}_0^+\mathcal{P}^+(z,0)\boldsymbol{E}^i_{\text{left}}(t), \\ \eta\boldsymbol{H}\left(z, t+\frac{z}{c}\right) = (\mathcal{I}+\mathcal{R}_0)^{-1}(\mathcal{I}-\mathcal{R}_0)\mathcal{T}_0^+\mathcal{P}^+(z,0)\mathbf{J}\boldsymbol{E}^i_{\text{left}}(t). \end{cases}$$

Equation (4.12) yields the split vector fields

$$\begin{cases} \boldsymbol{E}^+\left(z, t+\frac{z}{c}\right) = (\mathcal{I}+\mathcal{R}_0)^{-1}\mathcal{T}_0^+\mathcal{P}^+(z,0)\boldsymbol{E}^i_{\text{left}}(t), \\ \boldsymbol{E}^-\left(z, t+\frac{z}{c}\right) = (\mathcal{I}+\mathcal{R}_0)^{-1}\mathcal{R}_0\mathcal{T}_0^+\mathcal{P}^+(z,0)\boldsymbol{E}^i_{\text{left}}(t). \end{cases}$$

Note that the imbedding operator still is $\mathcal{R}_0$:

$$\boldsymbol{E}^-(z,t) = \mathcal{R}_0\boldsymbol{E}^+(z,t).$$

Finally, observe that the theory presented in this appendix can be used to study wave propagation in a finite collection of homogeneous bi-isotropic slabs placed one after another.

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