# Analysis of Rate Limiters using Integral Quadratic Constraints

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# ANALYSIS OF RATE LIMITERS USING INTEGRAL QUADRATIC CONSTRAINTS


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## Abstract

It is shown how the framework of integral quadratic constraints can be used to analyse systems with rate limiters, in spite of the fact that such systems can not be globally exponentially stable.

The analysis is based on computation of the $L_2$-gain in a feedback loop involving a saturation followed by an integrator.


## 1. Introduction

Stability criteria based on Lyapunov functions, dissipativity and absolute stability have been developed over several decades. However, a new perspective on the theory has recently emerged with the development of new numerical methods. For linear time-invariant systems with uncertainty, efficient computational tools have been developed based on the notion structured singular value, [Packard and Doyle, 1993]. For nonlinear and time-varying systems, the search for a quadratic Lyapunov function can be written as a convex optimization problem with linear matrix inequality (LMI) constraints. Such problems can be solved with great efficiency using interior point methods.

A large variety of results of this kind were recently unified and generalized using the notion integral quadratic constraint (IQC) [Megretski and Rantzer, 1997]. The general computational problem to find multipliers that prove stability was stated as an LMI.

This paper is devoted to the treatment of a less trivial nonlinearity namely a rate limiter, where an integrator appears in combination with a saturation. The unbounded integrator is an obstacle for direct application of the stability theorem for IQC's, but the problem is resolved by "encapsulating" the nonlinearity in an artificial feedback loop.

## Notation

The notation $\mathbf{L}_{2e}^n$ is used for the linear space of all functions $f : (0,\infty) \to \mathbf{R}^n$ which are square integrable on any finite interval. The subspace consisting of square integrable functions is denoted $\mathbf{L}_2^n$ and the corresponding norm is denoted $\|f\|$.

The set of proper rational transfer matrices $G = G(s)$ of size $k$ by $m$ is denoted by $\mathbf{RL}_\infty^{k\times m}$. This is a subspace of $\mathbf{CL}_\infty^{k\times m}$, which consists of all matrix functions that are bounded and continuous on the imaginary axis. The subset of stable functions $G \in \mathbf{RL}_\infty^{k\times m}$ is denoted by $\mathbf{RH}_\infty^{k\times m}$. Each element $G \in \mathbf{RL}_\infty^{k\times m}$ is associated with a corresponding causal LTI operator $G : \mathbf{L}_{2e}^m \to \mathbf{L}_{2e}^m$, defined by

$$(Gf)(t) = Df(t) + \int_0^t Ce^{\tau A}Bf(t-\tau)d\tau$$

where $G(s) = C(sI - A)^{-1}B + D$. An element $G \in \mathbf{RL}_\infty^{k\times m}$ is called *strictly proper* if $D = 0$.

The word "operator" will be used to denote an input/output system. Mathematically, it simply means any function (possibly multi-valued) from one signal space $\mathbf{L}_{2e}^k$ into another: an *operator* $\Delta : \mathbf{L}_{2e}^l \to \mathbf{L}_{2e}^m$ is defined by a subset $\mathcal{S}_\Delta \subset \mathbf{L}_{2e}^l \times \mathbf{L}_{2e}^m$ such that for every $v \in \mathbf{L}_{2e}^l$ there exists $w \in \mathbf{L}_{2e}^m$ with $(v,w) \in \mathcal{S}_\Delta$. The notation $w = \Delta(v)$ means that $(v,w) \in \mathcal{S}_\Delta$.

An operator $\Delta$ is said to be *causal* if the set of past projections $P_T w$ of possible outputs $w = \Delta(v)$ corresponding to a particular input $v$ does not depend on the future $v - P_T v$ of the input, i.e. $P_T\Delta = P_T\Delta P_T$ for all $T \geq 0$. The operator $\Delta$ is *bounded* if there exists $C$ such that

$$\|P_T w\| \leq C\|P_T v\| \qquad \forall T > 0, w = \Delta(v),\ v \in \mathbf{L}_{2e}^l$$

The *gain* $\|\Delta\|$ of $\Delta$ is then defined as the infimium of all $C$ for which the inequality holds.

## 2. Stability via Integral Quadratic Constraints

An interconnection of two operators means a relation of the form

$$\begin{cases} v = G(w) + f \\ w = \Delta(v) + e \end{cases} \tag{1}$$

We say that the interconnection of the two operators $G : \mathbf{L}_{2e}^m \to \mathbf{L}_{2e}^l$ and $\Delta : \mathbf{L}_{2e}^l \to \mathbf{L}_{2e}^m$ is *well posed* if the set of all solutions to (1) defines a causal operator $[G, \Delta] : (f, e) \mapsto (v, w)$. The interconnection is called *stable* if in addition $[G, \Delta]$ is bounded.

For $\Pi \in \mathbf{CL}_\infty^{l+m \times l+m}$ define $\sigma_\Pi$ to be the quadratic form

$$\sigma_\Pi(v, w) = \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega$$

The operator $\Delta : \mathbf{L}_{2e}^l \to \mathbf{L}_{2e}^m$ is said to satisfy the *integral quadratic constraint* (IQC) defined by $\Pi$ if

$$\sigma_\Pi(h) \geq 0 \qquad \forall h = (v, \Delta(v)) \in \mathbf{L}_2^{l+m}$$

The following result was proved in [Megretski and Rantzer, 1997].

Proposition 1
Let $G(s) \in \mathbf{RH}_\infty^{l \times m}$ and let $\Delta : \mathbf{L}_{2e}^l \to \mathbf{L}_{2e}^m$ be bounded and causal. Assume that

(*i*) for every $\tau \in [0, 1]$, the interconnection of $G$ and $\tau\Delta$ is well-posed.

(*ii*) for every $\tau \in [0, 1]$, the inequality $\sigma_\Pi(v, \Delta(v)) \geq 0$ holds for all $v \in \mathbf{L}_2^l[0, \infty)$.

(*iii*) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I \quad \forall \omega \in \mathbf{R}$$

Then the feedback interconnection of $G$ and $\Delta$ is stable. □

## 3. Rate limiters

Many systems of practical interest involve a pure integrator controlled by a saturated actuator. Unfortunately, direct application of Proposition 1 is impossible in this situation. For example, consider feedback interconnection of the pure integrator $G(s) = -1/s$ and $\Delta(y) = \text{sat}(y)$. The interconnection is not stable in the $L^2$-sense, because the operator $e \to y$ in

$$\dot{y} = -\text{sat}(y) - e, \qquad y(0) = 0$$

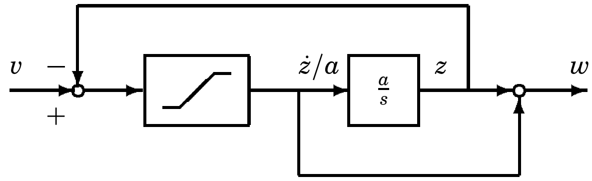


**Figure 1** Feedback "encapsulation" of saturation together with integrator

is not bounded. However, the system with $e \equiv 0$ is still asymptotically stable.

The system will now be analysed using a preliminary feedback loop. For this purpose, let the operator $w = \Gamma_{\text{sat}}^a(v)$ be defined by the relations

$$\begin{cases} \dot{z} = a\,\text{sat}(v - z) & z(0) = 0 \\ w = z + \text{sat}(v - z) \end{cases}$$

where $a > 0$. See Figure 1. Then $v, w \in \mathbf{L}_2$ if and only if $\xi = \text{sat}(y)$ with $\xi, \xi/s, y \in \mathbf{L}_2$ and

$$\begin{cases} y = v - \dfrac{a}{s+a} w \\ \xi = \dfrac{s}{s+a} w \end{cases} \qquad \begin{cases} w = \dfrac{s+a}{s} \xi \\ v = y + \dfrac{a}{s} \xi \end{cases}$$

The operator $\Gamma_{\text{sat}}^a$ is bounded and satisfies many useful IQC's:

Theorem 1
The operator $w = \Gamma_{\text{sat}}^a(v)$ is well-defined, causal and bounded. It satisfies the IQC's

$$0 \leq \int_{-\infty}^{\infty} \text{Re}\left[(\widehat{v} - \widehat{w}) H(i\omega) \frac{i\omega}{i\omega + a} \widehat{w}\right] d\omega \tag{2}$$

$$0 = \int_{-\infty}^{\infty} \text{Re}\left[\left(\frac{i\omega}{i\omega + a}\widehat{w}\right)^* \left(i\omega\widehat{v} - \frac{i\omega a}{i\omega + a}\widehat{w}\right)\right] d\omega \tag{3}$$

$$0 \leq \int_{-\infty}^{\infty} (2|\widehat{v}|^2 - |\widehat{w}|^2) d\omega \tag{4}$$

for $w, v, sv \in \mathbf{L}_2$ provided that $H$ has the form $H(i\omega) = h_0 + \sum_{k=1}^\infty h_k e^{-i\omega T_k} + \int_0^\infty h(t) e^{-i\omega t} dt$ with $h_0 \geq \sum_{k=1}^\infty |h_k| + \int_0^\infty |h(t)| dt$. □

**Remark 1** Convex combinations of the IQC's (2-4) can be used for stability verification in the usual way. However, it should be noted that quadratic forms in (2) and (3) need not be negative definite with respect to $w$, as required by Proposition 1. Hence, either attention should be restricted to convex combinations that satisfy this constraint, or the homotopy assumption of Theorem 2 needs to be addressed some other way. □

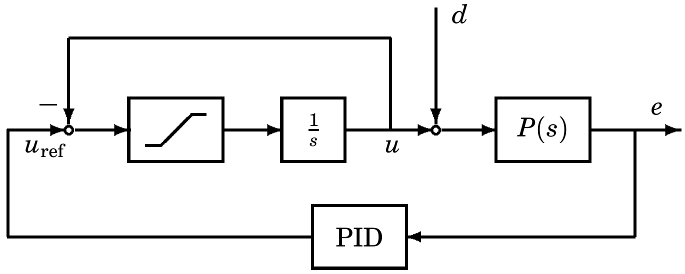


**Figure 2** PID control with rate limiter

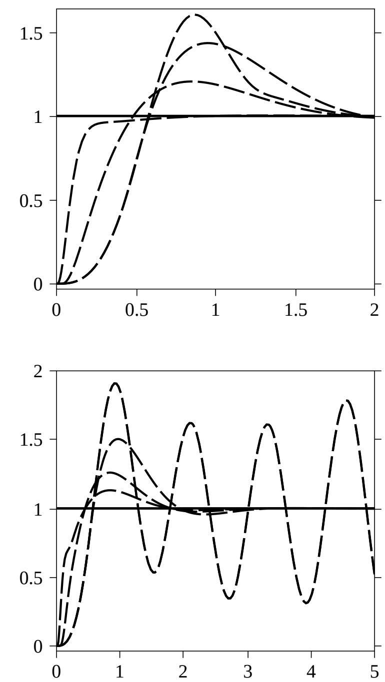

**Figure 3** Step responses with various rate limitations for $K = 40$ left and $K = 80$ right.

The outcome of Theorem 1 is that the stability theory based on integral quadratic constraints can be applied also in situations where saturations in combination with integrators excludes global exponential decay.

**Example 1** Rate limiters are common in aicraft applications. A very simple aircraft control loop can have the form

$$\begin{aligned} e &= P(u+d) \\ v &= Ce \\ \dot{u} &= \text{sat}(v-u) \end{aligned}$$

where $P$ is the plant, $C$ is the controller, $d$ is a disturbance, $v$ a reference value from the controller, while $u$ is the actual control signal with rate limitation $|\dot{u}| < 1$.

We will now use the previous results to prove that the control loop is stable for

$$\begin{aligned} P(s) &= \frac{1}{s^2+2s+11} \\ C(s) &= K\left(1+\frac{2.5}{s+0.01}+\frac{0.3s}{0.01s+1}\right) \end{aligned}$$

with $K = 40$ and compute an upper bound on the $\mathbf{L}_2$-induced gain from $d$ to $e$. Step responses with various saturation levels in the rate limiter are plotted in Figure 3, both for the stable case $K = 40$ and for the unstable case $K = 80$.

Note that $C(s)$ can be viewed as a PID controller, with leakage in the integrator and a time constant in the derivative parts. In presence of rate limitations, it is advisable to avoid instabilities by introducing an anti-windup scheme in the controller. However, for simplicity of presentation, we analyze the feedback system without anti-windup.

Define $w = u + \dot{u}$. Then

$$v = CP\frac{1}{s+1}w + CPd$$

The operator $w = \Gamma^1_{\text{sat}}(v)$ is bounded and satisfies the integral quadratic constraints (2-4). In particular, with $H(s) = 1 \pm (1 \pm s)^{-1}$, it satisfies (2). A convex combination of these IQC's proves stability and the gain bound $\|e\| \le 6.74\|d\|$. This has been found numerically using convex optimization in terms of linear matrix inequalities along the lines outlined in [Megretski and Rantzer, 1997]. □

## 4. Verification of well-posedness

There is an extensive literature on existence of solutions to differential equaitions and inclusions. A standard reference is the book by Filippov [Filippov, 1988]. Causality issues have been discussed in detail by Willems [Willems, 1971]. To make the presentation self-contained, we devote this section to the statement and proof of a criterion for well-posedness.

Two additional notions are needed. The operator $F$ is called *incremental* if for any $T > 0$ there exist $C_0, C_1, \tau > 0$, and $\theta < 1$ such that

$$\|P_{t+\tau}F(v)\| \le \theta\|P^t_{t+\tau}v\| + C_0 + C_1\|P_t v\| \qquad (5)$$

for all $t \in [0, T]$, $v \in \mathbf{L}_{2e}$.

We write $w_i \to^* w$ if $\sup\|w_i - w\| < \infty$ and $\langle g, w_i - w\rangle \to 0$ for every $g \in \mathbf{L}^n_2$. An operator $F$ is said to be *locally *-continuous* if for every $t > 0$ there exists $d > 0$ such that from every input-output sequence $w_i = F(v_i)$ with $P_{t-d}(w_i - w_0) = 0$,

$P_{t-d}(v_i - v_0) = 0$, and $P_{t+d}v_i \to^* P_{t+d}v$, one can extract a subsequence $w_{i(j)}$ such that $P_{t+d}w_{i(j)} \to^* w$ and $P_{t+d}w = P_{t+d}F(v)$.

Note that a composition of two incremental operators is incremental and a composition of two locally *-continuous operators is itself locally *-continuous.

THEOREM 2
Let $F : \mathbf{L}_{2e}^n \to \mathbf{L}_{2e}^n$ be a causal operator which is both locally *-continuous and incremental. Then the equation $w = F(w+v)$ has a solution for every $v \in \mathbf{L}_{2e}^n$, and the corresponding operator $v \mapsto w$ is causal and locally *-continuous.

Moreover, if $F$ is a composition of the form $\Delta \circ G$ or $G \circ \Delta$, where $G \in \mathbf{RH}_\infty$ is strictly proper and $\Delta$ is affinely bounded, then both $F$ and the operator $v \mapsto w$ are incremental. □

Theorem 2 is a general result which helps to establish well-posedness of various interconnections. The essential part of the proof is covered by the following result.

LEMMA 1
Let $F$ be a causal operator which is incremental and *-continuous. Then the equation $w = F(w)$ has a solution $w$ and the inequality

$$\|P_t w\| \le \frac{C_0}{C_1}\left(1 + \frac{C_1}{1-\theta}\right)^{1+\frac{t}{\tau}} \qquad (6)$$

where $\tau, C_0, C_1, \theta$ are the constants from (5), holds for every $w = F(w)$. □

*Proof.* We start by proving the inequality (6). If $w = F(w)$, then (5) with $t = (k-1)\tau$, $k = 1, 2, ...$ yields

$$\|P_{k\tau}^{(k-1)\tau}w\| \le \|P_{k\tau}F(w)\| \le \theta\|P_{k\tau}^{(k-1)\tau}w\| + C_0 + C_1\|P_{(k-1)\tau}w\|$$

In other words,

$$\mu_k \le a + b\sum_{l=1}^{k-1}\mu_l, \qquad (7)$$

where $\mu_k = \|P_{k\tau}^{(k-1)\tau}w\|$, $a = C_0/(1-\theta)$, and $b = C_1/(1-\theta)$. It is easy to check that the recursive inequality (7) yields

$$\mu_k \le a(b+1)^{k-1} \qquad \sum_{l=1}^{k}\mu_l \le \frac{a}{b}(b+1)^k,$$

which in turn implies (6).

To prove existence of a solution of $w = F(w)$, let $D_n$ for $n = 1, 2, ...$ be the operator of delay by $1/n$:

$$(D_n w)(t) = \begin{cases} w(t - 1/n) & t > 1/n \\ 0 & \text{otherwise} \end{cases}$$

Then the equation $w = D_n F(w)$, thanks to the presence of the delay, has a solution $w = w_n$ for any $n$. This solution is defined recursively, first on the interval $t \in (0, 1/n)$, then on the interval $t \in (1/n, 2/n)$, etc. Since (5) is satisfied for $F$, it will also be satisfied with the same constants for $F$ replaced by $D_n F$, because

$$\|P_{t+1/n}D_n v\| \le \|P_{t+1/n}v\| \qquad \forall\ t, v$$

Hence, the inequality (6) shows that $\sup_n \|P_T w_n\| < \infty$ for every $T > 0$ and therefore there exists a weakly convergent subsequence $P_T w_{n(i)} \to^* P_T w$ of $P_T w_n$.

Let the interval $[0, T]$ be covered by a finite number of intervals $(r_k, s_k) = (t_k - d(t_k), t_k + d(t_k))$, where $d(t) > 0$ is the number from the definition of local *-continuity. For $k = 1$ it follows from the *-continuity of $F$ that there exists a weakly convergent subsequence $P_{s_1}v_{n(i)} \to^* P_{s_1}v = P_{s_1}F(w)$, where $v_n$ are defined by $v_n = F(w_n)$. By (5), $\sup_n \|P_T v_n\| < \infty$ follows from the corresponding inequality for $w_n$. Hence, for every $0 < a < b < s_1$

$$\begin{aligned}\int_a^b (v - w)dt &= \lim_{n\to\infty}\int_a^b (v_n - w_n)dt \\ &= \lim_{n\to\infty}\int_a^b (v_n - D_n v_n)dt \\ &= \lim_{n\to\infty}\left(\int_a^b v_n dt - \int_{a-1/n}^{b-1/n} v_n dt\right) = 0\end{aligned}$$

and it follows that $P_{s_1}w = P_{s_1}v = P_{s_1}F(w)$.

The same argument can now be used repeatedly with $F$ replaced by

$$F_k(u) = P_{s_k}F\left(P_{r_k}w + P^{r_k}u\right) \qquad k = 2, 3, \ldots$$

to solve $P_{s_k}w = P_{s_k}F(w)$. This gives $P_T w = P_T F(w)$ and the prodecure can be repeated indefinitely in order to solve $w = F(w)$ over the whole real line. □

*Proof of Theorem 2.* Existence follows from Lemma 1 with $F$ replaced by $F_0(w) = F(w+v)$. In the same way, causality follows with $F$ replaced by $F_t(u) = F(P_t w + v + P^t u)$. The local *-continuity follows directly from the local *-continuity of $F$.

Let $G(s) = \int_0^\infty e^{-st}g(t)dt$ and $G_\tau(s) = \int_0^\tau e^{-st}g(t)dt$. That the compositions $\Delta \circ G$ and $G \circ \Delta$ are incremental then follows from the inequalities

$$\begin{aligned}
&\|P_{t+\tau}\Delta(Gw)\| \\
&\quad\le \|\Delta(P_{t+\tau}(Gw))\| \\
&\quad\le c_0 + c_1\|P_{t+\tau}Gw\| \\
&\quad= c_0 + c_1 \left\|P_{t+\tau}\left((P_\tau g) * P^t_{t+\tau}w + g * P_t w\right)\right\| \\
&\quad\le c_0 + c_1\|G_\tau\|_\infty \cdot \|P^t_{t+\tau}w\| + c_1\|G\|_\infty \cdot \|P_t w\|
\end{aligned}$$

$$\begin{aligned}
&\|P_{t+\tau}G\Delta(w)\| \\
&\quad= \|P_{t+\tau}\left((P_\tau g) * P^t_{t+\tau}\Delta(w) + g * P_t\Delta(w)\right)\| \\
&\quad\le \|G_\tau\|_\infty \cdot \|\Delta(P^t_{t+\tau}w)\| + \|G\|_\infty \cdot \|\Delta(P_t w)\| \\
&\quad\le \|G_\tau\|_\infty(c_0 + c_1\|P^t_{t+\tau}w\|) + \|G\|_\infty(c_0 + c_1\|P_t w\|)
\end{aligned}$$

where $\theta = c_1\|G_\tau\|_\infty < 1$ when $\tau$ is sufficiently small. To see that the corresponding operator $v \mapsto w$ is incremental, rewrite the incrementality inequality for $F$ as

$$\begin{aligned}
\|P_{t+\tau}w\| &= \|P_{t+\tau}F(w+v)\| \\
&\le \theta\|P^t_{t+\tau}w\| + \theta\|P^t_{t+\tau}v\| + C_0 + C_1\|P_t(w+v)\| \\
\|P_{t+\tau}w\| &\le \frac{1}{1-\theta}\left(\theta\|P^t_{t+\tau}v\| + C_0 + C_1\|P_t w\| + C_1\|P_t v\|\right)
\end{aligned}$$

Here $\theta/(1-\theta) < 1$ when $\tau$ is selected sufficiently small and the term $C_1\|P_t w\|$ can be removed by applying the inequality recursively over the squence of intervals $[0,\tau]$, $[\tau, 2\tau]$, $[2\tau, 3\tau]$, ... □

## 5. Proofs

*Proof of Theorem 1.* Theorem 2 shows that the operator $\Gamma^a_{\text{sat}}$ is well-defined and causal. The parameter $a$ only defines the time scale and does not affect the gain, so consider the case $a = 1$ without loss of generality. Let $V(z) \ge 0$ be the Lyapunov function defined by

$$\frac{dV}{dz}(z) = \begin{cases} 4z & \text{for } |z| \le 1 \\ (1+|z|)^2 z/|z| & \text{for } |z| \ge 1 \end{cases} \qquad V(0) = 0$$

First, we will verify that

$$-\frac{dV}{dz}(z)\,\text{sat}(v-z) + 2|v|^2 - |z + \text{sat}(v-z)|^2 \ge 0 \tag{8}$$

Given a fixed $z$, consider the minimum of the left hand side in (8). There are two possibilities. Either the saturation occurs at the optimum. Then all terms except $|v|^2$ are locally independent of $v$, the minimum must be at $v = 0$ and the minimal value is nonnegative. The other possibility is that saturation does not occur. Then the left hand side is quadratic in $v$ and the minimum zero is attained at $v = z - 1$ if $z < -1$, at $v = 2z$ if $|z| \le 1$ and at $v = z + 1$ if $|z| > 1$.

Integrating (8) over the time interval $[0, T]$ gives

$$\int_0^T (2|v|^2 - |z + \text{sat}(v-z)|^2)dt \ge V(z(T)) \ge 0$$

This proves that $\|\Delta^a_{\text{sat}}\| \le \sqrt{2}$.

The opposite inequality $\|\Gamma^a_{\text{sat}}\| \ge \sqrt{2}$ follows by considering the inputs

$$v(t) = \begin{cases} 1+t & \text{for } 0 \le t \le T \\ 0 & \text{for } t > T \end{cases}$$

where $T \to \infty$. □

## 6. Conclusions

It has been shown how the framework of integral quadratic constraints can be used to analyse systems with rate limiters, using a gain bound computed for a preliminary feedback loop.

We believe that that this approach can be extended to a large number of other cases: Once a gain bound has been obtained for a given component or feedback loop, this gain bound can be used in a general computational framework of integral quadratic constraints.

## 7. Acknowledgements

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## 8. References

Filippov, A. F. (1988): *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publishers.

Megretski, A. and A. Rantzer (1997): “System analysis via Integral Quadratic Constraints.” *IEEE Transactions on Automatic Control*, **47:6**, pp. 819–830.

Packard, A. and J. Doyle (1993): “The complex structured singular value.” *Automatica*, **29:1**, pp. 71–109.

Willems, J. (1971): *The Analysis of Feedback Systems*. MIT Press.

Yakubovich, V. (1982): “On an abstract theory of absolute stability of nonlinear systems.” *Vestnik Leningrad Univ. Math.*, **10**, pp. 341–361. Russian original published in 1977.