Robustness of a Design Method Based on Assignment of Poles and Zeros

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Abstract—**The sensitivity of a pole-zero placement design with respect to variations in the process model is discussed. Inequalities which guarantee stability and precision in the assigned poles are given.**

I. Introduction

It is an empirical fact that complex processes can often be controlled well by surprisingly simple regulators. When regulators are designed using analytical methods the complexity of the regulator is often determined by the complexity of the model. A high-order model will give a high-order regulator and vice versa. To obtain a simple regulator it is therefore important to base the design on a simplified model. It is then of interest to investigate the sensitivity of the closed-loop system to variations in the model used for the design. A problem of this type is formulated and solved in this paper.

The sensitivity of a closed-loop system with respect to perturbations in the open-loop transfer function is a classical problem [1] which recently has received new interest [2], [3]. The problem studied in this paper could be considered from this viewpoint. By making more assumptions, namely, that the closed-loop system is derived from a particular design technique, stronger results can, however, be obtained. The notations are given in Section II. A simple design method based on pole-zero assignment is used. This method is briefly described in Section III. The main results are given in Section IV. It is believed that results of the type given in this paper are useful for understanding design methods and their practical applications. The results given can be extended in many directions. Other design methods could be investigated [4]. Assumptions can be relaxed and multivariable systems could be considered.

II. Preliminaries

It is assumed that the systems considered are linear time-invariant and that they have one input and one output. The input–output characteristics of such systems can be described by analytic transfer functions. Both continuous-time and discrete-time systems are considered.

The *stability region* is a subset of the complex plane. For continuous-time systems it is the left half-plane excluding the imaginary axis. For discrete-time systems the stability region is the interior of the unit circle. A system is stable if all the poles of its transfer function are inside the stability region. The *instability region* is the complement of the stability region. From a practical point of view it is useful to introduce a restricted stability region $\mathfrak{T}$ which is strictly inside the stability region. For a continuous-time system the region $\mathfrak{T}$ could, e.g., be characterized by

$$3\pi/4 < \arg s < 5\pi/4$$
$$\operatorname{Re} s < -s_0 < 0.$$

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The image of this region under the transformation $z=\exp(sT)$ could be the restricted stability region for discrete-time systems.

The critical curve Γ is the imaginary axis for continuous-time systems and the unit circle for discrete-time systems.

III. Pole and Zero Placement

A simple servo design method is chosen as a typical representative for analytical design techniques. The design is based on classical pole-zero assignment as described in [5], [6], and [7]. A brief description together with some useful interpretations are given below.

The problem of designing a servo with a given closed-loop transfer function will now be described and solved.

Formulation

Consider a process characterized by the rational transfer function

$$G=\frac{B}{A} \tag{3.1}$$

where A and B are polynomials. It is assumed that A and B are coprime and that

$$\deg B<\deg A.$$

It is desired to find a controller such that the closed loop is stable and that the transfer function from the command input u_c to the output is given by

$$G_M=\frac{Q}{P} \tag{3.2}$$

where P and Q are coprime and

$$\deg P-\deg Q>\deg A-\deg B. \tag{3.3}$$

Design Procedure

A general linear regulator can be described by

$$Ru=Tu_c-Sy. \tag{3.4}$$

The regulator can be thought of as a combination of a feedback having the transfer function

$$G_{FB}=\frac{S}{R} \tag{3.5}$$

and a feedforward with the transfer function

$$G_{FF}=\frac{T}{R}. \tag{3.6}$$

The closed-loop transfer function relating y to u_c is given by

$$\frac{TB}{AR+BS}.$$

Since this transfer function should equal the desired closed-loop transfer function G_M given by (3.2) we get

$$\frac{TB}{AR+BS}=\frac{Q}{P}. \tag{3.7}$$

The design problem is thus equivalent to the algebraic problem of finding polynomials R, S, and T such that (3.7) holds. It follows from (3.7) that factors of B which are not also factors of Q must be factors of R. Since factors of B correspond to open-loop zeros, it means that open-loop zeros which are not desired closed-loop zeros must be canceled. Factor B as

$$B=B^+B^- \tag{3.8}$$

where all the zeros of B^+ are in the restricted stability region $\mathfrak{T}$ and all zeros of B^- outside $\mathfrak{T}$. This means that all zeros of B^+ correspond to well damped modes and all zeros of B^- correspond to unstable or poorly damped modes.

A necessary condition for solvability of the servo problem is thus that the specifications are such that

$$Q=Q_1B^-. \tag{3.9}$$

Since $\deg P$ is normally less than $\deg(AR+BS)$ it is clear that there are factors in (3.7) which cancel. In state space theory it can be shown that the regulator (3.4) corresponds to a combination of an observer and a state feedback (see [8]). It is natural to assume that the observer is designed in such a way that changes in command signals do not generate errors in the observer. This means that the factor which cancels in the right-hand side of (3.7) can be interpreted as the observer polynomial T_1.

The design procedure can be formulated as follows.

Data: Given the desired response specified by the polynomials P and Q, subject to $\deg P=\deg A$ and the conditions (3.3), (3.9), and the desired observer polynomial T_1. It is assumed that P and T_1 have all their zeros in $\mathfrak{T}$.

Step 1: Solve the equation

$$AR_1+B^-S=PT_1 \tag{3.10}$$

with respect to R_1 and S.

Step 2: The regulator which gives the desired closed-loop response is the given by (3.4) with

$$R=R_1B^+ \tag{3.11}$$

and

$$T=T_1Q_1. \tag{3.12}$$

Equation (3.10) can always be solved because it was assumed that A and B were coprime. This implies, of course, that A and B^- are also coprime. Equation (3.10) has infinitely many solutions. All solutions will give the specified closed-loop transfer function. The solutions will, however, differ with respect to disturbance rejection properties (see [8]).

Analysis

A direct calculation gives

$$\frac{TB}{AR+BS}=\frac{T_1Q_1B^+B^-}{B^+(AR_1+B^-S)}=\frac{T_1Q_1B^-}{PT_1}=\frac{Q}{P}$$

which shows that the regulator gives the desired closed-loop response. Notice that in this calculation we have divided with the factors B^+ and T_1. This is permitted since it was assumed that all their zeros are well damped.

A direct calculation shows that the closed-loop system has the characteristic polynomial B^+T_1P. The polynomial B^+ has all its zeros in the restricted stability region $\mathfrak{T}$ by definition. Since the observer polynomial T_1 and the polynomial P were chosen to have all their zeros in $\mathfrak{T}$ it follows that the closed-loop system has all its poles in $\mathfrak{T}$.

Interpretation as Model Following

The regulator (3.4) can be interpreted as a model following servo. It follows from (3.10), (3.11), and (3.12) that

$$\frac{T}{R}=\frac{T_1Q_1}{B^+R_1}=\frac{(AR_1+B^-S)Q_1}{PB^+R_1}=\frac{AQ_1}{B^+P}+\frac{SB^-Q_1}{B^+R_1P}=\frac{A}{B}\cdot\frac{Q}{P}+\frac{S}{R}\cdot\frac{Q}{P}.$$

The feedback law (3.4) can thus be written as

$$u=\frac{A}{B}y_c+\frac{S}{R}(y_c-y) \tag{3.14}$$

where

$$y_c=\frac{Q}{P}u_c.$$

The signal y_c can be interpreted as the output obtained when the command signal u_c is applied to the model Q/P. When the regulator (3.4) is rewritten as (3.14) it is clear that it can be thought of as composed of two parts, one feedforward term $(A/B)y_c=$

$(A/B)(Q/P)u_c$ and one feedback term $(S/R)(y_c-y)$. The feedforward is a combination of the ideal model and an inverse of the process model. The feedback term is obtained by feeding the error y_c-y through a dynamical system characterized by the operator S/R. It is thus clear that the regulator can be interpreted as a model following servo. Notice, however, that the system A/B is not realizable although the combination $AQ/(BP)$ is.

IV. Main Results

It follows from the discussion in Section III that an $(n-1)$th order regulator may be obtained when the design method is applied to an nth order system. Equation (3.10) will be poorly conditioned if the system has poles and zeros which are close together. It is an empirical fact that this often happens for models of high order. To use the analytical design method it is therefore necessary to base it on a low-order model. It is therefore of interest to investigate what happens when the design method is applied to a simplified model

$$G=\frac{B}{A}, \tag{4.1}$$

of a process with the transfer function G_0.

The stability of the closed-loop system will first be discussed. The sensitivity of the closed-loop poles are then considered.

Stability

A sufficient condition for stability is given by the following theorem.

Theorem 1: Consider the regulator (3.4) obtained by applying the pole-zero assignment design to the stable model $G=B/A$ with the specification that the closed-loop transfer function should be $G_M=Q/P$. Let the regulator control a stable system with the transfer function G_0. The closed-loop system is then stable if

$$|G-G_0|<\left|\frac{BPT}{AQS}\right|=\left|\frac{G}{G_M}\right|\cdot\left|\frac{G_{FF}}{G_{FB}}\right| \tag{4.2}$$

on the critical curve Γ and at $z=\infty$.

Proof: Consider the function

$$F=R+G_0S. \tag{4.3}$$

This function is regular outside the stability region because the system G_0 was assumed stable. The zeros of the function F are equal to the closed-loop poles. Solving (3.10) for R and insertion into (4.3) gives

$$F=PB^+T_1/A-BS/A+G_0S=PB^+T_1/A+S(G_0-G).$$

When $G=G_0$ the zeros of F are thus equal to the zeros of the polynomials B^+, P, and T_1. Since both the system and the model were assumed to be stable the functions PB^+T_1/A and $S(G_0-G)$ are both regular outside the stability region. The functions are thus regular on a contour which encloses the instability region. Notice that

$$\frac{BPT}{AQS}=\frac{B^+B^-PT_1Q_1}{AQ_1B^-S}=\frac{B^+PT_1}{AS}.$$

Condition (4.2) thus implies that

$$|S(G_0-G)|<|PB^+T_1/A|$$

on the critical curve and at infinity.

It now follows from Rouchè's theorem [9, p. 119] that the functions F and PB^+T_1/A have the same number of zeros in the instability region. It follows from the design procedure that the polynomial PB^+T_1 has all its zeros in the restricted stability region. The closed-loop system is thus stable. The equality in (4.2) follows from

$$\frac{BPT}{AQS}=\frac{BPTR}{AQRS}=\frac{GG_{FF}}{G_MG_{FB}}$$

where (3.5) and (3.6) have been used. □

Theorem 1 gives good insight into the sensitivity of the design to modeling errors. When a model has been obtained and a regulator has been designed, the right-hand side of (4.2) can be determined. The polynomials T, S and the transfer function G_{FF} and G_{FB} are then known. Bounds on the transfer function G which will give a stable closed-loop system are then given by (4.2). Notice that the bound is proportional to $|G_{FF}/G_{FB}|$. The requirements on model precision are thus less for designs which lead to a high ratio of feedforward to feedback as can be expected. For single-degree-of-freedom systems [10] $G_{FF}=G_{FB}$ and the bound is simplified further. Also notice that the bound is proportional to $|G/G_M|$. For a normal servo the desired transfer function G_M is unity for low frequencies. It then remains constant up to frequencies corresponding to the specified bandwidth where it starts to decrease. From the point of view of stability it is thus advantageous to have a high process gain. The ratio $|G/G_M|$ is normally large for low frequencies because the low-frequency gain of the process is typically larger than the desired low-frequency gain. Reasonable specifications are also often such that $|G/G_M|$ is constant for high frequencies. Since G_0 and G normally are small for high frequencies, this means that the inequality (4.2) can be satisfied even if G and G_0 deviates substantially at high frequencies. Normally, it is only in a fairly narrow frequency range around the bandwidth where (4.2) gives critical requirements on the model accuracy. This agrees well with empirical facts and explains qualitatively why simple models can be useful for pole-placement design. Notice also that it follows from (4.2) that the requirements on model precision will be reduced by reducing the bandwidth of desired closed-loop system.

Sensitivity of Closed-Loop Poles to Model Errors

So far the discussion has been focused on the stability problem. Having established that a model is sufficiently accurate to guarantee stability it is, of course, of interest to analyze the problem further and to investigate the requirements on model precision which are necessary to have the dominating poles close to their specified values.

In the proof of Theorem 1 it was shown that the closed-loop poles are the zeros of the function F defined by (4.3), i.e.,

$$F=PB^+T_1/A+S(G_0-G)=H+S(G_0-G).$$

When $G=G_0$ the system has thus poles at the zeros, p_i, of P, B_1, and T_1. Consider F as a function of z and G. A Taylor series expansion at $z=p_i$ and $G=G_0$ gives

$$F(z)\approx H(p_i)+H'(p_i)(z-p_i)-S(p_i)[G(p_i)-G_0(p_i)].$$

An approximative formula for the change of the pole p_i due to a modeling error is thus

$$z_i=p_i-[H'(p_i)]^{-1}S(p_i)[G_0(p_i)-G(p_i)].$$

If it is required that a pole p_i change by at most $\alpha|p_i|$ due to a modeling error the following inequality is obtained:

$$|G_0(p_i)-G(p_i)|\leq\alpha|H'(p_i)||S^{-1}(p_i)|\cdot|p_i|. \tag{4.4}$$

A requirement that certain dominant poles do not change too much will thus lead to a requirement that the values of the model pulse transfer function is close to the process pulse transfer function on the contour Γ and at the poles of interest. Such a requirement can of course also be satisfied by a fairly simple model provided that the number of dominant poles is not too large.

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