The servo problem for piecewise linear systems

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Published in:
Proceedings of the Fifteenth International Symposium on Mathematical Theory of Networks and Systems (MTNS)

2002

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Abstract

The servo problem for a wide class of nonlinear system is considered. A quantitative bound on system trajectories is derived. For piecewise linear systems the bound is shown to be computable in terms of linear matrix inequalities.

1 Introduction

Behavior of trajectories for piecewise linear systems, in presence of an input signal, is an important issue from a control theoretic point of view. Most analysis results on piecewise linear systems are oriented toward stability of the origin for the unforced system [1],[2],[3].. The convergence of trajectories of the *unforced piecewise linear system* as defined in [3] is not sufficient in general, to guarantee good behavior when input signals are applied to the system. Even if the unforced system is proved to be *stable*, applying an input might change the equilibrium points of the local linear system in such a way that the systems behavior becomes unsatisfactory.

The servo problem for a general nonlinear systems can be analysed in a framework presented in Figure 1. The problem is to obtain information about the differences between the system's trajectories (x) and a predetermined trajectory x_r in presence of an input signal r. The exogenous input considered in this framework will be the time derivative of r. Choosing $\mathcal{L}_2$ norm as measure for the signals, it is natural a choice of the $\mathcal{L}_2$ gain to characterize the systems behavior. Thus by computing the $\mathcal{L}_2$ gain from the input signal's derivative ($\dot{r}$) to the "distance" between system trajectories (x) and reference trajectories (x_r), one obtains information relating the convergence of the studied system's trajectories. In the literature on nonlinear systems, exists a qualitative result [4], [5], which roughly speaking states the following: if an autonomous nonlinear system depending on some parameter, is stable for different fixed values of this parameter, then slow variations of the parameter between these fixed values, results in a non-autonomous system that will stay in the neighborhood of the equilibria defined by the fixed parameters. Our contribution is to give a quantitative bound on the neighborhood of the equilibria when the variation of the parameter is a continuous function. In particular, for piecewise linear systems a computation method using convex optimization is proposed.

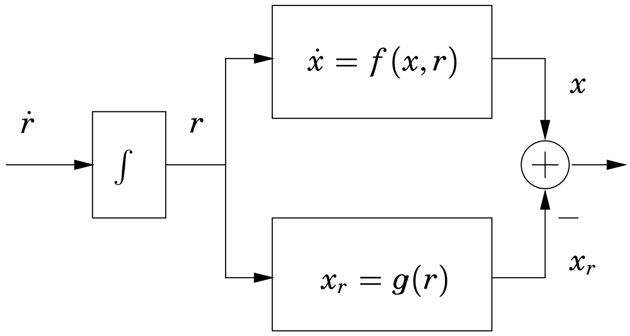

Figure 1: Computable bounds on the map from $\dot{r}$ to $|x - x_r|$ are derived in the paper

The layout of the paper is as follows : the second section presents the related problem for a linear system while the third section generalizes the problem for a nonlinear system. Section 4 treats the case of piecewise linear systems. Some conclusions are presented in Section 5.

2 The linear case

The computation of $\mathcal{L}_2$ gain for a linear system is a well known result. It makes use of the Riccati inequality, resulting in a convex optimization problem [6]. In Theorem 2.1, basically, this result is used for the computation of $\mathcal{L}_2$ gain between the reference signal's derivative and the difference $x - x_r$.

Theorem 2.1. *Consider the linear system:*

$$\dot{x} = Ax + Br, \;\; x(0) = 0 \tag{1}$$

such that A^{-1} exists. Furthermore, define

$$x_r \triangleq -A^{-1}Br \tag{2}$$

then the following statements are equivalent:

i) There exist $\gamma > 0, P > 0$ such that

$$\begin{bmatrix} A^TP + PA + I & PA^{-1}B \\ (A^{-1}B)^TP & -\gamma^2 I \end{bmatrix} < 0. \tag{3}$$

ii) For each solution of (1) with $r \in C^1$ and $r(0) = 0$ the following inequality holds

$$\int_0^\infty |x - x_r|^2 dt \leq \gamma^2 \int_0^\infty |\dot{r}|^2 dt \tag{4}$$

Proof. Define $\tilde{x} \triangleq x - x_r$. Then $\dot{\tilde{x}} = A\tilde{x} + A^{-1}B\dot{r}$ For this system standard results can be applied (see Corollary 12.3 [6]).

3 The generic nonlinear case

In case of a general nonlinear system with a time varying input, is more difficult to draw conclusions about trajectory convergence. Still, it is possible to find an upper bound on the $\mathcal{L}_2$ gain from the input signal's derivative to the "distance" from the system's state to a defined trajectory.

Theorem 3.1. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz. For every $r \in \mathcal{R} \subset \mathbb{R}^m$ let $x_r \in \mathbb{R}^n$ be a unique solution to $0 = f(x_r, r)$.*

If there exists $\gamma > 0$ and a non-negative C^1 function V, with $V(x_r, r) = 0$ and

$$\begin{bmatrix} \frac{\partial V}{\partial x} f(x,r) + |x - x_r|^2 & \frac{1}{2}\frac{\partial V}{\partial r} \\ \frac{1}{2}\left(\frac{\partial V}{\partial r}\right)^T & -\gamma^2 I \end{bmatrix} < 0 \tag{5}$$

for all $(x, r) \in \mathcal{S}$, then for each solution to

$$\dot{x} = f(x,r), \;\; x(0) = x_{r_0}, \; r(0) = r_0 \tag{6}$$

such that $r(t) \in \mathcal{R}$ and $(x(t), r(t)) \in \mathcal{S}$ for all t, it holds that

$$\int_0^T |x - x_r|^2 dt \le \gamma^2 \int_0^T |\dot{r}|^2 dt \tag{7}$$

Proof. Multiplying (5) from left and right with $\begin{bmatrix} \mathbf{1} & \dot{r}^T \end{bmatrix}$ one obtains:

$$\frac{\partial V}{\partial x} f(x,r) + |x - x_r|^2 + \frac{\partial V}{\partial r}\dot{r} - \gamma^2 |\dot{r}|^2 < 0$$

that is

$$\frac{dV}{dt} + |x - x_r|^2 - \gamma^2 |\dot{r}|^2 < 0$$

which in turns by integration on $[0,\ T]$ gives

$$V(x(T), r(T)) + \int_0^T |x - x_r|^2 \mathrm{dt} - \gamma^2 \int_0^T |\dot{r}|^2 \mathrm{dt} < 0$$

and inequality (7) results since $V(x,r) \ge 0$.

Remark 3.1. *Consider a linear system as in (1) with x_r defined by (2). Furthermore, consider a Lyapunov function of the form $V(x,r) = (x - x_r)^T P (x - x_r)$. Then*

$$\frac{\partial V}{\partial x} f(x,r) \overset{(2)}{=} (x - x_r)^T (A^T P + PA)(x - x_r)$$

$$\frac{\partial V}{\partial r} = 2(x - x_r)^T P A^{-1} B$$

and the matrix in (5) becomes:

$$\begin{bmatrix} x - x_r & 0 \\ 0 & \mathbf{1} \end{bmatrix}^T \begin{bmatrix} A^T P + PA + I & PA^{-1}B \\ (A^{-1}B)^T P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x - x_r & 0 \\ 0 & \mathbf{1} \end{bmatrix}$$

ones negative definiteness is given by (3).

Remark 3.2. *The matrix inequality (5) using Schur complement can be written as*

$$\frac{\partial V}{\partial x} f(x,r) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial r} \left(\frac{\partial V}{\partial r}\right)^T \\ + \frac{1}{2}(x - x_r)^T (x - x_r) \leq 0$$

which is the Hamilton-Jacobi inequality for the system

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} f(x,r) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \; x - x_r \end{cases} \tag{8}$$

(see Theorem 6.5 in [4]).

Similarly to Theorem 3.1, an upper bound on the instantaneous value of $|x - x_r|$ can be obtained. The following result is analogous to Theorem 3.1.

Theorem 3.2. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz. For every $r \in \mathcal{R} \subset \mathbb{R}^m$, let $x_r \in \mathbb{R}^n$ be a unique solution to $0 = f(x_r, r)$.*
If there exist $\gamma, c, p > 0$ and a C^1 function V with $V(x,r) \geq c|x - x_r|^p$ and

$$\begin{bmatrix} \frac{\partial V}{\partial x} f(x,r) + \lambda V & \frac{1}{2}\frac{\partial V}{\partial r} \\ \frac{1}{2}\left(\frac{\partial V}{\partial r}\right)^T & -\gamma^2 I \end{bmatrix} < 0 \tag{9}$$

for all $(x,r) \in \mathcal{S}$, then for each solution to

$$\dot{x} = f(x,r), \;\; x(0) = x_{r_0}, \; r(0) = r_0 \tag{10}$$

such that $r(t) \in \mathcal{R}$ and $(x(t), r(t)) \in \mathcal{S}$, it holds that

$$|x(T) - x_r(T)|^p \leq \frac{\gamma^2}{c} \int_0^T |\dot{r}|^2 e^{-\lambda(T-t)} dt \tag{11}$$

Proof. Multiplying (5) from left and right with $\begin{bmatrix} \mathbf{1} & \dot{r}^T \end{bmatrix}$ one obtains:

$$\frac{\partial V}{\partial x} f(x,r) + \frac{\partial V}{\partial r} \dot{r} + \lambda V - \gamma^2 |\dot{r}|^2 < 0$$

thus on $\mathcal{S}$ yields:

$$\frac{dV}{dt} + \lambda V - \gamma^2 |\dot{r}|^2 < 0$$

which by multiplication with $e^{-\lambda(T-t)} > 0$ gives

$$\frac{dV}{dt} e^{-\lambda(T-t)} + \lambda V e^{-\lambda(T-t)} - \gamma^2 |\dot{r}|^2 e^{-\lambda(T-t)} < 0 \\ \Leftrightarrow \frac{d}{dt} V e^{-\lambda(T-t)} - \gamma^2 |\dot{r}|^2 e^{-\lambda(T-t)} < 0$$

then by integrating on $[0, T]$ and using that $V(x(0), r(0)) = 0$, results

$$c^2|x(T) - x_r(T)|^p \leq V(x(T), r(T)) <$$
$$< \gamma^2 \int_0^T |\dot{r}|^2 e^{-\lambda(T-t)} \mathrm{dt}$$

thus inequality (11) holds.

Remark 3.3. *Consider a linear system as in (1) with x_r defined by (2) and $\mathcal{S} = \mathbb{R}^n \times \mathbb{R}^m$. Furthermore, consider a Lyapunov function of the form $V(x, r) = (x - x_r)^T P (x - x_r)$ and $p = 2$. Then the Lyapunov function's positivity condition in Theorem 3.2 translates to:*

$$P - c^2 I > 0 \Leftrightarrow \begin{bmatrix} P & I \\ I & \frac{1}{c^2} I \end{bmatrix} > 0$$

while (9) becomes:

$$\begin{bmatrix} A^T P + PA + \lambda P & PA^{-1}B \\ (A^{-1}B)^T P & -\gamma^2 I \end{bmatrix} < 0$$

Obviously, for a generic nonlinear system as considered in (6) it might be difficult to find a $V(x, r)$ such that (5) or (9) is fulfiled. In case of piecewise linear systems convex optimization can be used in the analysis.

4 Piecewise linear system

Consider now a particular kind of nonlinear systems, a piecewise linear system, of the form:

$$\dot{x} = A_i x + B_i r, \quad x(t) \in X_i \tag{12}$$

with $\{X_i\}_{i \in I} \subseteq \mathbb{R}^n$ a partition of the state space into a number of convex polyhedral cells with disjoint interior. Suppose that for any constat $r \in \mathcal{R}$ the piecewise linear system has a unique equilibrium point.

Furthermore, consider symmetric matrices S_{ij} that satisfy the inequality:

$$\begin{bmatrix} x - x_r \\ r \end{bmatrix}^T S_{ij} \begin{bmatrix} x - x_r \\ r \end{bmatrix} > 0, \ x \in X_i, \ r \in \mathcal{R}_j \tag{13}$$

Define

$$\overline{B}_j \triangleq \begin{bmatrix} A_j^{-1} B_j \\ 1 \end{bmatrix}, \qquad \overline{I} \triangleq \begin{bmatrix} I_n & 0 \\ 0 & 0_m \end{bmatrix} \tag{14}$$

$$\overline{A}_{ij} \triangleq \begin{bmatrix} A_i & -A_i A_j^{-1} B_j + B_i \\ 0 & 0 \end{bmatrix} \tag{15}$$

The following proposition is useful for application of Theorem 3.1 and Theorem 3.2.

Proposition 4.1. *Let $f(x,r) = A_i x + B_i r$, $x_r = -A_j^{-1} B_j r$ with $x(0) = x_r(0)$, $r(0) = r_0$. If there exist $\gamma > 0$, $P > 0$ such that $\bar{P} = diag\{P, 0\}$ satisfies*

$$\begin{bmatrix} \overline{A}_{ij}^T \overline{P} + \overline{PA}_{ij} + S_{ij} + \overline{I} & \overline{PB}_j \\ \overline{B}_j^T \overline{P} & -\gamma^2 I \end{bmatrix} < 0,\ i \neq j \tag{16}$$

$$\begin{bmatrix} A_j^T P + PA_j + I & PA_j^{-1} B_j \\ (A_j^{-1} B_j)^T P & -\gamma^2 I \end{bmatrix} < 0 \tag{17}$$

then $V(x,r) = (x - x_r)^T P (x - x_r)$ satisfies (5) for all $x \in X_i$, $r(t) \in \mathcal{R}_j$.

Remark 4.1. *In particular, in the case when $\dot{r}(t) = 0$, for $t > T$, by finding a finite $\gamma > 0$ it is shown that all trajectories of the nonlinear system (12) will converge to x_r.*

Remark 4.2. *When the local linear systems contain affine terms the argument vector of the Lyapunov function will be extended to $\begin{bmatrix} \tilde{x} & r & 1 \end{bmatrix}$. Similarly, when partitions that do not contain the origin are to be described, the argument vector will be augmented.*

The conservatism of the theorems can be reduced by considering piecewise quadratic Lyapunov function. In this case the Lyapunov function will be piecewise C^1 instead of C^1. Imposing that is non-increasing at the points of discontinuity, the results yield (see [3]).

Remark 4.3. *The variation in the affine term due to r, can be viewed as parametric uncertainty in the system. Thus the theorem can be used to prove robust stability for a piecewise linear system, with uncertain affine terms in the local linear systems.*

Example 4.1. *Consider the system of the form:*

$$\dot{x} = Ax + B(r - \varphi(Cx))$$

where A is Hurwitz. The nonlinearity is defined as:

$$\varphi(x) = \begin{cases} x, & x < 1 \\ 1, & x \geq 1 \end{cases}$$

This system can be described by the following piecewise linear system.

$$\dot{x} = \begin{cases} Ax - B + Br, & Cx \geq 1 \\ (A - BC)x + Br, & Cx < 1 \end{cases} \tag{18}$$

The state space partitions of such a system (where the subsystems are of second order and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$) is shown in Figure 2. Here $X_1 = \{x | Cx \geq 1\}$ and $X_2 = \{x | Cx < 1\}$.

The numerical values are:

$$A = \begin{bmatrix} -0.5 & 1 \\ -1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

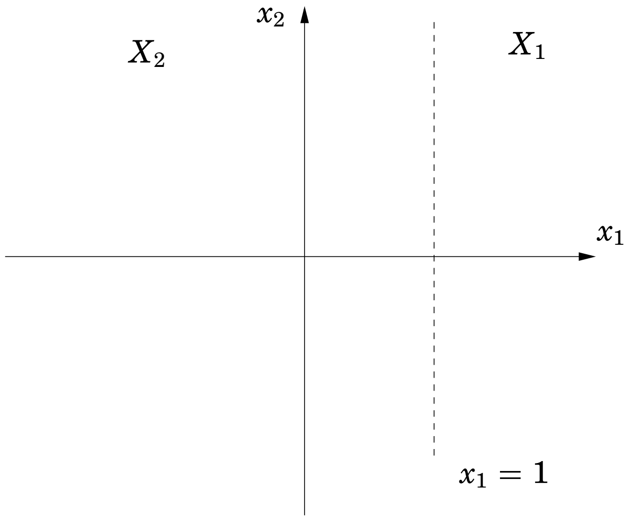

Figure 2: State space partitions of the system in Example 4.1

Then the sets $\mathcal{R}_1 = (\frac{4}{3}, \infty)$ and $\mathcal{R}_2 = (-\infty, \frac{4}{3})$ follow from simple computations. Consider first the equilibria $x_r \in X_2$, that is $r(t) \in \mathcal{R}_2$ for all t.
The LMI's resulting from Theorem 3.1 turn out to be infeasible, suggesting that a quadratic Lyapunov function might be to conservative. Therefore a piecewise quadratic Lyapunov function is tried (see [3]) :

$$V(x,r) = \begin{cases} \begin{bmatrix} x - x_r \\ r \\ 1 \end{bmatrix}^T P_1 \begin{bmatrix} x - x_r \\ r \\ 1 \end{bmatrix}, \ x \in X_1 \\ (x - x_r)^T P_2 (x - x_r), \ x \in X_2 \end{cases}$$

Minimizing γ subject to the LMI constraints, one obtains the Lyapunov function's matrices:

$$P_1 = \begin{bmatrix} 5.0749 & -0.8930 & -6.6918 & 8.9351 \\ -0.8930 & 5.1082 & 0.0703 & 0.2583 \\ -6.6918 & 0.0703 & -12.1141 & 16.2238 \\ 8.9351 & 0.2583 & 16.2238 & -2.2493 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 20.69 & -0.63 \\ -0.63 & 5.1 \end{bmatrix}$$

and $\gamma = 7.182$.

Consider now the equilibria $x_r \in X_1$, that is $r(t) \in \mathcal{R}_1$ for all t.

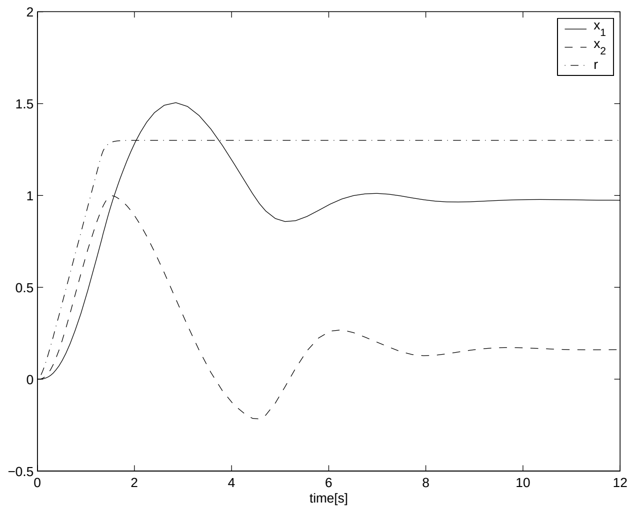

Figure 3: Simulation results for the system in Example 4.1

Consider the Lyapunov function:

$$V(x,r) = \begin{cases} (x-x_r)^T P_1 (x-x_r), \ x \in X_1 \\ \begin{bmatrix} x-x_r \\ r \\ 1 \end{bmatrix}^T P_2 \begin{bmatrix} x-x_r \\ r \\ 1 \end{bmatrix}, \ x \in X_2 \end{cases}$$

Solving the constrained minimization problem, one obtains the Lyapunov function's matrices:

$$P_2 = \begin{bmatrix} 18.36 & -2.55 & 47.17 & -68.64 \\ -2.55 & 4.22 & -13.32 & 8.21 \\ 47.17 & -13.32 & 146.8 & -173.34 \\ -68.64 & 8.21 & -173.34 & 317.36 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 3.874 & -0.503 \\ -0.503 & 4.225 \end{bmatrix}$$

and $\gamma = 8.3221$.

Thus for every (x,r) *starting in* $X_1 \times \mathcal{R}_1$ *respectively in* $X_2 \times \mathcal{R}_2$*, trajectory convergence, in the sense of Theorem 3.1, is guaranteed by the finite* γ*'s.*

In Figure 3 state trajectories x_1 *and* x_2 *are presented when* $r(t) \in \mathcal{R}_2$*. Notice that* x_1 *is passing through region* X_1*.*

As seen above, S-procedure is used (S_i) *to describe the state-space partition of (12), and in the same time describe the set of considered r's. More details on how to find* *such matrices can be found in [3]. The used matrices are: for* X_1

$$S_{12} = \begin{bmatrix} 0 & 0 & -8.562 & 11.772 \\ 0 & 0 & 0 & 0 \\ -8.562 & 0 & -12.844 & 16.259 \\ 11.772 & 0 & 16.259 & -20.528 \end{bmatrix}$$

and for X_2

$$S_{21} = \begin{bmatrix} 0 & 0 & -52.66 & -34.957 \\ 0 & 0 & 0 & 0 \\ -52.66 & 0 & -315.965 & 100.217 \\ -34.957 & 0 & 100.217 & -336.966 \end{bmatrix}$$

5 Conclusions

Trajectory convergence in presence of constant and time varying inputs has been studied. Quantitative result has been established for a sufficient condition regarding trajectory convergence for a class of nonlinear systems, where one of the parameters (r) is time varying. This result has been used for piecewise linear systems, where Proposition 4.1 in combination with Theorem 3.1, give a tool for computing an upper bound on the $\mathcal{L}_2$ gain from $\dot{r}$ to $x - x_r$, characterizing the servo problem for such systems.

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