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Sootla, Aivar

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LUND UNIVERSITY

PO Box 117  
221 00 Lund  
+46 46-222 00 00



# Nu-gap Model Reduction in the Frequency Domain

Aivar Sootla

**Abstract**—In this paper a model reduction algorithm in the nu-gap metric is considered. The metric was originally developed to evaluate robustness of a controller for a given plant. In fact, the nu-gap metric induces the weakest topology in which stability is a robust property. All in all the nu-gap metric is perhaps the best metric to evaluate the distance between two systems in a closed loop setup. In the field of distributed control, if approximation of the subsystems is considered, such a metric can be vital for modeling purposes. The presented algorithm of model reduction in the nu-gap metric is based on semidefinite programming methods and exploits the frequency domain representation of the systems. Therefore it may be easily extended to incorporate into the optimization procedure constraints on a specific frequency region of a particular interest or the closed loop performance.

## I. INTRODUCTION

Model reduction of linear systems in  $\mathcal{H}$  spaces is well studied in the literature and reasonable suboptimal methods have been derived. The methods delivering the best approximation quality are the, so called, Singular Value Decomposition (SVD) methods, as balanced truncation ([1]) and optimal Hankel model reduction ([2]). The methods also guarantee stability, however, computationally very expensive. Krylov methods ([3], [4], [5]), on the other hand, are considerably cheaper, however, can not guarantee stability, in general. Krylov/SVD ([6]) is a trade-off framework, which is cheaper than SVD, and can guarantee stability. Another trade-off framework was developed in [7], [8], [9] it is based on frequency response matching and the semidefinite optimization techniques.

All of the above mentioned methods measure the error in  $\mathcal{H}_\infty$  or  $\mathcal{H}_2$  spaces, meaning that it is a measure on the distance in the open loop setup. In the closed loop setup these norms usually do not reflect the distance adequately. The first attempt to introduce a more reliable metric in the closed loop setup was the introduction of the gap metric in [10], followed by many papers including [11] and [12]. In the latter the  $\nu$ -gap metric was introduced and it is the only metric for which “... any plant at a distance less than  $\beta$  from the nominal will be stabilized by any compensator stabilizing the nominal with a stability margin  $\beta$ . Furthermore, any plant at a distance greater than  $\beta$  from the nominal will be destabilized by some compensator that stabilizes the nominal with a stability margin of at least  $\beta$ ” ([13]). The stability margin here is  $b_{P,C}$  and defined therein. Moreover, the  $\nu$ -gap induces the weakest topology in which stability is a robust property. To some extent we can evaluate the stability of a closed loop without considering the stabilized plant. Therefore the  $\nu$ -gap metric

may be a crucial tool in distributed system modeling, where the evaluation of the entire system can be computationally overwhelming.

Early work in  $\nu$ -gap model reduction includes [14] and [15], which use the state-space representation of the systems and an iterative LMI program. The algorithm presented in this paper uses semidefinite programming as a tool and coprime fractions of the original system. As opposed to [15], the frequency domain data is used to obtain a reduced model. The algorithm was also extended to account for the overall performance in controller-plant loops, which is presented in Section IV. Although the algorithm is derived for the scalar valued transfer functions, with extra restrictions it may be extended to the matrix valued transfer function using the techniques described in [9], [16].

*Notation.*  $\mathcal{R}^{m_1 \times m_2}$  denotes the space of discrete-time  $m_1$  by  $m_2$  matrix valued rational transfer functions. Operation  $\sim$  denotes a complex conjugate on the unit circle i.e.,  $G^\sim(e^{j\omega}) = G^T(e^{-j\omega})$ , where  $j$  is a complex identity.  $G(\omega)$  stands for the frequency response of  $G(e^{j\omega})$  to  $\omega \in [0, \pi]$ . The infinity norm is computed as  $\|G\|_\infty = \sup_\omega |G(\omega)|$ , where  $G(\omega)$  is a scalar-valued function. The Hankel norm of a transfer function is denoted as  $\|\cdot\|_H$  (for the definition see, [17]). Function  $\eta(G)$  stands for the number of poles of  $G$  outside the unit circle and finally

$$[G, K] = \begin{pmatrix} G \\ I \end{pmatrix} (I - KG)^{-1} \begin{pmatrix} -K & I \end{pmatrix}.$$

## II. PRELIMINARIES

Firstly, it may be useful to illustrate why and in which situations the  $\nu$ -gap metric is employable. Consider a toy example borrowed from [18, pp. 349-350].

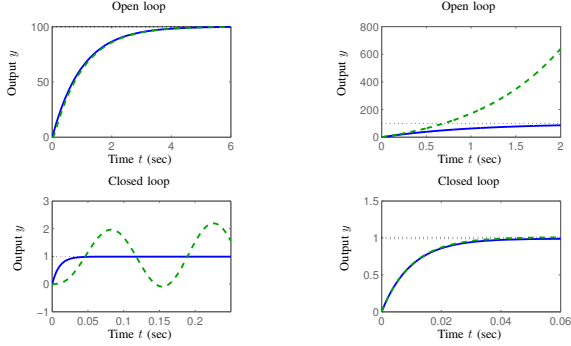
*Example 1: Distance between systems.* Given the systems

$$P_1 = \frac{100}{s+1} \quad P_2 = \frac{100}{(s+1)(0.0025s+1)^2} \quad P_3 = \frac{100}{s-1}$$

investigate the step responses in open loop and closed loop (simple negative feedback) settings in Fig. 1. The plants  $P_1$  and  $P_2$  are close in the  $\mathcal{H}_\infty$  norm, on the other hand plants  $P_1$  and  $P_3$  are close in the  $\nu$ -gap metric. Notice, that  $P_2$  is unstable in the closed loop setup, however stable in the open loop.  $P_3$  on the other hand is stable in the closed loop setting, but unstable in the open loop and  $P_1$  is stable in both settings. Fig. 1(a) shows that the open loop step responses of  $P_1$  and  $P_2$  are almost identical, however the closed loop ones are not. Fig. 1(b) shows the opposite situation:  $P_1$  is close to  $P_2$  in the closed loop, which is open loop unstable. Therefore the  $\nu$ -gap showed itself as a better measure on the

A. Sootla is with Automatic Control, LTH, Lund University, Ole Römers väg 1, SE 223 63, Lund, Sweden aivar@control.lth.se

distance than the  $\mathcal{H}_\infty$  norm in this particular closed loop setup.



(a) Plants  $P_1$  (solid blue) and  $P_2$  (b) Plants  $P_1$  (solid blue) and  $P_3$  (dashed green) (dashed green)

Fig. 1. Determining when the distance between two systems is small. The figures in (a) show that the open step responses of the plants  $P_1$  and  $P_2$  are close, but the closed loop responses are not. The figures in (b) show the opposite situation: the plants  $P_1$  and  $P_3$  are close in closed loop but different different in open loop.

There exist a few equivalent definitions of the metric. The one chosen in this paper is more convenient for our goal. Denote  $b, a$  a normalized left coprime factorization (NRCF) of  $G_1 = a^{-1}b$ , and  $p, q$  a right coprime factorization (RCF), not necessarily normalized, of  $G_2 = pq^{-1}$ .

*Definition 1 ( $\nu$ -gap metric):* Define a function  $\delta_\nu(\cdot, \cdot) : \mathcal{R}^{m_1 \times m_2} \times \mathcal{R}^{m_1 \times m_2} \rightarrow \mathbb{R}$  as follows

$$\delta_\nu(G_1, G_2) = \begin{cases} \delta_{\mathcal{L}_2}(G_1, G_2) & \text{if } \eta([G_2, -G_1^*]) = \\ & = \eta([G_1, -G_1^*]) \\ 1 & \text{otherwise} \end{cases}$$

where

$$\delta_{\mathcal{L}_2}(G_1, G_2) = \sqrt{1 - \left\| \begin{pmatrix} p \\ q \end{pmatrix} (a^*q + b^*p)^{-1} \right\|_\infty^{-2}}$$

The constraint  $\eta([G_2, -G_1^*]) = \eta([G_1, -G_1^*])$  is usually called a winding number condition in the literature. Note, that throughout the paper it is assumed that  $G_1$  and  $G_2$  are scalar, not necessarily stable, transfer function. Therefore  $a, b, p$  and  $q$  are scalar transfer functions as well.

Finally, we are ready to formulate the problem as an optimization one. Given asymptotically stable  $a$  and  $b$ , such that  $b^*b + a^*a = I$  and  $b/a$  not necessarily stable, solve

$$\gamma_{opt} = \min_{\gamma > 0, p, q} \gamma \text{ s. t.: } \left\| \begin{pmatrix} p \\ q \end{pmatrix} (a^*q + b^*p)^{-1} \right\|_\infty \leq \gamma$$

$$\eta\left(\left[\frac{p}{q}, -\left(\frac{b}{a}\right)^*\right]\right) = \eta\left(\left[\frac{b}{a}, -\left(\frac{b}{a}\right)^*\right]\right)$$

Notice, that  $\delta_\nu(b/a, p/q) \leq \sqrt{1 - 1/\gamma_{opt}^2}$  by construction. However, the obtained program is not generally convex even for the scalar valued function due to the winding number condition and the computation of  $\mathcal{H}_\infty$  norm. Therefore, a convexification is required. To authors best knowledge, using

the described formulation a meaningful convex relaxation is impossible. Therefore the two-step techniques proposed in [19] (developed in [8]) will be employed to address the problem. First, a so called ‘‘central’’ transfer function will be computed and then around it the solution will be constructed using semidefinite programming.

### III. MODEL REDUCTION IN THE $\nu$ -GAP METRIC

#### A. Main Result

As the main result the algorithm of approximation in the  $\nu$ -gap metric is presented in this subsection. The details and theoretical justification follow in the sequel.

Consider a scalar valued discrete-time transfer function  $G$  with a normalized coprime factorization (NCF)  $\begin{pmatrix} b & a \end{pmatrix}$ , where  $b$  and  $a$  are stable transfer functions of the same order  $l$  as the system  $G$ . Note,  $G$  is not necessarily a stable system. Compute its  $k$ -th order approximation  $\hat{G}$  in the  $\nu$ -gap metric as:

- 1) Compute a  $k$ -th order approximation  $\begin{pmatrix} n/\theta & m/\theta \end{pmatrix}$  of a transfer matrix  $\begin{pmatrix} b & a \end{pmatrix}$  using any stability preserving model reduction method, where  $m, n$  and  $\theta$  are finite impulse response (FIR) filters of order  $k$
- 2) Fix a ‘‘central’’ transfer function  $\phi = ma^* + nb^*$  and solve the following semidefinite program:

$$\gamma_\nu^N = \min_{p, q} \gamma \quad \text{subject to}$$

$$\forall \omega \in [0, \pi] : \text{Re}((q(\omega)a^*(\omega) + p(\omega)b^*(\omega))/\phi(\omega)) > 0$$

$$\forall i = 1, \dots, N : \omega_i \in [0, \pi], \left| \frac{p(\omega_i)}{q(\omega_i)} / \phi(\omega_i) \right| <$$

$$\gamma \text{Re}((q(\omega_i)a^*(\omega_i) + p(\omega_i)b^*(\omega_i))/\phi(\omega_i))$$

$$p(\omega) = \sum_{i=0}^k p_i e^{-ij\omega} \quad q(\omega) = \sum_{i=0}^k q_i e^{-ij\omega} \quad (1)$$

- 3) The reduced order plant  $\hat{G}$  is computed as  $p/q$ .
- 4) If required, compute an NCF  $m/\theta, n/\theta$  of  $\hat{G}$  and repeat steps 2 – 4.

Denote  $\gamma_\nu^\infty$  the solution to the relaxed problem in the step 2 of the Algorithm 1, in the case when  $N = \infty$  and  $\{\omega_i\}_{i=1}^\infty$  are dense in  $[0, \pi]$ , and  $\gamma_\nu^c$ , if all the constraints are enforced for all the frequencies  $\omega$ . It can be shown that  $\lim_{N \rightarrow \infty} \gamma_\nu^N = \gamma_\nu^\infty$ , moreover, since  $\{\omega_i\}$  is dense in  $[0, \pi]$ ,  $\gamma_\nu^c$  is equal to  $\gamma_\nu^\infty$ . Therefore with a big enough  $N$  it is possible to approximate  $\gamma_\nu^c$  with a good accuracy.

*Theorem 1:* Consider the Algorithm 1 with a full sampling, i.e. the constraints are enforced for all the frequencies  $\omega$ , where  $\gamma_\nu^c, p$ , and  $q$  is the output of the algorithm. Then  $\delta_\nu(G, p/q) \leq \sqrt{1 - (1/\gamma_\nu^c)^2}$ .

## B. Theoretical justification and details

In Section II a minimization program was constructed to approximate a system in the  $\nu$ -gap metric:

$$\gamma_{opt} = \min_{\gamma > 0, p, q} \gamma \quad (2)$$

$$\left\| \begin{pmatrix} p \\ q \end{pmatrix} (a^*q + b^*p)^{-1} \right\|_{\infty} \leq \gamma \quad (3)$$

$$\eta \left( \begin{bmatrix} p \\ q \\ -\frac{b^*}{a^*} \end{bmatrix} \right) = \eta \left( \begin{bmatrix} b \\ a \\ -\frac{b^*}{a^*} \end{bmatrix} \right) \quad (4)$$

Both constraints (3,4) are not convex. The technique applied to the program was introduced in [8]. Firstly, consider another program with only (3) as a constraint, rewriting (3) in the process as an infinite number of constraints:

$$\gamma_1 = \min_{\gamma > 0, p, q} \gamma$$

$$\forall \omega \in [0, \pi] : \left| \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} \right| < \gamma |q(\omega)a^*(\omega) + p(\omega)b^*(\omega)|$$

As in [8] introduce a new variable  $\phi$  into the program as:

$$\gamma_2 = \min_{\gamma > 0, p, q, \phi} \gamma$$

$$\forall \omega \in [0, \pi] : \left| \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} / \phi(\omega) \right| <$$

$$\gamma \text{Re}((q(\omega)a^*(\omega) + p(\omega)b^*(\omega)) / \phi(\omega))$$

The programs are equivalent, which is understood as equality  $\gamma_1 = \gamma_2$ . It can be shown that an optimal choice of  $\phi(\omega)$  is  $q(\omega)a^*(\omega) + p(\omega)b^*(\omega)$ . Basically,  $\phi(\omega)$  is an initial guess on  $q(\omega)a^*(\omega) + p(\omega)b^*(\omega)$ . Introducing an extra variable  $\phi$  does not solve all the problems. However, by computing  $\phi$  in advance and fixing it in the minimization the program becomes quasi-convex.

Surely, there is a question of conservatism of the positive real condition. Is the described set big enough to provide any improvement at all? This question was studied in [19] and [20], besides the mentioned work [8]. The results for low orders as 2 and 3 are colorfully illustrated in [19]. Indeed, given two polynomials  $\theta$  and  $\xi$  this condition describes all positive real transfer functions with a fixed polynomial  $\theta$  and the set of all possible  $\xi$  is shown in numerical examples to be sufficiently big comparing to the set of all stable  $\xi$ . However, no theoretical results were provided in any work.

In  $\nu$ -gap reduction algorithm  $\phi$  should be an initial guess on  $qa^* + pb^*$ . Given an initial point  $p^0$  and  $q^0$ , the algorithm can be iterated, therefore the choice of the starting point is the most important part. Reasonable  $p^0$  and  $q^0$ , can be obtained by order reduction of the normalized coprime factors of  $G$   $a$  and  $b$ .

*Remark 1:* Normalized coprime factors are the central point of the  $\nu$ -gap metric theory. Moreover, using the  $\mathcal{H}_{\infty}$  the distance between two NCFs it is possible to bound the  $\nu$ -gap metric between those, as it is shown in [21], [17]. Therefore, by applying the described algorithm the quality of approximation in the metric is improved comparing to the NCF quality.

Remarkably, by convexifying the condition (3) the condition (4) was incorporated into the convexified norm constraint. A proof of this fact is summed up in the following lemma.

*Lemma 2:* Assume that  $a, b$  are the normalized coprime factors of  $G$ ,  $p$  and  $q$  are coprime factors of  $\widehat{G}$ , and  $\phi$  is chosen according to Remark 1. If  $\text{Re}((qa^* + pb^*)/\phi) > 0$  for all frequencies  $\omega$  then the winding number condition  $\eta([p/q, -b^*/a^*]) = \eta([b/a, -b^*/a^*])$  is satisfied.

*Proof:* Recall, that  $\phi = p^0b^* + q^0a^*$  where  $\eta([p^0/q^0, -b^*/a^*]) = \eta([b/a, -b^*/a^*])$ . Therefore we only need to show that  $\eta([p^0/q^0, -b^*/a^*]) = \eta([p/q, -b^*/a^*])$ . Which is shown in a straight forward manner from the condition  $\text{Re}((qa^* + pb^*)/\phi) > 0$ . Indeed, the positive real condition  $\text{Re}(c/d) > 0$  is equivalent to “ $c$  has the same number of unstable zeros as  $d$ ” given that the poles for  $c$  and  $d$  are equal. It is true if the number of zeros of  $c$  and  $d$  is equal, which is the case in the Algorithm 1. Therefore  $\eta([p^0/q^0, -b^*/a^*]) = \eta([p/q, -b^*/a^*])$  and finally  $\eta([p/q, -b^*/a^*]) = \eta([b/a, -b^*/a^*])$ . ■

*Remark 2:* In the proof it has been assumed that the number of zeros of  $qa^* + pb^*$  and  $\phi = q^0a^* + p^0b^*$  is equal. In semidefinite programming obtaining a  $qa^* + pb^*$  which has fewer zeros than  $\phi = q^0a^* + p^0b^*$  is equivalent to obtaining a matrix which is rank deficient. The set of rank deficient matrices is a null measure subset of the space of full-rank matrices. Therefore it is highly unlikely to obtain a rank deficient matrix in the semidefinite programming. With a similar reasoning it can be stated that  $p$  and  $q$  are in fact coprime.

Finally, the quasi-convex semidefinite program of approximation in the  $\nu$ -gap metric for a given  $\phi$  may be deduced:

$$\gamma_{\nu}^c = \min_{p, q} \gamma \quad \text{subject to}$$

$$\forall \omega \in [0, \pi] : \left\| \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} / \phi(\omega) \right\|_2 <$$

$$\gamma \text{Re}((q(\omega)a^*(\omega) + p(\omega)b^*(\omega)) / \phi(\omega)) \quad (5)$$

*Theorem 1:* Consider the Algorithm 1 with a full sampling, i.e. the constraints are enforced for all the frequencies  $\omega$ , where  $\gamma_{\nu}^c$ ,  $p$ , and  $q$  is the output of the algorithm. Then  $\delta_{\nu}(G, p/q) \leq \sqrt{1 - (1/\gamma_{\nu}^c)^2}$ .

*Proof:* Shown by construction using Lemma 2. ■

## C. Tractable Algorithm and Implementation

The first step of the algorithm is choosing an appropriate “central”  $\phi$ . According to the Remark 1 such a  $\phi$  may be produced as follows

- 1) Compute an approximation  $(n/\theta \quad m/\theta)$  of  $(b \quad a)$ , which is a NCF of the full model  $G$ . Note that  $m, n$  and  $\theta$  are FIR filters.
- 2) Fix  $\phi = nb^* + ma^*$ .

Surely, there is a number of other ways to choose  $\phi$  (e.g. the choice  $\phi = \theta$  also provided excellent results) and author did not provide a rigorous proof, that this particular choice of  $\phi$  will always deliver results. However, the intuition behind

the choice, which is described in the Remark 1 is reasonable and provides reasonable results in numerical experiments.

The program (5) is quasi-convex and can be solved using standard tools. The second order cone constraint can be easily transformed into an LMI using the Schur complement, providing a semidefinite constraint instead. A frequency dependent semidefinite constraint may be imposed for all frequencies at ones using the KYP lemma, e.g. a formulation from [22]. To provide a computationally cheaper program the constraints may be enforced on a frequency grid  $\{\omega_i\}_{i=1}^N$ , where  $N$  is big enough to avoid over-fit.

If required  $\text{Re}((q(\omega)a^*(\omega) + p(\omega)b^*(\omega))/\phi(\omega)) > 0$  can be enforced for all frequencies to ensure the winding number condition (e.g. using the KYP lemma). The total cost of the algorithm will not be drastically affected by adding this feature.

The algorithm is implemented using the interior-point solvers SEDUMI ([23]) and SDPT<sup>3</sup> ([24]) and the parser YALMIP ([25])

#### D. Computational Complexity

There are two main contributors computational complexity: computation of normalized coprime factors and optimization problem. Computation of NCFs is done using Riccati equations and therefore complexity is  $O(l^3)$  floating point operations (flops), where  $l$  is the order of the equation (of the full order model  $G$ ).

The optimization cost of a semidefinite program differs depending on the tolerance level, number of decision variables and if the constraints are enforced for all frequencies or just on a grid. The cost of one iteration when solved with SEDUMI does not exceed  $O(N_1^2 N_2^{2.5} + N_1^{3.5})$  flops, where  $N_1$  being the number of decision variables and  $N_2$  the number of constraints. If constraints are enforced on a frequency grid, then  $N_1 = O(k)$  and  $N_2 = N$ . Computing the frequency samples costs in general  $O(l^3)$  and can be lowered to  $O(l \log(l))$  in certain cases ([26], [27]). If all the constraints are enforced using the KYP lemma for all frequencies, then  $N_1 = O(l^2 k^2)$ . In both cases  $k$  is the order of approximation and  $N$  is the number or frequency points in the grid. The number of iterations is bounded by solvers tolerance  $\varepsilon$  as  $O(\sqrt{N_1} \log \frac{1}{\varepsilon})$ , although in practice more than 50 iterations is rarely required.

Since the program is quasi-convex it is solved using bisection. The tolerance of bisection should be higher than in the similar model reduction methods. Indeed, low values of  $\delta_\nu$  correspond to the values of  $\gamma$  very close to 1, for instance if  $\delta_\nu = 0.05$  then  $\gamma = 1.00125$ . Therefore the tolerance of approximation has to be modified accordingly depending on an application.

## IV. EXAMPLES

Throughout the section METHOD 1 will denote approximation in the  $\nu$ -gap metric and METHOD 2 will denote approximation in the  $\nu$ -gap with a fixed performance degradation level, which will be introduced shortly.

*Example 2: Approximation of a Flexible Beam Model.* In this example it is possible to see the effect of the approximation in the  $\nu$ -gap metric. A continuous time model of a flexible beam is described in [28]. As may be seen in Table I, METHOD 1 always provides a better  $\nu$ -gap match than Hankel approximations. For orders 2 and 4 considerable improvement was not achieved, since any method can only match peaks in the frequency response (every peak corresponds to a pair of complex conjugate poles). For orders 1 and 3 there is extra freedom in the choice of poles of the system which is exploited by the METHOD 1. Fig. 2 depicts the frequency responses of the reduced order and full order models.

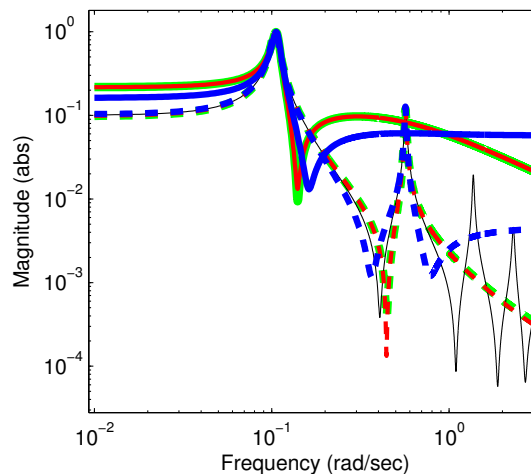


Fig. 2. Frequency responses of the original beam model (thin solid black), Hankel approximation (thick green lines), Hankel approximation of normalized coprime factors (thick red lines) and the proposed method (thick blue lines). Hankel approximations almost coincide. Order 4 approximations are dashed, order 3 approximations are thick solid.

TABLE I  
APPROXIMATION ERRORS IN  $\delta_\nu \cdot 10^{-2}$  OF VARIOUS METHODS IN  
EXAMPLE 2

Reduction Orders	1	2	3	4
HMR	67.2	11.4	11.5	1.88
HMR of NCFs	60.7	11.3	11.5	1.88
METHOD 1	37.9	11.2	6.2	1.87

#### A. Application. Controller Reduction

The controller reduction is a very complicated problem since a designer must keep in mind both robustness and performance criteria to obtain a reasonable controller. Good surveys of methods using coprime factorization and frequency weighted approaches, which tackles both criteria are given in [17] and [29].  $\nu$ -gap does not account for the performance of the closed loop system, therefore it is desirable to include the constraints on performance into the optimization problem. Using the semidefinite program as a tool makes it possible. As a basis the method briefly described in [8] is used. As example, consider a closed loop

transfer function  $H(G, K)$  where  $G$  is a plant and  $K$  is a controller and  $H$  is a so called gang-of-four:

$$H(G, K) = \begin{pmatrix} \frac{G}{1+GK} & \frac{-KG}{1+GK} \\ \frac{1}{1+GK} & \frac{-K}{1+GK} \end{pmatrix}$$

Another closed-loop function may be approached in a similar manner, for example,  $T(G, K) = GK/(1 + KG)$ . Denote  $b/a$  an NCF of  $K$  and  $c/d$  an NCF of  $G$ .

$$\begin{aligned} & \min_{\gamma_r > 0, p, q} \gamma_r \quad \text{subject to} \\ & \left| \left( H(\omega)e(\omega) - \begin{pmatrix} q(\omega)c(\omega) & -p(\omega)c(\omega) \\ q(\omega)d(\omega) & -p(\omega)d(\omega) \end{pmatrix} \right) / \psi(\omega) \right| < \\ & \quad < \gamma_p \text{Re}(e(\omega)) \quad \forall \omega \\ & \left| \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} / \phi(\omega) \right| < \gamma_r \text{Re}(f(\omega)) \quad \forall \omega \\ & \text{Re}(f(\omega)) > 0 \quad \text{Re}(e(\omega)) > 0 \quad \forall \omega \\ & e(\omega) = (q(\omega)d(\omega) + p(\omega)c(\omega)) / \psi(\omega) \\ & f(\omega) = (q(\omega)a^*(\omega) + p(\omega)b^*(\omega)) / \phi(\omega) \end{aligned}$$

where  $\gamma_p$  is a pre-determined performance degradation level and  $\phi$  is an initial guess on  $qa^* + pb^*$ , and  $\psi$  is an initial guess on  $q(\omega)d(\omega) + p(\omega)c(\omega)$ .

TABLE II

APPROXIMATION ERRORS OF VARIOUS METHODS IN THE  $\nu$ -GAP METRIC AND EFFECT ON THE CLOSED LOOP PERFORMANCE IN EXAMPLE 4

DISTANCE BETWEEN $K$ AND $\hat{K}$ IN THE $\nu$ -GAP METRIC ( $\cdot 10^{-2}$ )						
Reduction Orders	2	3	4	5	6	7
HMR	72.9	83.14	24.58	18.88	12.67	6.77
HMR of NCF	67.01	70.48	13.47	4.96	6.29	2.67
METHOD 1	38.03	7.49	3.35	3.23	3.22	1.25
METHOD 2	44.56	7.84	4.63	4.05	4.27	1.42

DISTANCE BETWEEN $T(G, K)$ AND $T(G, \hat{K})$ IN $\mathcal{H}_\infty$						
Reduction Orders	2	3	4	5	6	7
HMR	3.64	26.02	1.09	0.84	0.35	0.31
HMR of NCF	2.37	2.34	0.92	0.34	0.38	0.19
METHOD 1	2.05	0.58	0.27	0.26	0.26	0.11
METHOD 2	1.53	0.43	0.21	0.19	0.19	0.08

*Example 3: Altitude Controller of a Flexible Spacecraft*  
The controller was designed in [30] using loop shaping procedures. Controller has two unstable poles, which is not a particularly robust solution to the design problem. Therefore slight changes in controller may result in an unstable close loop. Using the  $\nu$ -gap reduction method it was possible to obtain a stable closed loop only for order 2 with  $\delta_\nu$  error of 0.014, however the performance did not match at all. Moreover, a controller of order 4 with  $\delta_\nu$  distance to the original one equal to 0.005, was destabilizing to the control loop. Conventional methods (such as weighted reduction and reduction of normalized coprime factors) always provided a destabilizing controller.

*Example 4: Approximating a Youla Controller.* Consider gang-of-four  $H(G, K)$ , where the plant  $G$  is controlled in a robust manner by a controller  $K$ . The 152-nd order controller  $K$  was obtained in [31] using Youla parameterization, therefore the controller itself is stable and so is the third order

plant  $G$ . For every order the level  $\gamma_p$  will be fixed to 75 % of the performance obtained by the METHOD 1. The results are presented in Table 4.

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#### V. CONCLUSION

A linear system approximation method in the  $\nu$ -gap metric is presented in the paper. Such a method may be very useful for modeling of structured or multi-agent systems. Approximation is obtained using semidefinite programming and a normalized coprime factorization of the original model. The method can be applied to controller reduction with taking into account the performance of closed loop.

Future work on the method includes comparison with the method from [15], which uses the LMI formulation of the problem and the state-space data of the systems. It also relies on an iterative procedure to obtain a solution. The second direction of the future work is a MIMO extension using the techniques in the mentioned [9], [16].

The current algorithm has some advantages comparing to [15] regardless of the future comparison. Incorporating extra constraints, adding frequency-dependent weights and/or restricting the objective to a specific frequency interval is straight-forward using the frequency domain representation.

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