Nu-gap Model Reduction in the Frequency Domain

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2011

2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

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Abstract— **In this paper a model reduction algorithm in the nu-gap metric is considered. The metric was originally developed to evaluate robustness of a controller for a given plant. In fact, the nu-gap metric induces the weakest topology in which stability is a robust property. All in all the nu-gap metric is perhaps the best metric to evaluate the distance between two systems in a closed loop setup. In the field of distributed control, if approximation of the subsystems is considered, such a metric can be vital for modeling purposes. The presented algorithm of model reduction in the nu-gap metric is based on semidefinite programming methods and exploits the frequency domain representation of the systems. Therefore it may be easily extended to incorporate into the optimization procedure constraints on a specific frequency region of a particular interest or the closed loop performance.**

I. Introduction

Model reduction of linear systems in $\mathcal{H}$ spaces is well studied in the literature and reasonable suboptimal methods have been derived. The methods delivering the best approximation quality are the, so called, Singular Value Decomposition (SVD) methods, as balanced truncation ([1]) and optimal Hankel model reduction ([2]). The methods also guarantee stability, however, computationally very expensive. Krylov methods ([3], [4], [5]), on the other hand, are considerably cheaper, however, can not guarantee stability, in general. Krylov/SVD ([6]) is a trade-off framework, which is cheaper than SVD, and can guarantee stability. Another trade-off framework was developed in [7], [8], [9] it is based on frequency response matching and the semidefinite optimization techniques.

All of the above mentioned methods measure the error in $\mathcal{H}_\infty$ or $\mathcal{H}_2$ spaces, meaning that it is a measure on the distance in the open loop setup. In the closed loop setup these norms usually do not reflect the distance adequately. The first attempt to introduce a more reliable metric in the closed loop setup was the introduction of the gap metric in [10], followed by many papers including [11] and [12]. In the latter the ν-gap metric was introduced and it is the only metric for which *"... any plant at a distance less than β from the nominal will be stabilized by any compensator stabilizing the nominal with a stability margin β. Furthermore, any plant at a distance* **greater** *than β from the nominal will be* **destabilized** *by some compensator that stabilizes the nominal with a stability margin of at least β"* ([13]). The stability margin here is $b_{P,C}$ and defined therein. Moreover, the ν-gap induces the weakest topology in which stability is a robust property. To some extent we can evaluate the stability of a closed loop without considering the stabilized plant. Therefore the ν-gap metric may be a crucial tool in distributed system modeling, where the evaluation of the entire system can be computationally overwhelming.

Early work in ν-gap model reduction includes [14] and [15], which use the state-space representation of the systems and an iterative LMI program. The algorithm presented in this paper uses semidefinite programming as a tool and coprime fractions of the original system. As opposed to [15], the frequency domain data is used to obtain a reduced model. The algorithm was also extended to account for the overall performance in controller-plant loops, which is presented in Section IV. Although the algorithm is derived for the scalar valued transfer functions, with extra restrictions it may be extended to the matrix valued transfer function using the techniques described in [9], [16].

Notation. $\mathcal{R}^{m_1\times m_2}$ denotes the space of discrete-time m_1 by m_2 matrix valued rational transfer functions. Operation $^\sim$ denotes a complex conjugate on the unit circle i.e., $G^\sim(e^{\jmath\omega}) = G^T(e^{-\jmath\omega})$, where $\jmath$ is a complex identity. $G(\omega)$ stands for the frequency response of $G(e^{\jmath\omega})$ to $\omega \in [0, \pi]$. The infinity norm is computed as $\|G\|_\infty = \sup_\omega |G(\omega)|$, where $G(\omega)$ is a scalar-valued function. The Hankel norm of a transfer function is denoted as $\|\cdot\|_H$ (for the definition see, [17]). Function $\eta(G)$ stands for the number of poles of G outside the unit circle and finally

$$[G, K] = \begin{pmatrix} G \\ I \end{pmatrix} (I - KG)^{-1} \begin{pmatrix} -K & I \end{pmatrix}.$$

II. Preliminaries

Firstly, it may be useful to illustrate why and in which situations the ν-gap metric is employable. Consider a toy example borrowed from [18, pp. 349-350].

Example 1: Distance between systems. Given the systems

$$P_1 = \frac{100}{s+1} \quad P_2 = \frac{100}{(s+1)(0.0025s+1)^2} \quad P_3 = \frac{100}{s-1}$$

investigate the step responses in open loop and closed loop (simple negative feedback) settings in Fig. 1. The plants P_1 and P_2 are close in the $\mathcal{H}_\infty$ norm, on the other hand plants P_1 and P_3 are close in the ν-gap metric. Notice, that P_2 is unstable in the closed loop setup, however stable in the open loop. P_3 on the other hand is stable in the closed loop setting, but unstable in the open loop and P_1 is stable in both settings. Fig. 1(a) shows that the open loop step responses of P_1 and P_2 are almost identical, however the closed loop ones are not. Fig. 1(b) shows the opposite situation: P_1 is close to P_2 in the closed loop, which is open loop unstable. Therefore the ν-gap showed itself as a better measure on the

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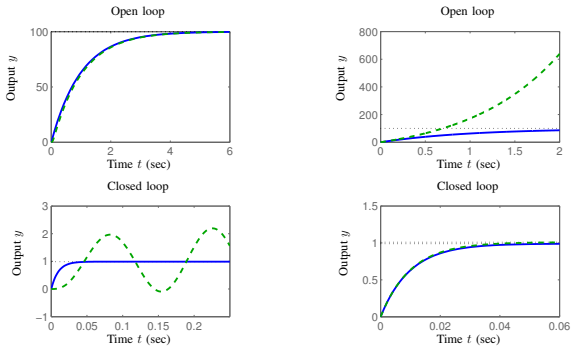
distance than the $\mathcal{H}_\infty$ norm in this particular closed loop setup.



(a) Plants P_1 (solid blue) and P_2 (dashed green)

(b) Plants P_1 (solid blue) and P_3 (dashed green)

Fig. 1. Determining when the distance between two systems is small. The figures in (a) show that the open step responses of the plants P_1 and P_2 are close, but the closed loop responses are not. The figures in (b) show the opposite situation: the plants P_1 and P_3 are close in closed loop but different different in open loop.

There exist a few equivalent definitions of the metric. The one chosen in this paper is more convenient for our goal. Denote b, a a normalized left coprime factorization (NRCF) of $G_1 = a^{-1}b$, and p, q a right coprime factorization (RCF), not necessarily normalized, of $G_2 = pq^{-1}$.

Definition 1 (ν-gap metric): Define a function $\delta_\nu(\cdot,\cdot) : \mathcal{R}^{m_1\times m_2} \times \mathcal{R}^{m_1\times m_2} \to \mathbb{R}$ as follows

$$\delta_\nu(G_1,G_2) = \begin{cases} \delta_{\mathcal{L}_2}(G_1,G_2) & \text{if } \eta([G_2,-G_1^*]) = \\ & = \eta([G_1,-G_1^*]) \\ 1 & \text{otherwise} \end{cases}$$

where

$$\delta_{\mathcal{L}_2}(G_1,G_2) = \sqrt{1-\left\|\begin{pmatrix}p\\q\end{pmatrix}(a^*q+b^*p)^{-1}\right\|_\infty^{-2}}$$

The constraint $\eta([G_2,-G_1^*]) = \eta([G_1,-G_1^*])$ is usually called a winding number condition in the literature. Note, that throughout the paper it is assumed that G_1 and G_2 are scalar, not necessarily stable, transfer function. Therefore a, b, p and q are scalar transfer functions as well.

Finally, we are ready to formulate the problem as an optimization one. Given asymptotically stable a and b, such that $b^*b + a^*a = I$ and b/a not necessarily stable, solve

$$\gamma_{opt} = \min_{\gamma>0,p,q} \gamma \text{ s. t.: } \left\|\begin{pmatrix}p\\q\end{pmatrix}(a^*q+b^*p)^{-1}\right\|_\infty \le \gamma$$
$$\eta\left(\left[\frac{p}{q},-\left(\frac{b}{a}\right)^*\right]\right) = \eta\left(\left[\frac{b}{a},-\left(\frac{b}{a}\right)^*\right]\right)$$

Notice, that $\delta_\nu(b/a, p/q) \le \sqrt{1-1/\gamma_{opt}^2}$ by construction. However, the obtained program is not generally convex even for the scalar valued function due to the winding number condition and the computation of $\mathcal{H}_\infty$ norm. Therefore, a convexification is required. To authors best knowledge, using the described formulation a meaningful convex relaxation is impossible. Therefore the two-step techniques proposed in [19] (developed in [8]) will be employed to address the problem. First, a so called "central" transfer function will be computed and then around it the solution will be constructed using semidefinite programming.

III. Model Reduction in the ν-gap Metric

A. Main Result

As the main result the algorithm of approximation in the ν-gap metric is presented in this subsection. The details and theoretical justification follow in the sequel.

Consider a scalar valued discrete-time transfer function G with a normalized coprime factorization (NCF) $\begin{pmatrix}b & a\end{pmatrix}$, where b and a are stable transfer functions of the same order l as the system G. Note, G is not necessarily a stable system. Compute its k-th order approximation $\widehat{G}$ in the ν-gap metric as:

Algorithm 1:
1) Compute a k-th order approximation $\begin{pmatrix}n/\theta & m/\theta\end{pmatrix}$ of a transfer matrix $\begin{pmatrix}b & a\end{pmatrix}$ using any stability preserving model reduction method, where m, n and θ are finite impulse response (FIR) filters of order k
2) Fix a "central" transfer function $\phi = ma^* + nb^*$ and solve the following semidefinite program:

$$\begin{gathered}\gamma_\nu^N = \min_{p,q}\gamma \quad \text{subject to} \\ \forall\,\omega\in[0,\pi]:\ \mathrm{Re}((q(\omega)a^*(\omega)+p(\omega)b^*(\omega))/\phi(\omega)) > 0 \\ \forall\, i = 1,\dots,N:\ \omega_i\in[0,\pi], \left|\begin{pmatrix}p(\omega_i)\\q(\omega_i)\end{pmatrix}/\phi(\omega_i)\right| < \\ \gamma\mathrm{Re}((q(\omega_i)a^*(\omega_i)+p(\omega_i)b^*(\omega_i))/\phi(\omega_i)) \\ p(\omega) = \sum_{i=0}^k p_i e^{-ij\omega} \quad q(\omega) = \sum_{i=0}^k q_i e^{-ij\omega}\end{gathered} \tag{1}$$

3) The reduced order plant $\widehat{G}$ is computed as p/q.
4) If required, compute an NCF $m/\theta, n/\theta$ of $\widehat{G}$ and repeat steps $2-4$.

Denote γ_ν^∞ the solution to the relaxed problem in the step 2 of the Algorithm 1, in the case when $N = \infty$ and $\{\omega_i\}_{i=1}^\infty$ are dense in $[0,\pi]$, and γ_ν^c, if all the constraints are enforced for all the frequencies ω. It can be shown that $\lim_{N\to\infty}\gamma_\nu^N = \gamma_\nu^\infty$, moreover, since $\{\omega_i\}$ is dense in $[0,\pi]$, γ_ν^c is equal to γ_ν^∞. Therefore with a big enough N it is possible to approximate γ_ν^c with a good accuracy.

Theorem 1: Consider the Algorithm 1 with a full sampling, i.e. the constraints are enforced for all the frequencies ω, where γ_ν^c, p, and q is the output of the algorithm. Then $\delta_\nu(G, p/q) \le \sqrt{1-(1/\gamma_\nu^c)^2}$.

B. Theoretical justification and details

In Section II a minimization program was constructed to approximate a system in the ν-gap metric:

$$\gamma_{opt} = \min_{\gamma>0,p,q} \quad \gamma \tag{2}$$

$$\left\| \begin{pmatrix} p \\ q \end{pmatrix} (a^*q + b^*p)^{-1} \right\|_\infty \leq \gamma \tag{3}$$

$$\eta\left(\left[\frac{p}{q}, -\frac{b^*}{a^*}\right]\right) = \eta\left(\left[\frac{b}{a}, -\frac{b^*}{a^*}\right]\right) \tag{4}$$

Both constraints (3,4) are not convex. The technique applied to the program was introduced in [8]. Firstly, consider another program with only (3) as a constraint, rewriting (3) in the process as an infinite number of constraints:

$$\gamma_1 = \min_{\gamma>0,p,q} \quad \gamma$$
$$\forall\, \omega \in [0,\pi] : \left| \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} \right| < \gamma |q(\omega)a^*(\omega) + p(\omega)b^*(\omega)|$$

As in [8] introduce a new variable ϕ into the program as:

$$\gamma_2 = \min_{\gamma>0,p,q,\phi} \quad \gamma$$
$$\forall\, \omega \in [0,\pi] : \left| \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} / \phi(\omega) \right| <$$
$$\gamma \mathrm{Re}((q(\omega)a^*(\omega) + p(\omega)b^*(\omega))/\phi(\omega))$$

The programs are equivalent, which is understood as equality $\gamma_1 = \gamma_2$. It can be shown that an optimal choice of $\phi(\omega)$ is $q(\omega)a^*(\omega) + p(\omega)b^*(\omega)$. Basically, $\phi(\omega)$ is an initial guess on $q(\omega)a^*(\omega)+p(\omega)b^*(\omega)$. Introducing an extra variable ϕ does not solve all the problems. However, by computing ϕ in advance and fixing it in the minimization the program becomes quasi-convex.

Surely, there is a question of conservatism of the positive real condition. Is the described set big enough to provide any improvement at all? This question was studied in [19] and [20], besides the mentioned work [8]. The results for low orders as 2 and 3 are colorfully illustrated in [19]. Indeed, given two polynomials θ and ξ this condition describes all positive real transfer functions with a fixed polynomial θ and the set of all possible ξ is shown in numerical examples to be sufficiently big comparing to the set of all stable ξ. However, no theoretical results were provided in any work.

In ν-gap reduction algorithm ϕ should be an initial guess on $qa^* + pb^*$. Given an initial point p^0 and q^0, the algorithm can be iterated, therefore the choice of the starting point is the most important part. Reasonable p^0 and q^0, can be obtained by order reduction of the normalized coprime factors of G a and b.

Remark 1: Normalized coprime factors are the central point of the ν-gap metric theory. Moreover, using the $\mathcal{H}_\infty$ the distance between two NCFs it is possible to bound the ν-gap metric between those, as it is shown in [21], [17]. Therefore, by applying the described algorithm the quality of approximation in the metric is improved comparing to the NCF quality.

Remarkably, by convexifying the condition (3) the condition (4) was incorporated into the convexified norm constraint. A proof of this fact is summed up in the following lemma.

Lemma 2: Assume that a, b are the normalized coprime factors of G, p and q are coprime factors of $\widehat{G}$, and ϕ is chosen according to Remark 1. If $\mathrm{Re}((qa^* + pb^*)/\phi) > 0$ for all frequencies ω then the winding number condition $\eta([p/q, -b^*/a^*]) = \eta([b/a, -b^*/a^*])$ is satisfied.

Proof: Recall, that $\phi = p^0b^* + q^0a^*$ where $\eta\left(\left[p^0/q^0, -b^*/a^*\right]\right) = \eta([b/a, -b^*/a^*])$. Therefore we only need to show that $\eta\left(\left[p^0/q^0, -b^*/a^*\right]\right) = \eta([p/q, -b^*/a^*])$. Which is shown in a straight forward manner from the condition $\mathrm{Re}((qa^* + pb^*)/\phi) > 0$. Indeed, the positive real condition $\mathrm{Re}(c/d) > 0$ is equivalent to "c *has the same number of unstable zeros as* d" given that the poles for c and d are equal. It is true if the number of zeros of c and d is equal, which is the case in the Algorithm 1. Therefore $\eta\left(\left[p^0/q^0, -b^*/a^*\right]\right) = \eta([p/q, -b^*/a^*])$ and finally $\eta([p/q, -b^*/a^*]) = \eta([b/a, -b^*/a^*])$. ■

Remark 2: In the proof it has been assumed that the number of zeros of $qa^* + pb^*$ and $\phi = q^0a^* + p^0b^*$ is equal. In semidefinite programming obtaining a $qa^* + pb^*$ which has fewer zeros than $\phi = q^0a^* + p^0b^*$ is equivalent to obtaining a matrix which is rank deficient. The set of rank deficient matrices is a null measure subset of the space of full-rank matrices. Therefore it is highly unlikely to obtain a rank deficient matrix in the semidefinite programming. With a similar reasoning it can be stated that p and q are in fact coprime.

Finally, the quasi-convex semidefinite program of approximation in the ν-gap metric for a given ϕ may be deduced:

$$\gamma_\nu^c = \min_{p,q} \gamma \quad \text{subject to}$$
$$\forall\, \omega \in [0,\pi] : \left\| \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} / \phi(\omega) \right\|_2 < \tag{5}$$
$$\gamma \mathrm{Re}((q(\omega)a^*(\omega) + p(\omega)b^*(\omega))/\phi(\omega))$$

Theorem 1: Consider the Algorithm 1 with a full sampling, i.e. the constraints are enforced for all the frequencies ω, where γ_ν^c, p, and q is the output of the algorithm. Then $\delta_\nu(G, p/q) \leq \sqrt{1 - (1/\gamma_\nu^c)^2}$.

Proof: Shown by construction using Lemma 2. ■

C. Tractable Algorithm and Implementation

The first step of the algorithm is choosing an appropriate "central" ϕ. According to the Remark 1 such a ϕ may be produced as follows

1) Compute an approximation $\begin{pmatrix} n/\theta & m/\theta \end{pmatrix}$ of $\begin{pmatrix} b & a \end{pmatrix}$, which is a NCF of the full model G. Note that m, n and θ are FIR filters.
2) Fix $\phi = nb^* + ma^*$.

Surely, there is a number of other ways to choose ϕ (e.g. the choice $\phi = \theta$ also provided excellent results) and author did not provide a rigorous proof, that this particular choice of ϕ will always deliver results. However, the intuition behind

the choice, which is described in the Remark 1 is reasonable and provides reasonable results in numerical experiments.

The program (5) is quasi-convex and can be solved using standard tools. The second order cone constraint can be easily transformed into an LMI using the Schur complement, providing a semidefinite constraint instead. A frequency dependent semidefinite constraint may be imposed for all frequencies at ones using the KYP lemma, e.g. a formulation from [22]. To provide a computationally cheaper program the constraints may be enforced on a frequency grid $\{\omega_i\}_{i=1}^{N}$, where N is big enough to avoid over-fit.

If required $\mathrm{Re}((q(\omega)a^*(\omega) + p(\omega)b^*(\omega))/\phi(\omega)) > 0$ can be enforced for all frequencies to ensure the winding number condition (e.g. using the KYP lemma). The total cost of the algorithm will not be drastically affected by adding this feature.

The algorithm is implemented using the interior-point solvers SeDuMi ([23]) and SDPT^3 ([24]) and the parser YALMIP ([25])

D. Computational Complexity

There are two main contributors computational complexity: computation of normalized coprime factors and optimization problem. Computation of NCFs is done using Riccati equations and therefore complexity is $O(l^3)$ floating point operations (flops), where l is the order of the equation (of the full order model G).

The optimization cost of a semidefinite program differs depending on the tolerance level, number of decision variables and if the constraints are enforced for all frequencies or just on a grid. The cost of one iteration when solved with SeDuMi does not exceed $O(N_1^2 N_2^{2.5} + N_1^{3.5}))$ flops, where N_1 being the number of decision variables and N_2 the number of constraints. If constraints are enforced on a frequency grid, then $N_1 = O(k)$ and $N_2 = N$. Computing the frequency samples costs in general $O(l^3)$ and can be lowered to $O(l\log(l))$ in certain cases ([26], [27]). If all the constraints are enforced using the KYP lemma for all frequencies, then $N_1 = O(l^2k^2)$. In both cases k is the order of approximation and N is the number or frequency points in the grid. The number of iterations is bounded by solvers tolerance ε as $O(\sqrt{N_1}\log\frac{1}{\varepsilon})$, although in practice more than 50 iterations is rarely required.

Since the program is quasi-convex it is solved using bisection. The tolerance of bisection should be higher than in the similar model reduction methods. Indeed, low values of δ_ν correspond to the values of γ very close to 1, for instance if $\delta_\nu = 0.05$ then $\gamma = 1.00125$. Therefore the tolerance of approximation has to be modified accordingly depending on an application.

IV. Examples

Throughout the section Method 1 will denote approximation in the ν-gap metric and Method 2 will denote approximation in the ν-gap with a fixed performance degradation level, which will be introduced shortly.

Example 2: Approximation of a Flexible Beam Model. In this example it is possible to see the effect of the approximation in the ν-gap metric. A continuous time model of a flexible beam is described in [28]. As may be seen in Table I, Method 1 always provides a better ν-gap match than Hankel approximations. For orders 2 and 4 considerable improvement was not achieved, since any method can only match peaks in the frequency response (every peak corresponds to a pair of complex conjugate poles). For orders 1 and 3 there is extra freedom in the choice of poles of the system which is exploited by the Method 1. Fig. 2 depicts the frequency responses of the reduced order and full order models.

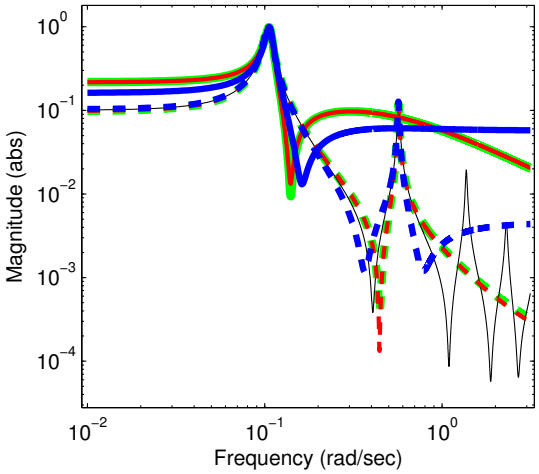


Fig. 2. Frequency responses of the original beam model (thin solid black), Hankel approximation (thick green lines), Hankel approximation of normalized coprime factors (thick red lines) and the proposed method (thick blue lines). Hankel approximations almost coincide. Order 4 approximations are dashed, order 3 approximations are thick solid.

TABLE I
Approximation errors in $\delta_\nu \cdot 10^{-2}$ of various methods in Example 2

Reduction Orders	1	2	3	4
HMR	67.2	11.4	11.5	1.88
HMR of NCFs	60.7	11.3	11.5	1.88
Method 1	37.9	11.2	6.2	1.87

A. Application. Controller Reduction

The controller reduction is a very complicated problem since a designer must keep in mind both robustness and performance criteria to obtain a reasonable controller. Good surveys of methods using coprime factorization and frequency weighted approaches, which tackles both criteria are given in [17] and [29]. ν-gap does not account for the performance of the closed loop system, therefore it is desirable to include the constraints on performance into the optimization problem. Using the semidefinite program as a tool makes it possible. As a basis the method briefly described in [8] is used. As example, consider a closed loop

transfer function $H(G,K)$ where G is a plant and K is a controller and H is a so called gang-of-four:

$$H(G,K) = \begin{pmatrix} \frac{G}{1+GK} & \frac{-KG}{1+GK} \\ \frac{1}{1+GK} & \frac{-K}{1+GK} \end{pmatrix}$$

Another closed-loop function may be approached in a similar manner, for example, $T(G,K) = GK/(1+KG)$. Denote b/a an NCF of K and c/d an NCF of G.

$$\min_{\gamma_r>0,p,q} \gamma_r \quad \text{subject to}$$

$$\left| \left(H(\omega)e(\omega) - \begin{pmatrix} q(\omega)c(\omega) & -p(\omega)c(\omega) \\ q(\omega)d(\omega) & -p(\omega)d(\omega) \end{pmatrix} \right) / \psi(\omega) \right| < \\ < \gamma_p \mathrm{Re}(e(\omega)) \ \forall \omega$$

$$\left| \begin{pmatrix} p(\omega) \\ q(\omega) \end{pmatrix} / \phi(\omega) \right| < \gamma_r \mathrm{Re}(f(\omega)) \ \forall \omega$$

$$\mathrm{Re}(f(\omega)) > 0 \quad \mathrm{Re}(e(\omega)) > 0 \ \forall \omega$$

$$e(\omega) = (q(\omega)d(\omega) + p(\omega)c(\omega))/\psi(\omega)$$

$$f(\omega) = (q(\omega)a^*(\omega) + p(\omega)b^*(\omega))/\phi(\omega)$$

where γ_p is a pre-determined performance degradation level and ϕ is an initial guess on $qa^* + pb^*$, and ψ is an initial guess on $q(\omega)d(\omega) + p(\omega)c(\omega)$.

TABLE II

APPROXIMATION ERRORS OF VARIOUS METHODS IN THE ν-GAP METRIC AND EFFECT ON THE CLOSED LOOP PERFORMANCE IN EXAMPLE 4

DISTANCE BETWEEN K AND $\widehat{K}$ IN THE ν-GAP METRIC ($\cdot 10^{-2}$)

Reduction Orders	2	3	4	5	6	7
HMR	72.9	83.14	24.58	18.88	12.67	6.77
HMR of NCF	67.01	70.48	13.47	4.96	6.29	2.67
METHOD 1	38.03	7.49	3.35	3.23	3.22	1.25
METHOD 2	44.56	7.84	4.63	4.05	4.27	1.42

DISTANCE BETWEEN $T(G,K)$ AND $T(G,\widehat{K})$ IN $\mathcal{H}_\infty$

Reduction Orders	2	3	4	5	6	7
HMR	3.64	26.02	1.09	0.84	0.35	0.31
HMR of NCF	2.37	2.34	0.92	0.34	0.38	0.19
METHOD 1	2.05	0.58	0.27	0.26	0.26	0.11
METHOD 2	1.53	0.43	0.21	0.19	0.19	0.08

Example 3: Altitude Controller of a Flexible Spacecraft The controller was designed in [30] using loop shaping procedures. Controller has two unstable poles, which is not a particularly robust solution to the design problem. Therefore slight changes in controller may result in an unstable close loop. Using the ν-gap reduction method it was possible to obtain a stable closed loop only for order 2 with δ_ν error of 0.014, however the performance did not match at all. Moreover, a controller of order 4 with δ_ν distance to the original one equal to 0.005, was destabilizing to the control loop. Conventional methods (such as weighted reduction and reduction of normalized coprime factors) always provided a destabilizing controller.

Example 4: Approximating a Youla Controller. Consider gang-of-four $H(G,K)$, where the plant G is controlled in a robust manner by a controller K. The 152-nd order controller K was obtained in [31] using Youla parameterization, therefore the controller itself is stable and so is the third order plant G. For every order the level γ_p will be fixed to 75 $\%$ of the performance obtained by the METHOD 1.The results are presented in Table 4.

ACKNOWLEDGMENT

This work was sponsored by LCCC–Lund Center for Control of Complex Engineering Systems funded by the Swedish Research Council. The author would like to thank Kin Cheong Sou and Andrey Ghulchak for fruitful discussions. The author also acknowledges the great atmosphere during the theme semester at LTH and LCCC workshops which lead to this publication. Specifically, authors thanks Henrik Sandberg for pointing out the significance of the problem.

V. CONCLUSION

A linear system approximation method in the ν-gap metric is presented in the paper. Such a method may be very useful for modeling of structured or multi-agent systems. Approximation is obtained using semidefinite programming and a normalized coprime factorization of the original model. The method can be applied to controller reduction with taking into account the performance of closed loop.

Future work on the method includes comparison with the method from [15], which uses the LMI formulation of the problem and the state-space data of the systems. It also relies on an iterative procedure to obtain a solution. The second direction of the future work is a MIMO extension using the techniques in the mentioned [9], [16].

The current algorithm has some advantages comparing to [15] regardless of the future comparison. Incorporating extra constraints, adding frequency-depended weights and/or restricting the objective to a specific frequency interval is straight-forward using the frequency domain representation.

REFERENCES

[1] B. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *Automatic Control, IEEE Transactions on*, 26(1):17 – 32, feb. 1981.

[2] Keith Glover. All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds. *Control, International Journal*, 39:1115–1193, 1984.

[3] A. C. Antoulas, D. C. Sorensen, and S. Gugercin. A survey of model reduction methods for large-scale systems. *Contemporary Mathematics*, 280:193–219, 2001.

[4] Roland W. Freund. Model reduction methods based on Krylov subspaces. *Acta Numerica*, 12:267–319, january 2003.

[5] S. Gugercin, A. C. Antoulas, and C. Beattie. $\mathcal{H}_2$ model reduction for large-scale linear dynamical systems. *SIAM Journal on Matrix Analysis and Applications*, 30(2):609–638, 2008.

[6] Athanasios C. Antoulas. *Approximation of Large-Scale Dynamical Systems (Advances in Design and Control) (Advances in Design and Control)*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2005.

[7] Kin Cheong Sou, A. Megretski, and L. Daniel. A quasi-convex optimization approach to parameterized model order reduction. *Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on*, 27(3):456–469, March 2008.

[8] Aivar Sootla. Hankel-type model reduction based on frequency response matching. In *Proceedings of 49th Conference on Decision and Control*, Atlanta, GA, December 2010.

[9] Aivar Sootla, Anders Rantzer, and Georgios Kotsalis. Multivariable optimization-based model reduction. *Automatic Control, IEEE Transactions on*, 54(10):2477–2480, Oct. 2009.

[10] George Zames and Ahmed K. El Sakkary. Unstable systems and feedback: The gap metric. In *Proc. Allerton Conf*, pages 380–385, 1980.

[11] M. Vidyasagar. The graph metric for unstable plants and robustness estimates for feedback stability. *Automatic Control, IEEE Transactions on*, 29(5):403 – 418, May 1984.

[12] G. Vinnicombe. Frequency domain uncertainty and the graph topology. *Automatic Control, IEEE Transactions on*, 38(9):1371 –1383, sep. 1993.

[13] Glen Vinnicombe. *Measuring Robustness of Feedback Systems*. PhD thesis, University of Cambridge, Dept. of Engineering, 1993.

[14] M. Cantoni. On model reduction in the ν-gap metric. In *Decision and Control, 2001. Proceedings of the 40th IEEE Conference on*, volume 4, pages 3665 –3670 vol.4, 2001.

[15] G. Buskes and M. Cantoni. Reduced order approximation in the ν-gap metric. In *Decision and Control, 2007 46th IEEE Conference on*, pages 4367 –4372, 2007.

[16] Aivar Sootla and Kin Cheong Sou. Frequency domain model reduction method for parameter-dependent systems. In *Proceedings of the American Control Conference*, July 2010.

[17] Kemin Zhou, John C. Doyle, and Keith Glover. *Robust and optimal control*. Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1996.

[18] Karl Johan Åström and Richard M. Murray. *Feedback Systems: An Introduction for Scientists and Engineers*. Princeton University Press, Princeton, NJ, USA, 2008.

[19] Didier Henrion, Micheal Šebek, and Vladimir Kučera. Positive polynomials and robust stabilization with fixed-order controllers. *Automatic Control, IEEE Transactions on*, 48(7):1178 – 1186, jul. 2003.

[20] Fuwen Yang, Mahbub Gani, and Didier Henrion. Fixed-order robust H_∞ controller design with regional pole assignment. *Automatic Control, IEEE Transactions on*, 52(10):1959 –1963, oct. 2007.

[21] Glen Vinnicombe. *Uncertainty and Feedback, H_∞ Loop-Shaping and the ν-Gap Metric*. World Scientific Publishing Company, 2000

[22] Anders Rantzer. On the Kalman-Yakubovich-Popov lemma. *Syst. Control Lett.*, 28(1):7–10, 1996.

[23] Jos F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12:625–653, 1999.

[24] R. H. Tütüncü, K. C. Toh, and M. J. Todd. SDPT3 - a MATLAB software package for semidefinite-quadratic-linear programming, version 3.0. *Optimization Methods and Software*, 11:545–581, 1999.

[25] Johan Löfberg. Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[26] Tarek Moselhy, Xin Hu, and Luca Daniel. pFFT in FastMaxwell: a fast impedance extraction solver for 3D conductor structures over substrate. In *DATE '07: Proceedings of the conference on Design, automation and test in Europe*, pages 1194–1199, San Jose, CA, USA, 2007. EDA Consortium.

[27] Zhenhai Zhu, Ben Song, and Jacob White. Algorithms in FastImp: a fast and wideband impedance extraction program for complicated 3-D geometries. In *Proceedings of the 40th annual Design Automation Conference*, pages 712–717, New York, NY, USA, 2003. ACM.

[28] Younes Chahlaoui and Paul Van Dooren. A collection of benchmark examples for model reduction of linear time invariant dynamical systems. Slicot working note 2002-2, Univesité Catholique de Louvain, February 2002.

[29] G. Obinata and B.D.O. Anderson. *Model Reduction for Control System Design*. London,UK:Springer-Verlag, 2001.

[30] K. Glover and D. C. McFarlane. *Robust Controller Design Using Normalized Coprime Factor Plant Descriptions*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 1989.

[31] Olof Garpinger. Design of robust PID controllers with constrained control signal activity. Licentiate Thesis LUTFD2/TFRT--3245--SE, Department of Automatic Control, Lund University, Sweden, March 2009.