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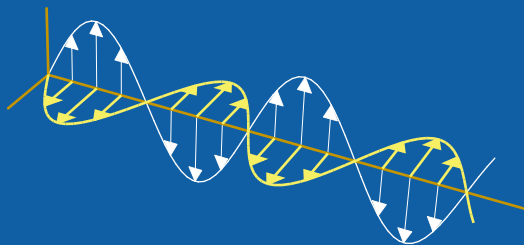1

Reconstruction of nonlinear material properties for homogeneous, isotropic slabs using electromagnetic waves

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Abstract

This paper addresses the inverse problem of reconstructing a medium's instantaneous, nonlinear response to electromagnetic excitation. Using reflection and transmission data for an almost arbitrary incident field on a homogeneous slab, we are able to obtain the nonlinear constitutive relations for both electric and magnetic fields, with virtually no assumptions made on the specific form of the relations. It is shown that for a nonmagnetic material, reflection data suffices to obtain the electrical nonlinear response. We also show that the algorithms are well posed. Numerical examples illustrate the analysis presented in this paper.

1 Introduction

There has been an increased interest in nonlinear electromagnetic materials recently, much powered by the progresses in nonlinear optics. Especially the nonlinear effects in optical fibers have been a great inspiration, *i.e.*, the experimental verification of soliton solutions [13, 14, 21], and the use of different field-dependent scattering mechanisms for amplification of a propagating signal [1]. Some chaotic effects have also been studied, as in [10].

The research in this field is largely conducted in the frequency domain, where the nonlinearities manifest in the generation of multiple frequencies. In this paper, we study nonlinear effects in the time domain, where the nonlinearities rather cause a change of shape in a propagating pulse. This change of shape may ultimately turn into a shock solution, where the pulse becomes discontinuous after a finite propagation time, although this problem is not addressed in this paper.

We study a material which has an instantaneous, nonlinear response, *i.e.*, we do not consider memory effects of any kind. We further assume the material to be passive, isotropic and homogeneous, and solve the problem of reconstructing the constitutive relations. Then we are able to reconstruct the nonlinear relation between $\boldsymbol{E}$ and $\boldsymbol{D}$ as well as between $\boldsymbol{H}$ and $\boldsymbol{B}$ with reflection and transmission data from a finite slab for an (almost) arbitrary input signal. Since no further assumptiosn have to be made regarding the specific form of the constitutive relations, the reconstruction is model independent.

Previous work in the field include the propagation of pulses in nonlinear slabs, where the paper by Kazakia and Venkataraman deserves special attention [18]. They have obtained an analytical solution for the propagation of a step function through a slab with some special constitutive functions. The wave propagation in more complicated nonlinear materials have appeared, *i.e.*, mixed nonlinearities [19], bi-anisotropic and bi-isotropic media [5], and nonlinearities in chiral media [2, 23].

Though much work has been done on the direct problem of wave propagation, our solution of the inverse problem of reconstructing the material seems to be novel. It extends and improves the results in [20], where the inverse problem is solved for a nonmagnetic material, based on measurements inside the material.

In Section 2 we formulate the stratified Maxwell equations, introduce the constitutive relations for the studied materials and try to interpret the dynamics in terms

known from the linear case. The main theory is contained in Section 3, where we formulate the necessary boundary conditions and state the solution to our inverse problems. Some numerical results are contained in Section 4.

2 Prerequisites

2.1 The Maxwell equations in one spatial dimension

In a source-free environment the Maxwell equations are

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) + \partial_t \boldsymbol{B}(\boldsymbol{r}, t) = 0$$
$$\nabla \times \boldsymbol{H}(\boldsymbol{r}, t) - \partial_t \boldsymbol{D}(\boldsymbol{r}, t) = 0.$$

Since we wish to study a homogeneous medium, it is sufficient to observe variations for only one direction. We thus assume that the fields depend on only one spatial variable, say z, in a Cartesian coordinate system (x, y, z). Then the curl operator can be written $\nabla \times = \hat{z} \times \mathbf{I} \partial_z = \mathbf{J} \partial_z$, where $\mathbf{J}$ denotes a rotation $\pi/2$ around the z-axis, and the Maxwell equations become

$$\mathbf{J} \cdot \partial_z \boldsymbol{E}(z, t) + \partial_t \boldsymbol{B}(z, t) = 0$$
$$\mathbf{J} \cdot \partial_z \boldsymbol{H}(z, t) - \partial_t \boldsymbol{D}(z, t) = 0.$$

We now assume the fields to be linearly polarized and the material to be isotropic, i.e., the $\boldsymbol{D}$ and $\boldsymbol{B}$ fields are parallell to the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields, respectively, which vary only in amplitude. This means we can write the Maxwell equations in a scalar form,[1]

$$\partial_z E(z, t) + \partial_t B(z, t) = 0$$
$$\partial_z H(z, t) + \partial_t D(z, t) = 0,$$

where E, B, H and D denotes the field amplitudes (which may be negative). The geometry of the scattering situation studied in this paper is depicted in Figure 1.

2.2 Constitutive relations, passive materials

We consider the field strengths E and H to be the primary fields, and the flux densities D and B as effects of these. If we assume that the material responds instantaneous to excitation, we are studying the following situation:

$$D(z, t) = \varepsilon_0 F_{\mathrm{e}}(E(z, t))$$
$$B(z, t) = \frac{1}{c_0} F_{\mathrm{m}}(\eta_0 H(z, t)),$$

where the constants c_0 (speed of light in vacuum), ε_0 (permittivity of vacuum), and η_0 (wave impedance of vacuum) are explicit for convenience. The functions $F_{\mathrm{e}}(E)$

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[1]We can also choose the equations $\partial_z E - \partial_t B = 0$ and $\partial_z H - \partial_t D = 0$, but it changes nothing.
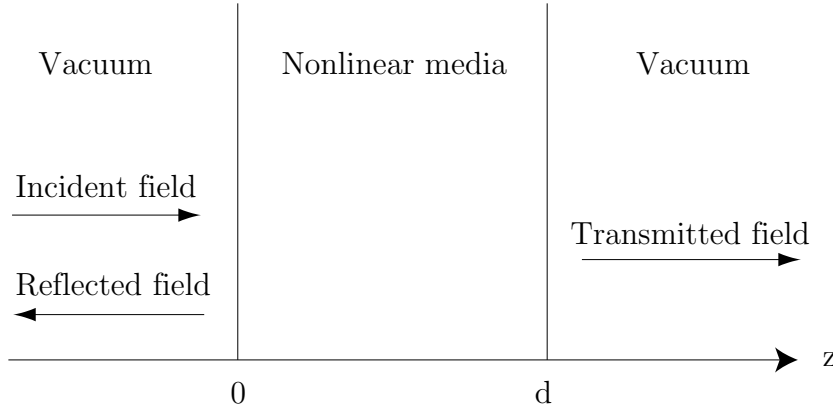
Figure 1: The scattering geometry studied in this paper.

and $F_{\mathrm{m}}(\eta_0 H)$ are generalizations of the linear optical responses, $\varepsilon_r E$ and $\mu_r \eta_0 H$, respectively. This type of nonlinear constitutive response with a similar dynamic is investigated in [3, 11, 20]. In nonlinear optics similar relations are often used, although frequently in the context of the frequency domain, see for instance [1, 4].

Some thermodynamic restrictions can be put on the constitutive relations, see [6], but these deal mainly with the symmetry of cross terms, which we do not take into account here. Reference [20] discusses the restrictions on the functions F_{e} and F_{m} in order to model passive media, though they call it dissipative.[2] The result is that for a passive, nonmagnetic material, $F_{\mathrm{e}}'(x) \geq a > 0$ is a *sufficient* condition. In this paper we generalize this to materials which also have $F_{\mathrm{m}}'(x) \geq b > 0$, and call these *positive passive*.

When demanding isotropy, we have the implication that a change of sign in the electric and magnetic fields leads to a change of sign in the electric and magnetic fluxes, *i.e.*, $(E, H) \rightarrow (-E, -H) \Rightarrow (D, B) \rightarrow (-D, -B)$. This is also true for crystals with an *inversion symmetry*, see [4] for further discussions on symmetries. This property implies that the constitutive functions should be odd, which will be important in the future.

Eliminating the D and B fields using the constitutive relations, the scalar Maxwell equations become

$$\begin{cases} \partial_z E + \dfrac{1}{c_0} F_{\mathrm{m}}'(\eta_0 H)\partial_t \eta_0 H = 0 \\ \partial_z \eta_0 H + \eta_0 \varepsilon_0 F_{\mathrm{e}}'(E)\partial_t E = 0 \end{cases} \Rightarrow \begin{cases} \partial_z E + \dfrac{1}{c_0} F_{\mathrm{m}}'\partial_t \eta_0 H = 0 \\ \partial_z \eta_0 H + \dfrac{1}{c_0} F_{\mathrm{e}}'\partial_t E = 0, \end{cases} \tag{2.1}$$

where we have dropped the arguments for simplicity.

---

[2]With a passive material we mean that the electromagnetic energy produced in a region is nonpositive for all times, *i.e.*, the material is not active.

2.3 The dynamics as a symmetric system, physical interpretation

Though it is possible to directly introduce the well known Riemann invariants $\frac{1}{2}(\int_0^E \sqrt{F_e'(x)}\,dx \pm \int_0^{\eta_0 H} \sqrt{F_m'(x)}\,dx)$ as in [3, 11], we wish to follow a different approach, where we try to interprete our variables and make comparisons to the linear case. We start by formulating the dynamics as

$$\begin{pmatrix} F_e'\partial_t E \\ F_m'\partial_t \eta_0 H \end{pmatrix} + c_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_z \begin{pmatrix} E \\ \eta_0 H \end{pmatrix} = 0,$$

which after division by the square root of the derivative of F_e and F_m leads to

$$\begin{pmatrix} \sqrt{F_e'}\partial_t E \\ \sqrt{F_m'}\partial_t \eta_0 H \end{pmatrix} + c_0 \begin{pmatrix} 0 & \frac{1}{\sqrt{F_e'F_m'}} \\ \frac{1}{\sqrt{F_e'F_m'}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{F_e'}\partial_z E \\ \sqrt{F_m'}\partial_z \eta_0 H \end{pmatrix} = 0.$$

We now introduce the functions,

$$g_e(E) = \int_0^E \sqrt{F_e'(x)}\,dx$$

$$g_m(\eta_0 H) = \int_0^{\eta_0 H} \sqrt{F_m'(x)}\,dx.$$

These functions can be thought of as the generalizations of the linear expressions $\sqrt{\varepsilon_r}E$ and $\sqrt{\mu_r}\eta_0 H$. The product of the derivative of the functions, $g_e'g_m'$, which appears in the wave speed below, can be viewed as the generalization of $\sqrt{\varepsilon_r \mu_r}$, the relative refractive index. Furthermore, for an isotropic, positive passive material, the g-functions are odd and monotonous, since the integrands are always even and positive. With these functions we can write the dynamics as

$$\partial_t \begin{pmatrix} g_e(E) \\ g_m(\eta_0 H) \end{pmatrix} + \frac{c_0}{g_e'(E)g_m'(\eta_0 H)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_z \begin{pmatrix} g_e(E) \\ g_m(\eta_0 H) \end{pmatrix} = 0,$$

which in the new variables $u_1 = g_e(E)$ and $u_2 = g_m(\eta_0 H)$ is the symmetric system

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + c(u_1, u_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_z \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0, \tag{2.2}$$

where the wave speed c is

$$c(u_1, u_2) = \frac{c_0}{g_e'(g_e^{-1}(u_1))g_m'(g_m^{-1}(u_2))} = c_0 \left(\frac{d}{du_1} g_e^{-1}(u_1) \right) \left(\frac{d}{du_2} g_m^{-1}(u_2) \right). \tag{2.3}$$

This result generalizes the nonmagnetic case given in Reference [20].

3 Methods to solve the inverse problem

In this section we demonstrate the methods used to solve the propagation problem and to resolve the boundary conditions. We also state our inverse problems of reconstructing the materials constitutive relations.

3.1 Wave splitting

The symmetric system (2.2) can be written as a system of one-dimensional wave equations with the wave splitting, see References [8, 9, 20],

$$
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
$$

This change of variables is exactly the introduction of the Riemann invariants of the one-dimensional Maxwell equations, which was mentioned in Section 2.3. The dynamics (2.2) now becomes

$$
\partial_t \begin{pmatrix} u^+ \\ u^- \end{pmatrix} + c(u^+ + u^-, u^+ - u^-) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_z \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = 0, \qquad (3.1)
$$

with c defined by (2.3). This is a system of one-dimensional wave equations, which couple only through the wave speed c.

Analytical solutions for the wave propagation have been found in [18, 22] for some special constitutive relations. These solutions could be used to benchmark an algorithm for the wave propagation, though this is not performed in this work.

3.2 Propagation along characteristics

We can solve the propagation problem of the system (3.1) via the method of characteristics. A characteristic curve for this kind of differential equations is one on which the dependent variables are constant. We study the development of the variables $u^\pm(z,t)$ on the path $(z,t) = (\zeta^\pm(\tau), \tau)$, where $\zeta^\pm(\tau) = \zeta_0 \pm \int_0^\tau c(\boldsymbol{u}') \, d\tau'$. The notation $c(\boldsymbol{u}')$ is short hand for $c(\boldsymbol{u}(\zeta^\pm(\tau'), \tau'))$, and $\boldsymbol{u} = (u^+, u^-)$. The variation of $u^\pm(z,t)$ along these curves are

$$
\frac{d}{d\tau} u^\pm(\zeta^\pm(\tau), \tau) = \frac{\partial u^\pm}{\partial t} + \frac{d\zeta^\pm(\tau)}{d\tau} \frac{\partial u^\pm}{\partial z} = \frac{\partial u^\pm}{\partial t} \pm c(\boldsymbol{u}) \frac{\partial u^\pm}{\partial z} = 0,
$$

since $u^\pm$ satisfy the differential equations $u_t \pm c u_z = 0$. Thus, we conclude that u^+ is constant along the characteristic path $\zeta(\tau) = \zeta_0 + \int_0^\tau c(\boldsymbol{u}') \, d\tau'$, and u^- is constant along the characteristic path $\zeta(\tau) = \zeta_0 - \int_0^\tau c(\boldsymbol{u}') \, d\tau'$.

This means we can find the values of the fields at a point (z,t) if we can trace the characteristics to some boundary where they are known. If only one of the waves is present, it is particularly simple; then the characteristics are straight lines, with a slope given by the boundary values, see for instance [20].

3.3 Boundary conditions

Since we want to study propagation in a nonlinear slab, we must solve the problem of satisfying the boundary conditions. In this paper, we are studying a slab imbedded in vacuum. The generalization to more general linear materials should be obvious from the method used.

The solution is based on the wave splitting, which allows us to determine in which direction the energy of the fields are travelling. In the surrounding vacuum, the splitting is made by definition of incident, reflected and transmitted field. The boundary conditions we have to satisfy are the usual, *i.e.*, continuity of the tangential electrical and magnetic field strengths. Since we are assuming normal incidence, this means continuity of the total fields E and H. Inside the slab, the electric and magnetic fields can be expressed as

$$E_{\text{slab}} = g_{\text{e}}^{-1}(u^+ + u^-)$$
$$\eta_0 H_{\text{slab}} = g_{\text{m}}^{-1}(u^+ - u^-).$$

In vacuum, the magnetic field strength is related to the electric field strength via $\eta_0 H^\pm = \pm E^\pm$, where the $\pm$ indicate right(left) propagating fields.

It is possible to define local reflection and transmission coefficients which look exactly like the linear ones, where the square root of the relative permittivity is replaced by $g_{\text{e}}'(E) = \sqrt{F_{\text{e}}'(E)}$ and the square root of the relative permeability by $g_{\text{m}}'(\eta_0 H) = \sqrt{F_{\text{m}}'(\eta_0 H)}$, see for instance [18]. The problem with this approach is that the fields E and H will depend on the sought reflected and transmitted fields. We prefer to simply state the equations we have to solve for the desired fields.

3.3.1 The left boundary

In vacuum, $z < 0$, we have an incident field from the left E^i, and a reflected field into vacuum, E^r. In the slab two fields are present: a right propagating field u^+, and a left propagating field u^-. The continuity of electric and magnetic fields implies that

$$\begin{cases} E^i + E^r = g_{\text{e}}^{-1}(u^+ + u^-) \\ E^i - E^r = g_{\text{m}}^{-1}(u^+ - u^-) \end{cases} \Leftrightarrow \begin{cases} g_{\text{e}}(E^i + E^r) = u^+ + u^- \\ g_{\text{m}}(E^i - E^r) = u^+ - u^-. \end{cases} \tag{3.2}$$

This gives two, generally nonlinear, equations from which the desired fields u^+ and E^r can be determined:

$$\begin{cases} 2E^i = g_{\text{e}}^{-1}(u^+ + u^-) + g_{\text{m}}^{-1}(u^+ - u^-) \\ 2u^- = g_{\text{e}}(E^i + E^r) - g_{\text{m}}(E^i - E^r). \end{cases}$$

The incident field is, of course, supposed to be known, but also the left propagating field u^- can be thought of as known. This is because this field can be traced back in time via a characteristic curve into the slab, and is therefore, from a computational point of view, known. We have noted earlier that for a positive passive material, the g-functions are monotonous, which means that their inverses are too. The above equations are therefore monotonous in the sought variables, and easy to solve numerically.

3.3.2 The right boundary

At the right boundary, $z = d$, we have just a transmitted field in the vacuum, but we still have both right and left propagating fields in the slab. Continuity of the

fields now gives

$$\begin{cases} E^t = g_e^{-1}(u^+ + u^-) \\ E^t = g_m^{-1}(u^+ - u^-) \end{cases} \Leftrightarrow \begin{cases} g_e(E^t) = u^+ + u^- \\ g_m(E^t) = u^+ - u^-. \end{cases} \tag{3.3}$$

From this we get the following equations to determine u^- and E^t:

$$\begin{cases} 2u^+ = g_e(E^t) + g_m(E^t) \\ 2u^- = g_e(E^t) - g_m(E^t). \end{cases}$$

We can consider the field u^+ as known, since it can be traced back in time into the slab. The same conclusions as above about the solvability of these equations apply here.

3.4 Inverse problems

The objective of this paper is to find methods from which the material properties can be obtained from measurements outside the slab, *i.e.*, the incident, reflected, and transmitted fields.

3.4.1 Reflection

If we can ignore the left-propagating field at the left boundary, *i.e.*, $u^- = 0$, the boundary conditions (3.2) become

$$E^i + E^r = g_e^{-1}(u^+)$$
$$E^i - E^r = g_m^{-1}(u^+).$$

A situation where this approximation applies is a half space or a sufficiently thick slab, in which the reflection from the right boundary, $z = d$, does not appear until after some later time. We thus have a relation between the measurable quantities $E^i + E^r$ and $E^i - E^r$,

$$g_e(E^i + E^r) = g_m(E^i - E^r),$$

and the composite function $g_e^{-1}(g_m(\cdot))$ (or its inverse $g_m^{-1}(g_e(\cdot))$) can be determined. The derivative of this function can be shown to correspond to the wave impedance. $E = \pm g_e^{-1}(g_m(\eta_0 H))$ are the electric fields which combined with $\eta_0 H$ gives a left(right) propagating wave in the slab.

In nonlinear optics, the materials can often be considered as nonmagnetic. This implies $g_m(x) = x$, and we can easily determine the electric response function g_e, from which we get F_e or the wave speed c. The strength of the input signal E^i directly determines which range of F_e we can reconstruct.
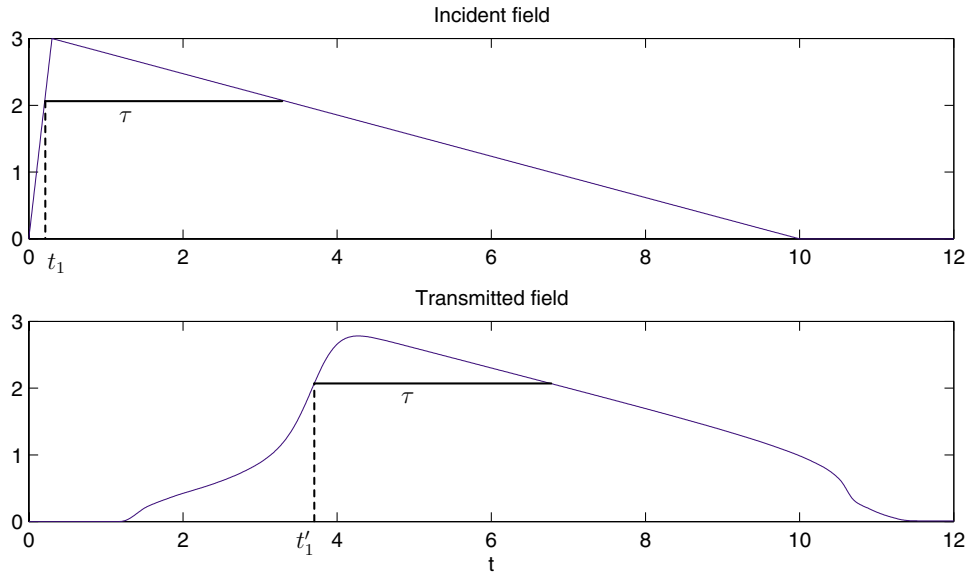
Figure 2: Method for extracting the travel time for different field amplitudes. Since equal amplitudes travel with equal speed, they arrive with the same time separation and the travel time is $t_1' - t_1$.

3.4.2 Transmission

If we neglect the fields that are reflected at the right boundary, $z = d$, we are considering a problem where the wave speed depends on only one variable, and the u^+-fields propagate independently of the u^--fields. This means that the characteristic curves for the right-going fields are straight lines, which can be used to our advantage. Since the left propagating wave induced by an internal reflection is rather small compared with the direct wave, this is an acceptable approximation.

We assume that the right propagating field has a pulse shape of some sort, *i.e.*, for $z = 0+$, there are two times for which the right propagating field u^+ assumes the same value, $u^+(0, t_1) = u^+(0, t_1 + \tau)$. Since the wave speed depends only on u^+ when we neglect left propagating field, these two points will appear with the same time separation on the right side of the slab $(z = d-)$, $u^+(d, t_1') = u^+(d, t_1' + \tau)$. This can be used to find the propagation time, $t_1' - t_1$, and thereby the wave speed $c(u^+)$.

One complication is that we can only measure the fields outside the slab, but using the boundary conditions (3.2) and (3.3),

$$\begin{cases} 2u^+ = g_e(E^i + E^r) + g_m(E^i - E^r) \\ 2u^+ = g_e(E^t) + g_m(E^t), \end{cases}$$

we find that there is a one-to-one correspondence between the incident field strength and the u^+-level, and between the transmitted field strength and the u^+-level. This means that if $E^i(t_1) = E^i(t_1+\tau)$, then there is a time t_1' for which $E^t(t_1') = E^t(t_1'+\tau)$, and we have found our transmission time $t_1' - t_1$.

In other words, we take a segment of a certain length of the time axis, and fit this into the curves $E^i(t)$ and $E^t(t)$. The time difference between the fits is the travel time for this particular amplitude, see Figure 2. This does not work with shock solutions, but the only consequence is that we cannot get any information on the travel time for the amplitudes over which the shock occurs.

We have the following relationships determined by reflection data and transmission time:

$$E^i + E^r = g_{\mathrm{e}}^{-1}(g_{\mathrm{m}}(E^i - E^r))$$
$$c(E^i + E^r, E^i - E^r) = \frac{c_0}{g_{\mathrm{e}}'(E^i + E^r)g_{\mathrm{m}}'(E^i - E^r)}.$$

If we denote the measurable quantities $E^i + E^r$ and $E^i - E^r$ by e and h, we have the experimentally determined functions

$$e(h) = g_{\mathrm{e}}^{-1}(g_{\mathrm{m}}(h))$$
$$c(e, h) = \frac{c_0}{g_{\mathrm{e}}'(e)g_{\mathrm{m}}'(h)}.$$

The derivative of e with respect to h is $\frac{de}{dh} = \frac{g_{\mathrm{m}}'(h)}{g_{\mathrm{e}}'(e)}$, corresponding to the wave impedance. We can thus find $g_{\mathrm{e}}'(e)^2 = F_{\mathrm{e}}'(e)$ and $g_{\mathrm{m}}'(h)^2 = F_{\mathrm{m}}'(h)$ by combining these relations:

$$F_{\mathrm{e}}'(e) = \frac{c_0 \frac{dh}{de}}{c(e, h(e))} \quad \Rightarrow \quad F_{\mathrm{e}}(e) = \int_0^{h(e)} \frac{c_0 dh'}{c(e(h'), h')}$$
$$F_{\mathrm{m}}'(h) = \frac{c_0 \frac{de}{dh}}{c(e(h), h)} \quad \Rightarrow \quad F_{\mathrm{m}}(h) = \int_0^{e(h)} \frac{c_0 de'}{c(e', h(e'))}.$$

From these expressions we conclude that there is a one-to-one correspondence between $F_{\mathrm{e,m}}$ and $c(e, h)$ once the relation between e and h is given. Since this is given by $g_{\mathrm{e}}(e) - g_{\mathrm{m}}(h) = 0$, and $g_{\mathrm{e,m}}$ are monotonous functions, this is a one-to-one relation. With shockfree propagation of a pulseshaped signal, the transmitted signal should also be pulseshaped, see *e.g.*, the example in Figure 2. Then the wavespeed $c(e, h(e)) = c(e(h), h)$ must be unique, and we conclude that the reconstructed functions are unique, always exist, and depend continuously on the data. Thus the algorithm is well posed.

3.5 Implementation of the forward problem

In order to obtain the reflected and transmitted fields from the slab, an FDTD-algorithm in MATLAB has been used. The algorithm is based on interpolating the wave speed and fields between two neighboring points in the grid with a linear function, and tracing the characteristics back one time step, see Figure 3. The tracing is made by searching for the point in the grid for which the interpolated wave speed points to the new grid point. The method is described in [12].

This method does not handle discontinuous solutions, but rather smears the discontinuity over 10-20 grid points. Since we never use shock solutions in our
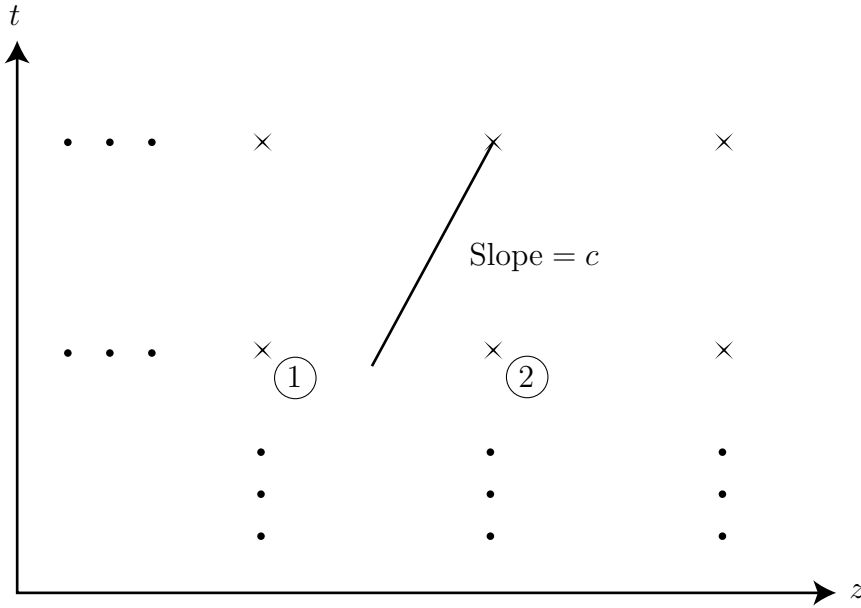
Figure 3:   Backtracing of characteristics in the calculation grid of the forward problem. The new u^+-value is linearly interpolated from the two grid points between which the characteristic crosses the previous time step. With a suitable choice of Δt and Δz these are always the two points 1 and 2 shown.

reconstruction algorithm, this is not a problem. When tested, the travel times for the shocks seem to be correct, though.

The spacetime is scaled by the wave speed in vacuum, so that we may consider a dimensionless spacetime with a wave speed in vacuum equal to one. The slab studied has thickness 1, and is discretized with 100 grid points. The step size in time is chosen the same as that for space. This guarantees that when tracing the characteristic back in time, we stay within the nearest grid points in space. The electric and magnetic fields are scaled with a factor such that the nonlinear effects are appreciably strong with numerical field strengths of a few units. This means that some constants multiplying the fields in the constitutive relations have to be scaled also, see Reference [20].

4   Numerical results

4.1   Reflection

When implementing this reconstruction, it is difficult not committing the inverse crime, *i.e.*, using the same algorithm for both simulating data and reconstructing the constitutive functions, leading to a perfect match, see Reference [7].

It is therefore meaningless to present *any* results for reconstruction with pure reflection data, unless some measured data is available, which is not the case at the present time. The reconstruction is anyway used in the transmission reconstruction, where we get good results.

4.2 Transmission

Simulations have been run, giving reflection and transmission data for a given input signal. When using our reconstructing algorithm, it is vital to construct an input signal which does not evolve into a shock. Since the shocks form at the back edge of a pulse [20], this is achieved by using an incident field which rises very rapidly and decays very slowly. Moreover, if the pulse rises fast enough so that its peak value is obtained before the reflected field at the back has returned, we also get an exact map of the relation $g_e(E) = g_m(\eta_0 H)$, since then $g_e(E) - g_m(\eta_0 H) = u^- = 0$.

It should be stressed that it is not necessary to make the measurement of reflected and transmitted fields simultaneously. This is because the reflected field is only used to establish the relation between incident and reflected fields.

The reconstruction is based on neglecting the field reflected from the back edge. To investigate the validity of this approximation the following test has been made. The left propagating field was neglected in the solution of the forward problem, which gives perfect reconstruction. Then we used the full forward problem, and the reconstruction was comparable to the first case. This shows that our approximation of the travel time is good, at least for the materials studied in this paper. In Appendix A we show that the approximation is good provided the nonlinearity is weak, *i.e.*, if the second derivative of the constitutive relations is small.

Figures 4 and 5 show the calculated fields and the reconstructed constitutive functions for a material with a nonlinear saturated Kerr effect, *i.e.*, a material which saturates as a linear material at some field strength [20]. The fields are calculated using the full forward problem, *i.e.*, the left propagating field u^- in the slab is present. In Appendix A we show that the relative error in travel time for these fields is less than 10^{-3}.

5 Discussion and conclusions

It has been shown that it is possible to reconstruct the constitutive functions of a nonlinear slab, with the help of reflection and transmission data, not necessarily measured simultaneously. The algorithm is based on the fact that equal amplitudes travel with almost equal and constant speeds. When one of the constitutive functions is known, for instance for a nonmagnetic material, the other function is obtained with reflection data only. The algorithms seem to be robust and simple, and may be useful for measuring instantaneous nonlinear effects, with virtually no assumptions made on the specific form of the constitutive function, *i.e.*, the inverse algorithm is model independent.

Since the algorithm is based on shock free propagation, it is necessary to construct a suitable input signal. When measuring reflected and transmitted field simultaneously, the input signal should rise fast enough so that its maximum is reached before the first reflection from the back boundary turns up, and then decrease slow enough not to create a shock in the transmitted field. This may be a difficult field to create.

An interesting fact is that it is conceivable to have a material with nonlinear
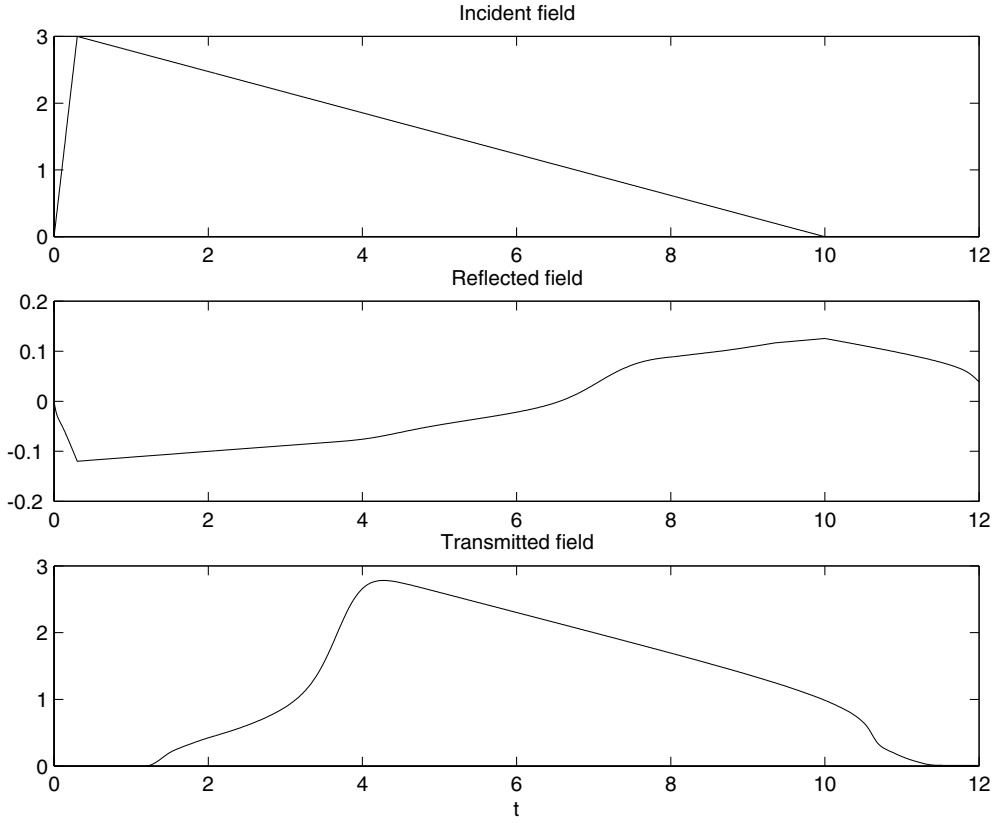
Figure 4: Incident, reflected and transmitted fields. $F_{\mathrm{e}}(E) = 1.5E + 2\frac{E^3}{E^2+1}$, $F_{\mathrm{m}}(H) = H + 2\frac{H^3}{H^2+1}$, *i.e.*, we have a saturated nonlinearity in both fields.

behavior in both electric and magnetic fields. If the media changes from being dominantly electric to being dominantly magnetic, or vice versa, we may get a very small reflection for a very strong incident wave. This might have some implications on the theory of nonreflecting materials, or provide a new kind of electric shutter.
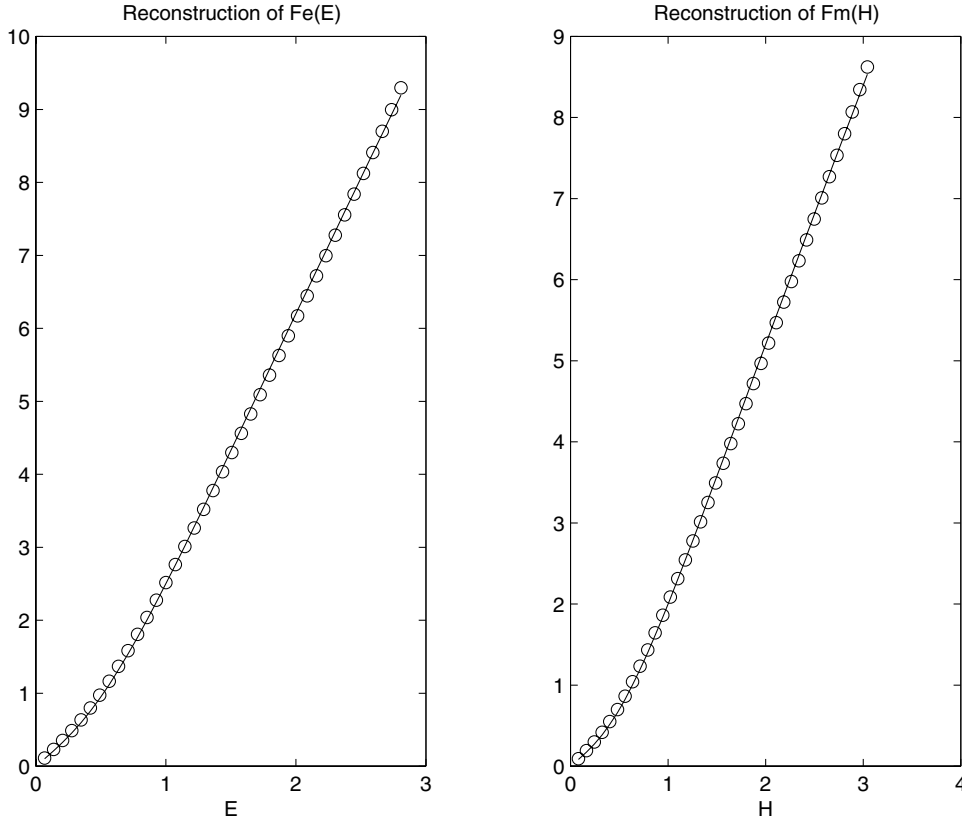
6   Acknowledgement

Appendix A   Estimate of error in travel time

To simplify the notation in this section, we will write H instead of $\eta_0 H$. When using the approximation of straight characteristics, we assume that the travel time can be written

$$\tau_1 = \frac{d}{c(u^+)} = \frac{d}{c(u^+, 0)} = \frac{d}{c_0}\left.\sqrt{F_{\mathrm{e}}'(E)F_{\mathrm{m}}'(H)}\right|_{g_{\mathrm{e}}(E)-g_{\mathrm{m}}(H)=0}.$$

Figure 5: Reconstructed functions, from the fields in figure 4. The circles are the reconstructed values, and the solid lines are the true functions.

The electric and magnetic fields are determined from reflection data, and can under suitable conditions be the fields that do not contain a left-going field in the slab. This is the condition $g_e(E) - g_m(H) = u^- = 0$. The *true* travel time however, is the integral along the characteristic

$$\tau_2 = \int_0^d \frac{dz'}{c(u^+, u^-)}.$$

We now wish to expand this expression in terms of the left-going amplitude u^-, since this should be small in comparison with the right-going amplitude u^+. This expansion is then compared with the travel time τ_1. We have

$$\frac{1}{c(u^+, u^-)} = \frac{1}{c(u^+, 0)} + \frac{\partial}{\partial u^-} \frac{1}{c(u^+, u^-)}\bigg|_{u^-=0} \cdot u^- + O((u^-)^2).$$

The derivative of the slowness is proportional to the derivative of $\sqrt{F'_e(E)F'_m(H)}$,

$$\frac{\partial}{\partial u^-} \sqrt{F'_e(E)F'_m(H)} = \frac{F''_e(E)}{2\sqrt{F'_e(E)}}\sqrt{F'_m(H)}\frac{dE}{du^-} + \sqrt{F'_e(E)}\frac{F''_m(H)}{2\sqrt{F'_m(H)}}\frac{dH}{du^-}$$

$$= \frac{F''_e(E)}{2F'_e(E)}\sqrt{F'_m(H)} - \sqrt{F'_e(E)}\frac{F''_m(H)}{2F'_m(H)},$$

where the last line follows from $E = g_e^{-1}(u^+ + u^-)$ and $H = g_m^{-1}(u^+ - u^-)$. This means that if the second derivatives of the constitutive relations are small, the error in travel time is guaranteed to be small. This corresponds to the *weak nonlinearity* approximation. From the relation $g_e(E) - g_m(H) = 0$ we have

$$\frac{dE}{dH} = -\frac{-g_m'(H)}{g_e'(E)} = \frac{\sqrt{F_m'(H)}}{\sqrt{F_e'(E)}}.$$

which helps us simplify the above expression to

$$\frac{1}{2}\sqrt{F_e'(E)}\left(\frac{dE}{dH}\frac{d}{dE}\ln F_e'(E) - \frac{d}{dH}\ln F_m'(H)\right) = \frac{1}{2}\sqrt{F_e'(E)}\frac{d}{dH}\ln\frac{F_e'(E)}{F_m'(H)}$$

$$= \sqrt{F_e'(E)}\frac{d}{dH}\ln\frac{dH}{dE} = \sqrt{F_e'(E)}\frac{1}{\frac{dH}{dE}}\frac{d}{dH}\frac{dH}{dE}$$

$$= \sqrt{F_e'(E)}\frac{dE}{dH}\frac{d}{dH}\frac{dH}{dE} = \sqrt{F_m'(H)}\frac{d}{dH}\frac{dH}{dE}.$$

This can be written in various ways, which all involve a twice differentiated relation between E and H. The error in travel time is

$$\tau_2 - \tau_1 = \int_0^d \frac{dz'}{c(u^+, u^-)} - \frac{d}{c(u^+)} = \int_0^d \left(\frac{1}{c(u^+, u^-)} - \frac{1}{c(u^+)}\right) dz',$$

and with our expansion of $1/c$ and the fact that u^+ is constant on its characteristic curve, we have the estimate

$$\frac{c_0}{d}|\tau_2 - \tau_1| < \left|\sqrt{F_m'(H)}\frac{d}{dH}\frac{dH}{dE}\right| \cdot |u^-|_{max},$$

which is interpreted as an error of travel time relative to propagation in vacuum. This is dimensionless and constitutes an upper bound on the relative error. A typical field strength in Figure 4 is $E = 2.00$, which implies $H = 2.17$. From the figure we conclude a maximum left propagating field strength of about 0.1, and we have a relative error of less than $0.0085 \cdot 0.1 < 10^{-3}$. Thus, we see that the error made when using the full forward problem, *i.e.*, using real data, is very small.

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