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Gattami, Ather; Bernhardsson, Bo

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Ather Gattami, Bo Bernhardsson
Department of Automatic Control
Lund University
Box 118, SE-221 00 Lund, Sweden
E-mail: ather@control.lth.se

Abstract— **We consider the problem of distributed decision making in a quadratic game between a *team* of players and nature. Each player has limited information that could be different from the other players in the team. We show that if there is a solution to the minimax team problem, then the linear policies are optimal, and we show how to find the linear optimal solution by solving a linear matrix inequality. The result is used to solve the distributed $\mathcal{H}_\infty$ control problem. It shows that information exchange with neighbours on the graph only, is enough to obtain a linear optimal policy.**

I. INTRODUCTION

We consider the problem of static minimax team decision. A team of players are to optimize a worst case scenario given limited information of nature's decision for each player. The problem can be considered as the *deterministic* analog of the stochastic team decision problems that were solved by Radner [7].

An initial step for solving the static deterministic problem was made by Didinsky and Basar [3], where they consider a team of two players using a stochastic framework. The solution given in [3] cannot easily be extended to more than two players, since it uses *common* information for the two players, a property that does not necessarily exist for more than two players. Also, the one step delay $\mathcal{H}_\infty$ control problem is solved in [3].

In this paper, we solve the static minimax (or deterministic) team decision problem completely for an arbitrary number of players, and show that the optimal solution is linear and can be found by solving a linear matrix inequality. Also, we show how to solve the dynamic finite-horizon $\mathcal{H}_\infty$ control problem, under some conditions that prevent *signaling*, which is analogous to the distributed finite-horizon stochastic LQG problem treated in Ho and Chu [4] and its generalization in Gattami [6]. For the infinite-horizon problem, similar conditions were obtained in [1] and [8]. We show that the information structure where subsystems on a graph are restricted to exchange information with neighbours only, is enough to obtain an optimal feedback law which turns out to be linear. This reveals a broader class of information structures that lead to tractable problems.

II. NOTATION

For a vector v, we denote the ith block component of v by v_i. The set of $n \times n$ symmetric matrices is denoted by $\mathbb{S}^n$. The pseudo-inverse of a matrix A is denoted by $A^\dagger$. We write $A \succeq 0$ ($A \succ 0$) to denote that A is positive semi-definite

(positive definite). For matrix A partitioned symmetrically in blocks, we denote the block in position (i, j) of A by $[A]_{ij}$.

III. THE MINIMAX TEAM DECISION PROBLEM

Consider the following team decision problem

$$\inf_\mu \sup_{x \neq 0} \frac{J(x, u)}{||x||^2}$$
$$\text{subject to} \quad y_i = C_i x \qquad\qquad (1)$$
$$u_i = \mu_i(y_i)$$
$$\text{for } i = 1, ..., N$$

where $u_i \in \mathbb{R}^{m_i}$, $m = m_1 + \cdots + m_N$, $C_i \in \mathbb{R}^{p_i \times n}$. $J(x, u)$ is a quadratic cost given by

$$J(x, u) = \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

where

$$\begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \in \mathbb{S}^{m+n}.$$

We will be interested in the case $Q_{uu} \succ 0$ (this can be generalized to $Q_{uu} \succeq 0$, but the presentation of the paper becomes more technical). The players $u_1, ..., u_N$ make up a *team*, which plays against *nature* represented by the vector x, using $\mu(x) = \left(\mu_1^T(C_1 x), \cdots, \mu_N^T(C_N x) \right)^T$.

Proposition 1: The value of the game in (1) is γ^* if and only if for any $\epsilon > 0$ there is a decision μ_ϵ such that

$$\gamma^* \leq \sup_{x \neq 0} \frac{J(x, \mu_\epsilon(x))}{||x||^2} < \gamma^* + \epsilon.$$

Proof: The statement follows immediately from the definition of the infimum. ∎

Proposition (1) shows that if γ^* is the value of the game in (1), then for any given real number $\gamma > \gamma^*$, there exists a policy μ such that $J(x, \mu(x)) - \gamma ||x||^2 \leq 0$ for all x. Hence, we can formulate the alternative team decision problem:

$$\inf_\mu \sup_{x \neq 0} J(x, u) - \gamma ||x||^2 \leq 0$$
$$\text{subject to} \quad y_i = C_i x \qquad\qquad (2)$$
$$u_i = \mu_i(y_i)$$
$$\text{for } i = 1, ..., N$$

The formulation above can be seen as the problem of looking for suboptimal solutions to the game given by (1). Clearly,

Proposition 1 shows that the value of the game resulting from the decision obtained in (2) approaches the optimal value in (1) as γ approaches γ^* (or as $\epsilon \to 0$). From now on we will consider the equivalent game given by (2). Introduce the matrix

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}.$$

C is a $p \times n$ matrix, where $p = p_1 + p_2 + \cdots + p_N$. For any given vector y, a vector x with $y = Cx$ can be written as $x = C^\dagger y + (I - C^\dagger C)\tilde{y}$, for some $\tilde{y}$. $\tilde{x} = (I - C^\dagger C)\tilde{y}$ can be seen as the unobservable part of x from the vector y, and $\hat{x} = C^\dagger y$ is the observable part. We will now show how to eliminate the unobservable part of x from our problem. Define

$$Q_\gamma = \begin{bmatrix} Q_{xx} - \gamma I & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix}, \tag{3}$$

and let V be given by

$$V = \begin{bmatrix} I - C^\dagger C & C^\dagger & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{4}$$

Then,

$$\begin{aligned} J(x,u) - \gamma\|x\|^2 &= \begin{bmatrix} x \\ u \end{bmatrix}^T Q_\gamma \begin{bmatrix} x \\ u \end{bmatrix} \\ &= \begin{bmatrix} C^\dagger y + (I - C^\dagger C)\tilde{y} \\ u \end{bmatrix}^T Q_\gamma \begin{bmatrix} C^\dagger y + (I - C^\dagger C)\tilde{y} \\ u \end{bmatrix} \\ &= \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix}^T V^T Q_\gamma V \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix}. \end{aligned} \tag{5}$$

Let $V^T Q_\gamma V$ be partitioned as

$$V^T Q_\gamma V = Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix}, \tag{6}$$

$$Z_{11} \in \mathbb{R}^{(n-p)\times(n-p)}, \quad Z_{22} \in \mathbb{R}^{p\times p}, \quad Z_{33} \in \mathbb{R}^{m\times m}.$$

Thus, we have

$$\begin{bmatrix} x \\ u \end{bmatrix}^T Q_\gamma \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix}^T Z \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix}.$$

Then, the game (2) can be equivalently formulated as

$$\inf_{\mu} \sup_{y_i = C_i x, x \neq 0} \sup_{\tilde{y}} \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix}^T Z \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix}$$

$$\text{subject to} \quad y_i = C_i x \tag{7}$$

$$u_i = \mu_i(y_i)$$

$$\text{for } i = 1, ..., N$$

Proposition 2: Let Z be the matrix given by (6). Then, the value of the game

$$\inf_{\mu} \sup_{y_i = C_i x, x \neq 0} \sup_{\tilde{y}} \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix}^T Z \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix}$$

$$\text{subject to} \quad y_i = C_i x$$

$$u_i = \mu_i(y_i)$$

$$\text{for } i = 1, ..., N$$

can be zero only if $Z_{11} \preceq 0$.

Proof: If $Z_{11} \not\preceq 0$, then $\tilde{y}$ can be chosen in the direction of the eigenvector corresponding to the positive eigenvalue of Z_{11}, which makes the value of the game arbitrarily large. Hence, a necessary condition for the game to have value zero is that $Z_{11} \preceq 0$. ∎

To ease the exposition of the paper, we will consider the case where $Z_{11} \prec 0$. The case where Z_{11} is semi-definite can be treated similarly, but is more technical, and therefore omitted here.

Proposition 3: If $Z_{11} \prec 0$, then

$$\sup_{\tilde{y}} \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix}^T Z \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix} = \tag{8}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} \left(\begin{bmatrix} Z_{22} & Z_{23} \\ Z_{32} & Z_{33} \end{bmatrix} - \begin{bmatrix} Z_{21} \\ Z_{31} \end{bmatrix} Z_{11}^{-1} \begin{bmatrix} Z_{21} \\ Z_{31} \end{bmatrix}^T \right) \begin{bmatrix} y \\ u \end{bmatrix}.$$

Proof: Completion of squares gives

$$\begin{aligned} \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix}^T Z \begin{bmatrix} \tilde{y} \\ y \\ u \end{bmatrix} &= \left(\tilde{y} + F \begin{bmatrix} y \\ u \end{bmatrix} \right)^T Z_{11} \left(\tilde{y} + F \begin{bmatrix} y \\ u \end{bmatrix} \right) + \\ &+ \begin{bmatrix} y \\ u \end{bmatrix} \left(\begin{bmatrix} Z_{22} & Z_{23} \\ Z_{32} & Z_{33} \end{bmatrix} - \begin{bmatrix} Z_{21} \\ Z_{31} \end{bmatrix} Z_{11}^{-1} \begin{bmatrix} Z_{21} \\ Z_{31} \end{bmatrix}^T \right) \begin{bmatrix} y \\ u \end{bmatrix} \end{aligned} \tag{9}$$

where F is given by

$$F = Z_{11}^{-1} \begin{bmatrix} Z_{21} \\ Z_{31} \end{bmatrix}^T. \tag{10}$$

Since $Z_{11} \prec 0$, the quadratic form in (9) is maximized for $\tilde{y} = -F \begin{bmatrix} y \\ u \end{bmatrix}$, which proves our proposition. ∎

Introduce now the matrix

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} Z_{22} & Z_{23} \\ Z_{32} & Z_{33} \end{bmatrix} - \begin{bmatrix} Z_{21} \\ Z_{31} \end{bmatrix} Z_{11}^{-1} \begin{bmatrix} Z_{21} \\ Z_{31} \end{bmatrix}^T. \tag{11}$$

Recall that $Z_{33} = Q_{uu} \succ 0$, and $Z_{11} \prec 0$, which implies that $Q_{22} \succ 0$. Now using Proposition (3), the game described by (7) reduces to

$$\inf_{\mu} \sup_{y_i = C_i x, x \neq 0} \begin{bmatrix} y \\ u \end{bmatrix}^T Q \begin{bmatrix} y \\ u \end{bmatrix}$$

$$\text{subject to} \quad y_i = C_i x \tag{12}$$

$$u_i = \mu_i(y_i)$$

$$\text{for } i = 1, ..., N$$

Hence, we consider the problem of finding policies $\mu_i(y_i)$ such that $u_i = \mu_i(y_i)$ and

$$\begin{bmatrix} Cx \\ u \end{bmatrix}^T Q \begin{bmatrix} Cx \\ u \end{bmatrix} \le 0$$

for all x. Now we are ready to state the main result of the paper where we show linearity of the optimal decisions:

Theorem 1: Let $Q_{22} \succ 0$ and $y_i = C_i x$, $i = 1, ..., N$. If there exist policies $\mu_i(y_i)$ such that

$$\sup_{x \ne 0} \begin{bmatrix} Cx \\ \mu(y) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} Cx \\ \mu(y) \end{bmatrix} \le 0, \qquad (13)$$

then there exist *linear* policies $\mu_i(y_i) = K_i y_i$ that satisfy (13).

Proof: Assume existence of policy μ that satisfies (13). If $y_i = C_i x = 0$ for some i, then the optimal decision for player i is to set $\mu_i(0) = 0$. To see this, take $y = 0$. Then

$$\begin{bmatrix} y \\ \mu(y) \end{bmatrix}^T Q \begin{bmatrix} y \\ \mu(y) \end{bmatrix} = \begin{bmatrix} 0 \\ \mu(0) \end{bmatrix}^T Q \begin{bmatrix} 0 \\ \mu(0) \end{bmatrix} = \mu^T(0) Q_{22} \mu(0).$$

Since $Q_{22} \succ 0$, we see that $\mu(0) = 0$ is the optimal decision. In particular, $\mu_i(0) = 0$ is the optimal decision for decision maker i.

Now suppose that $y_i \ne 0$ for $i = 1, 2, ..., N$. Define $K_i(y_i)$ as

$$K_i(y_i) = \frac{\mu_i(y_i) \cdot y_i^T}{\|y_i\|^2}, \qquad y_i \ne 0, \qquad (14)$$

for $i = 1, ..., N$. Also, define $K(x)$ as

$$K(x) = \begin{bmatrix} K_1(C_1 x) & 0 & \cdots & 0 \\ 0 & K_2(C_2 x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_N(C_N x) \end{bmatrix}. \quad (15)$$

It is easy to check that (13) is equivalent to

$$x^T C^T \begin{bmatrix} I \\ K(x) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} I \\ K(x) \end{bmatrix} Cx \le 0, \quad \forall x \ne 0. \tag{16}$$

Hence, we have obtained an equivalent problem for which the existence of policies μ_i is the same as the existence of matrix functions $K_1(y_1), ..., K_N(y_N)$ and $K(x)$ satisfying (15) and (16). Note that the problem of searching for linear policies corresponds to that of searching for constant matrices $K_i(C_i x) = K_i$. Furthermore, (16) is equivalent to the problem of finding a matrix function $M(X)$ such that

$$\mathbf{Tr}\, C^T \begin{bmatrix} I \\ M(X) \end{bmatrix}^T Q \begin{bmatrix} I \\ M(X) \end{bmatrix} CX \le 0, \quad \forall X = xx^T \ne 0. \tag{17}$$

To see this, take a matrix $M(X)$ satisfying (17), for $X = xx^T \ne 0$. Then, $K(x) = M(xx^T)$ satisfies (16). Conversely, given $K(x)$ satisfying (16), we can take $M(xx^T) = K(x)$ and (17) is satisfied.

Now if for a given matrix $X \ne 0$ and $M = M(X)$ the inequality in (17) is satisfied, then the same matrix M satisfies (17) with the matrix $X/\mathbf{Tr}\, X$ instead of X. Thus,

since we are considering matrices $X = xx^T \ne 0$, it is enough to consider matrices X with $\mathbf{Tr}\, X = 1$. Define the set

$$S_1 = \{X : x \in \mathbb{R}^n, X = xx^T, \mathbf{Tr}\, X = 1\}$$

Then (17) implies that

$$\max_{X \in S_1} \min_{M(X)} \mathbf{Tr}\, C^T \begin{bmatrix} I \\ M(X) \end{bmatrix}^T Q \begin{bmatrix} I \\ M(X) \end{bmatrix} CX \le 0. \quad (18)$$

We will now extend the set of matrices X from S_1 to the set

$$S = \{X : X \succeq 0, \mathbf{Tr}\, X = 1\}.$$

That is, we will consider the extended problem

$$\max_{X \in S} \min_{M(X)} \mathbf{Tr}\, C^T \begin{bmatrix} I \\ M(X) \end{bmatrix}^T Q \begin{bmatrix} I \\ M(X) \end{bmatrix} CX. \quad (19)$$

Clearly, we have that

$$\max_{X \in S_1} \min_{M(X)} \mathbf{Tr}\, C^T \begin{bmatrix} I \\ M(X) \end{bmatrix}^T Q \begin{bmatrix} I \\ M(X) \end{bmatrix} CX \le$$
$$\max_{X \in S} \min_{M(X)} \mathbf{Tr}\, C^T \begin{bmatrix} I \\ M(X) \end{bmatrix}^T Q \begin{bmatrix} I \\ M(X) \end{bmatrix} CX. \tag{20}$$

Let $M_*(X)$ be the optimal decision to the extended minimax problem (19), and suppose that

$$\max_{X \in S} \min_{M(X)} \mathbf{Tr}\, C^T \begin{bmatrix} I \\ M(X) \end{bmatrix}^T Q \begin{bmatrix} I \\ M(X) \end{bmatrix} CX = \alpha$$

for some real number α. This is equivalent to

$$\max_{X \in S} \min_{M(X)} \mathbf{Tr}\, \left\{ C^T \begin{bmatrix} I \\ M(X) \end{bmatrix}^T Q \begin{bmatrix} I \\ M(X) \end{bmatrix} CX - \alpha X \right\} =$$
$$= 0 \tag{21}$$

Note that

$$\max_{X \in S} \min_{M(X)} \mathbf{Tr}\, \left\{ C^T \begin{bmatrix} I \\ M(X) \end{bmatrix}^T Q \begin{bmatrix} I \\ M(X) \end{bmatrix} CX - \alpha X \right\}$$

is the dual to the following convex optimization problem (see Boyd *et al* [2]):

$$\min_{M,s}\ s$$
$$\text{subject to } C^T \begin{bmatrix} I \\ M \end{bmatrix}^T Q \begin{bmatrix} I \\ M \end{bmatrix} C - \alpha I \preceq sI. \tag{22}$$

Strong duality holds since the primal problem (22) is convex ($Q_{22} \succ 0$) and Slater's condition is satisfied, see [2]. Thus, existence of a decision matrix $M(X) = M_*(X)$ fulfilling (21) implies existence of a constant matrix $M(X) = K$ that fulfills

$$\max_{X} \mathbf{Tr}\, \left\{ C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX - \alpha X \right\} = 0. \quad (23)$$

Now take any positive semi-definite matrix X of rank $k \le n$ and $\mathbf{Tr}\, X = 1$. Then, we can write X as

$$X = \sum_{i=1}^k \lambda_i X_i,$$

where $X_i = x_i x_i^T$, $\|x_i\| = 1$, $x_i^T x_j = 0$ for $i \neq j$, $\lambda_i > 0$, and $\sum_{i=1}^{k} \lambda_i = 1$ (see Horn and Johnson [5], pp.457). Let $X_* = \sum_{i=1}^{k} \lambda_i X_i$ be

$$X_* = \arg\max_{X \in S} \mathbf{Tr} \left\{ C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX - \alpha X \right\}.$$

This gives together with equation (23):

$$\mathbf{Tr}\, C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX_* = \mathbf{Tr}\, \alpha X_* = \alpha.$$

Let X_j be the matrix for which

$$\mathbf{Tr}\, C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX_i$$

is maximized among $X_1, ..., X_k$. Then

$$\begin{aligned}
\alpha &= \mathbf{Tr}\, C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX_* \\
&= \sum_{i=1}^{k} \lambda_i \mathbf{Tr}\, C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX_i \\
&\leq \sum_{i=1}^{k} \lambda_i \mathbf{Tr}\, C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX_j \\
&= \mathbf{Tr}\, C^T \begin{bmatrix} I \\ K \end{bmatrix}^T Q \begin{bmatrix} I \\ K \end{bmatrix} CX_j \leq 0.
\end{aligned}$$

Hence, we have proved that the worst case is attained for a matrix X with rank 1, and the extension of the set S_1 to the set S does *not* increase the cost. We conclude that the optimal decision can be taken to be a linear decision with $\mu(y) = Ky$, and the proof is complete. ∎

IV. COMPUTATION OF THE OPTIMAL TEAM DECISION

In the previous section we showed that for the minimax team problem given by (2), the linear policy $u = KCx$ is optimal, where K is given by

$$K = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_N \end{bmatrix}. \tag{24}$$

Now the problem of finding linear policies satisfying (16) can be written as the following convex feasability problem:

$$\begin{aligned}
& \text{Find } K \\
& \text{such that } \begin{bmatrix} C \\ KC \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} C \\ KC \end{bmatrix} \preceq 0.
\end{aligned} \tag{25}$$

where $Q_{22} \succ 0$ The inequality in (25) can be written as where

$$R - (KC - LC)^T Q_{22} (KC - LC) \succeq 0, \tag{26}$$

$L = Q_{22}^{-1} Q_{21}$, and $R = -C^T Q_{11} C + C^T Q_{12} Q_{22}^{-1} Q_{21} C$. First note that a necessary condition for which (26) to be satisfied is that $R \succeq 0$. If $R \succeq 0$, then using the Schur

complement gives that inequality (26) can be written as an LMI

$$\begin{bmatrix} R & (KC - LC)^T \\ KC - LC & Q_{22}^{-1} \end{bmatrix} \succeq 0,$$

which can be computationally solved efficiently.

V. RELATION WITH THE STOCHASTIC MINIMAX TEAM DECISION PROBLEM

In this section we consider the stochastic minimax team decision problem

$$\min_K \max_{\mathbf{E}\|x\|^2 = 1} \mathbf{E} \left\{ x^T \begin{bmatrix} C \\ KC \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} C \\ KC \end{bmatrix} x \right\}.$$

Taking the expectation of the cost in the stochastic problem above yields the equivalent problem

$$\min_K \max_{\mathbf{Tr}X = 1} \mathbf{Tr} \begin{bmatrix} C \\ KC \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} C \\ KC \end{bmatrix} X$$

where X is a positive semi-definite matrix, and is the covariance matrix of x, i. e. $X = \mathbf{E}\, xx^T$. Hence, we see that the stochastic minimax team problem is equivalent to the deterministic minimax team problem, where nature maximizes with respect to all covariance matrices X of the stochastic variable x with variance $\mathbf{E}\|x\|^2 = \mathbf{E}\, x^T x = \mathbf{Tr}\, X = 1$.

VI. TEAM DECISION PROBLEMS AND SIGNALING

Consider a modified version of the static team problem posed in the previous section, where the observation y_i for every decision maker i is affected by the inputs of the other decision makers, that is

$$y_i = C_i x + \sum_j D_{ij} u_j,$$

where $D_{ij} = 0$ if decision maker j does not affect the observation y_i. The modified optimization problem becomes

$$\begin{aligned}
& \inf_\mu \sup_x \begin{bmatrix} x \\ u \end{bmatrix}^T Q \begin{bmatrix} x \\ u \end{bmatrix} \\
& \text{subject to } y_i = C_i x + \sum_j D_{ij} u_j \\
& u_i = \mu_i(y_i) \\
& \text{for } i = 1, ..., N.
\end{aligned} \tag{27}$$

The problem above is in general very complex if decision maker i does not have access to the information about the decisions u_i that appear in y_i. We say that the problem give rise to a *signaling incentive* for decision maker j. If we assume that decision maker i has the value of u_j available for every j such that $D_{ij} \neq 0$, then she can form the new output measurement given y_i

$$\bar{y}_i = y_i - \sum_j D_{ij} u_j = C_i x,$$

which transforms the problem to a static team problem without signaling, and the optimal solution is linear and can be found according to Theorem 1 and section IV. Note that
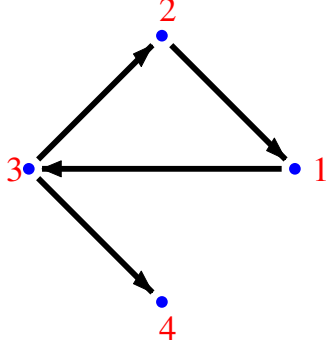
Fig. 1. The graph reflects the interconnection structure of the dynamics between four systems. The arrow from node 2 to node 1 indicates that system 1 affects the dynamics of system 2 directly.

if decision maker i has the information available that every decision maker j has, then the decision u_j is also available to decision maker i.

VII. DISTRIBUTED $\mathcal{H}_\infty$ CONTROL

In this section, we will treat the distributed linear quadratic $\mathcal{H}_\infty$ control problem with information constraints, which can be seen as a dynamic team decision problem. The idea is to transform the dynamic team problem to a static one, and then explore information structures for every time step.

Consider an example of four dynamically coupled systems according to the graph in Figure 1. The equations for the interconnected system are given by

$$
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}}_{x(k)} +
$$

$$
+ \underbrace{\begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \\ u_4(k) \end{bmatrix}}_{u(k)} + \underbrace{\begin{bmatrix} w_1(k) \\ w_2(k) \\ w_3(k) \\ w_4(k) \end{bmatrix}}_{w(k)}.
$$

$$(28)$$

For instance, the arrow from node 2 to node 1 in the graph means that the dynamics of system 2 are directly affected by system 2, which is reflected in the system matrix A, where the block $A_{21} \neq 0$. On the other hand, system 2 does not affect system 1 directly, which implies that $A_{12} = 0$. Because of the "physical" distance between the subsystems, there will be some constraints on the information available to each node.

The observation of system i at time k is given by

$$y_i(k) = C_i x_i(k),$$

where

$$
C_i = \begin{bmatrix} C_{i1} & 0 & 0 & 0 \\ 0 & C_{i2} & 0 & 0 \\ 0 & 0 & C_{i3} & 0 \\ 0 & 0 & 0 & C_{i4} \end{bmatrix}. \tag{29}
$$

Here, $C_{ij} = 0$ if system i does not have access to $y_j(k)$. The subsystems could exchange information about their outputs. Every subsystem recieves the information with some time delay, that is reflected by the interconnection structure. Let $\mathbb{I}_i^k$ denote the set of observations $y_j(n)$ and control signals $u_j(n)$ available to node i up to time k, $n \leq k$, $j = 1, ..., N$.

Consider the following (general) dynamic team decision problem:

$$
\inf_{\mu} \sup_{w} J(u, w)
$$
$$
\text{subject to} \quad x(k+1) = Ax(k) + Bu(k) + w(k)
$$
$$
y_i(k) = C_i x(k) \tag{30}
$$
$$
u_i(k) = \mu_i : \mathbb{I}_i^k \mapsto \mathbb{R}^{p_i}
$$
$$
\text{for } i = 1, ..., N.
$$

where

$$
J(u, w) = x^T(M) Q_f x^T(M) +
$$
$$
+ \sum_{k=0}^{M-1} \left\{ \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T Q \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} - \gamma \|w(k)\|^2 \right\}
$$
$$(31)$$
$$
Q = \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \in \mathbb{S}^{m+n},
$$

$Q_f \succeq 0$, $Q \succeq 0$, $Q_{uu} \succ 0$, $x(k) \in \mathbb{R}^n$, $y_i(k) \in \mathbb{R}^{m_i}$, $u_i(k) \in \mathbb{R}^{p_i}$.

Now write $x(k)$ and $y(k)$ as

$$
x(k) = A^t x(k-t) + \sum_{n=0}^{t-1} A^n B u(k-n-1) +
$$
$$
+ \sum_{n=0}^{t-1} A^n w(k-n-1),
$$
$$(32)$$
$$
y_i(k) = C_i A^t x(k-t) + \sum_{n=0}^{t-1} C_i A^n B u(k-n-1) +
$$
$$
+ \sum_{n=0}^{t-1} C_i A^n w(k-n-1).
$$

Note that the summation over n is defined to be zero when $t = 0$.

Theorem 2: Consider the optimization problem given by (30). The problem has no signaling incentive if and only if

$$
y_j(k) \in \mathbb{I}_i^k \Rightarrow
$$
$$
u_j(k-n-1) \in \mathbb{I}_i^k \text{ for } [C_i A^n B]_{ij} \neq 0 \tag{33}
$$

for all n such that $0 \leq n < t$, and $t = 0, ..., M-1$. In addition, an optimal solution to the optimization problem given by (30) is linear in the observations $\mathbb{I}_i^k$ if condition (33) is satisfied, and has a solution that can be found by solving a linear matrix inequality.

Proof: Introduce

$$\bar{x} = \begin{bmatrix} w(N-1) \\ w(N-2) \\ \vdots \\ w(0) \\ x(0) \end{bmatrix}, \quad \bar{u}_i = \begin{bmatrix} u_i(N) \\ u_i(N-1) \\ \vdots \\ u_i(0) \end{bmatrix},$$

Then, we can write the cost function $J(x, u)$ as

$$\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}^T \bar{Q} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}.$$

Consider the expansion given by (32). The problem here is that $y_i(k)$ depends on previous values of the control signals $u(n)$ for $n = 0, ..., k-1$. The components $u_j(k-n-1)$ that $y_i(k)$ depends on are completely determined by the structure of the matrix $[C_i A^n B]_{ij}$. This means that, to avoid signaling, it is enough for node i to have the information of $u_j(k-n-1)$ available at time k if the element $[C_i A^n B]_{ij} \neq 0$. Thus, we have proved the first statement of the theorem.

Now if condition (33) is satisfied, we can form the new output measurement

$$\check{y}_i(k) = y_i(k) - \sum_{n=0}^{k-1} C_i A^n B u(k-n-1)$$

$$= A^k x(0) + \sum_{n=0}^{k-1} C A^n w(k-n-1). \tag{34}$$

Let

$$\bar{y}_i(k) = \begin{bmatrix} \check{y}_i(k) \\ \check{y}_i(k-1) \\ \vdots \\ \check{y}_i(0) \end{bmatrix}.$$

With these new variables introduced, the optimization problem given by equation (30) reduces to the following static team decision problem:

$$\inf_{\mu} \sup_{\bar{x}} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}^T \bar{Q} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}$$
$$\text{subject to } u_i(k) = \mu_i(\bar{y}_i(k)) \tag{35}$$
$$\text{for } i = 1, ..., M.$$

and the optimal solution $\bar{u}$ is linear according to Theorem 1, and can be obtained by solving a linear matrix inequality as described in section IV, $\mathcal{QED}$. ∎

In fact, using the static team formulation reveals a much more general information structure. It turns out to be *enough to exchange information with the neighbours on the graph*. We illustrate this by an example:

Example 1: Consider the example presented at the beginning of this section. The dynamics of the second subsystem is given by

$$x_2(k+1) = A_{21} x_1(k) + A_{22} x_2(k) + B_2 u_2(k) + w_2(k).$$

If at time $k+1$, subsystem 2 has information about the state of its neighbours $x_1(k)$, then it has knowledge about the value of $w_2(k)$:

$$w_2(k) = x_2(k+1) - A_{21} x_1(k) - A_{22} x_2(k) - B_2 u_2(k).$$

Hence, if we restrict the control law $u_2(k+1)$ to be a function of $x_1(k)$, $x_2(k)$, $u_2(k)$ (information about the state of its neighbour and its own state and control input at time step k), and restrict it to be based only on the information about $w_2(k)$, then we can set $u_2(k+1) = \mu_2(w_2(k))$. The same information restriction can be similarly imposed on the other subsystems. Just as before, the dynamic $\mathcal{H}_\infty$ team problem can be reduced to the static team problem (35), where $u_i(k+1) = \mu_i(w_i(k))$. This problem has an optimal solution that is linear and can be found by solving a linear matrix inequality.

VIII. Conclusions and Future Work

We have fully solved the minimax (or deterministic) team decision problem completely for an arbitrary number of players, and show that the optimal solution is linear and can be found by solving a linear matrix inequality. Also, we show how to solve the finite-horizon $\mathcal{H}_\infty$ control problem, under some conditions that prevent *signaling*, which is analogous to the distributed stochastic LQG problem treated in Gattami [6]. It turns out that the information structure restricted to exchange information with neighbours only from one time step in the past, is enough to obtain a linear optimal feedback by solving a linear matrix inequality. This marks a starting point for a broader class of information structures that lead to tractable problems, which will be the subject for future work.

IX. Acknowledgements

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