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Gattami, Ather

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Ather Gattami
Department of Automatic Control
Lund University
Box 118, SE-221 00 Lund, Sweden
E-mail: ather@control.lth.se

***Abstract*— We consider the problem of stochastic control under constraints on the information structure. We review some old results from static team theory, where new simple proofs are given. The signalling phenomenon in distributed optimal linear quadratic Gaussian control plays a central role because of the complexity it implicates. Necessary and sufficient conditions for eliminating the signalling incentive are given in terms of the interconnection structure.**

I. Introduction

The problem of distributed control with information constraints is considered. For instance, information constraints appear naturally when making decisions over networks.
Early results considering team theory in [10], [17], [19], [21], and [22] showed the difficulty of the LQG problem with non-classical information structure. Recently, Bamieh *et al* [5] and Rotkowitz *et al* [18] showed that the distributed linear optimal control problem is convex if the rate of information propagation is faster than the dynamics.

In this paper, we consider the distributed linear quadratic Gaussian control problem and give a solution using statistical decision theory.

We give a mathematical definition of signalling in team decision problems, and give necessary and sufficient conditions that are determined by the system parameters (A, B, C) for elimination of the signalling incentive in optimal control. Under the conditions for eliminating the signalling incentive, the optimal distributed controller is shown to be linear.

The outline of the paper is as follows. In section II we introduce some notations that are used throughout the paper. In section III we summarize some results in static team theory, where new formulations and proofs are given. In section IV we give the necessary and sufficient conditions for elimination of the signalling incentive in terms of the interconnection structure. We conclude the main results of the paper in section V.

II. Notation

The ith component of a vector a is denoted by a_i. Column i of matrix M is denoted by M_i. The element in position j of a vector M_i is denoted by M_{ij}. For a matrix $M \in \mathbb{R}^{m\times n}$ we let $\text{vec}(M) = (M_1^T M_2^T \cdots M_n^T)^T$. Also, let $\text{diag}(A_i)$ denote a block diagonal matrix with the matrices A_i on its diagonal. $A \otimes B$ denotes the Kronecker product of the matrices A and B. We denote the set of $n \times n$ symmetric, positive semi-definite, and positive definite matrices by $\mathbb{S}^n$, $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$, respectively.

III. Team Decision Theory

In this section we will review some classical results in the theory of teams.

A. The Static Team Decision Problem

In the static team decision problem, one would like to solve the problem

$$\begin{aligned} &\text{minimize } \mathbf{E}\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ &\text{subject to } y_i = C_i x + v_i \\ &\qquad u_i = \mu_i(y_i) \\ &\qquad \text{for } i = 1, ..., N. \end{aligned} \tag{1}$$

Here, x and v are independent Gaussian variables taking values in $\mathbb{R}^m$, $\mathbb{R}^n$, $n = n_1 + ... + n_N$, and $x \sim N(0, V_{xx})$, $v \sim N(0, V_{vv})$. Also, let y_i and u_i be stochastic variables taking values in $\mathbb{R}^{n_i}$, $\mathbb{R}^{p_i}$, respectively,

$$\begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \in \mathbb{S}^{m+p}, \tag{2}$$

and $Q_{uu} \in \mathbb{S}^p_{++}$, $p = p_1 + ... + p_N$.

Assuming full state information about x to each *decision maker* u_i, the minimizing u would be found easily by completion of squares and is given by $u = Lx$, where L is the solution to

$$Q_{uu} L = -Q_{ux}.$$

Then, the cost function in (1) can be rewritten as

$$\begin{aligned} J(x, u) = &\mathbf{E}x^T (Q_{xx} - L^T Q_{uu} L)x + \\ &+ \mathbf{E}(u - Lx)^T Q_{uu} (u - Lx). \end{aligned} \tag{3}$$

Minimizing the cost function $J(x, u)$, is equivalent to minimizing $\mathbf{E}(u - Lx)^T Q_{uu} (u - Lx)$, since nothing

can be done about $\mathbf{E}(x^T(Q_{xx} - L^T Q_{uu} L)x)$ (the cost when u has full information).
The following result was in principal first shown by Radner [17], but we give a different formulation and proof:

Theorem 1: Let x and v_i be Gaussian variables with zero mean, taking values in $\mathbb{R}^m$, $\mathbb{R}^{n_i}$, $n = n_1 + ... + n_N$. Also, u_i is a stochastic variable taking values in $\mathbb{R}^{p_i}$, and $Q_{uu} \in \mathbb{S}^p_{++}$, $p = p_1 + ... + p_N$. The optimal solution u to the optimization problem

$$\begin{aligned} &\text{minimize } \mathbf{E}(u - Lx)^T Q_{uu}(u - Lx) \\ &\text{subject to } u_i = \mu_i(y_i) \\ &\qquad \text{for } i = 1, ..., N. \end{aligned} \tag{4}$$

is unique and linear in y.

Proof: Let H be the space of all measurable functions $g(y)$ from $\mathbb{R}^n$ to $\mathbb{R}^p$ for which $\mathbf{E}[g(y)^T Q_{uu} g(y)] < \infty$. Then H is a Hilbert space under the inner product

$$\langle g, h \rangle = \mathbf{E} g(y)^T Q_{uu} h(y),$$

and norm

$$||g(y)||^2 = \mathbf{E}[g(y)^T Q_{uu} g(y)].$$

Let $Y \subseteq \mathbb{R}^n$ be a space such that $z \in Y$ if z_i is a linear transformation of y_i, that is $z_i = A_i y_i$ for some real matrix $A_i \in \mathbb{R}^{p_i \times n_i}$. Clearly, Y is a linear subspace of $\mathbb{R}^n$. Now the optimization problem in equation (4) can be extended to

$$\begin{aligned} &\text{minimize } ||u - Lx||^2 \\ &\text{subject to } u = \mu : Y \to \mathbb{R}^p \end{aligned} \tag{5}$$

Finding the best *linear* optimal decision u^* to the extended problem is equivalent to finding the shortest distance from the subspace Y to the element $Lx \in \mathbb{R}^p$, where the minimizing u^* is the projection of Lx on Y, and hence unique. Also, since u^* is the projection, we have

$$0 = \langle u^* - Lx, u \rangle = \mathbf{E}(u^* - Lx)^T Q_{uu} u,$$

for all u. In particular, for $f_i = (0, 0, ..., z_i, 0, ..., 0)$, we have

$$\mathbf{E}(u^* - Lx)^T Q_{uu} f_i = \mathbf{E}[((u^* - Lx)^T Q_{uu})_i f_i] = 0.$$

The Gaussian assumption implies that f_i is independent of $((u^* - Lx)^T Q_{uu})_i$. Hence, for any decision u, *linear and nonlinear*, we have that

$$\begin{aligned} \langle u^* - Lx, u \rangle &= \mathbf{E}(u^* - Lx)^T Q_{uu} u \\ &= \sum_i \mathbf{E}[((u^* - Lx)^T Q_{uu})_i z_i] = 0. \end{aligned}$$

Finally, we get

$$\begin{aligned} ||u - Lx||^2 &= \langle u - Lx, u - Lx \rangle \\ &= \langle u^* - Lx + u - u^*, u^* - Lx + u - u^* \rangle \\ &= \langle u^* - Lx, u^* - Lx \rangle + \langle u - u^*, u - u^* \rangle + \\ &\quad + 2\langle u^* - Lx, u^* - u \rangle \\ &= \langle u^* - Lx, u^* - Lx \rangle + \langle u - u^*, u - u^* \rangle \\ &\geq \langle u^* - Lx, u^* - Lx \rangle \end{aligned}$$

with equality if and only if $u = u^*$. This concludes the proof.

■

The next theorem shows how to find the linear optimal control law $u = Ky$.

Theorem 2: Let x and v_i be independent Gaussian variables taking values in $\mathbb{R}^m$, $\mathbb{R}^{n_i}$, and $x \sim N(0, V_{xx})$, $v \sim N(0, V_{vv})$. Also, let y_i and u_i be stochastic variables taking values in $\mathbb{R}^{n_i}$, $\mathbb{R}^{p_i}$, respectively, and $Q_{uu} \in \mathbb{S}^p_{+}$. The linear optimal solution $u = Ky$ to the optimization problem

$$\begin{aligned} &\text{minimize } \mathbf{E}(u - Lx)^T Q_{uu}(u - Lx) \\ &\text{subject to } y_i = C_i x + v_i \\ &\qquad u_i = \mu_i(y_i) \\ &\qquad \text{for } i = 1, ..., N. \end{aligned} \tag{6}$$

is the solution to the linear system of equations

$$\sum_{j=1}^{N} (Q_{uu})_{ij} K_j (C_j V_{xx} C_i^T + (V_{vv})_{ji}) = -(Q_{ux})_i V_{xx} C_i^T,$$

for $i = 1, ..., N$,

$$C = \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix}, \quad K_i \in \mathbb{R}^{p_i \times n_i}, \quad K = \text{diag}(K_i).$$

Proof: The problem of finding the optimal linear feedback law $u_i = K_i y_i$ can be written as

$$\begin{aligned} &\text{minimize } \mathbf{Tr}[\mathbf{E} Q_{uu}(u - Lx)(u - Lx)^T] \\ &\text{subject to } u = K(Cx + v) \end{aligned} \tag{7}$$

Now

$$\begin{aligned} f(K) &= \mathbf{Tr}[\mathbf{E} Q_{uu}(u - Lx)(u - Lx)^T] \\ &= \mathbf{Tr}[\mathbf{E} Q_{uu}(KCx + Kv - Lx)(KCx + Kv - Lx)^T] \\ &= \mathbf{Tr}[Q_{uu}(K(CV_{xx}C^T + V_{vv})K^T - 2LV_{xx}C^T K^T \\ &\quad + LV_{xx}L^T)] \\ &= \mathbf{Tr}[\sum_{i,j=1}^{N} (Q_{uu})_{ij} K_j (C_j V_{xx} C_i^T + (V_{vv})_{ji}) K_i^T - \\ &\quad - 2 \sum_{i,j=1}^{N} (Q_{uu})_{ij} L_j V_{xx} C_i^T K_i^T] + \mathbf{Tr}[Q_{uu} L V_{xx} L^T]. \end{aligned} \tag{8}$$

The minimizing K is obtained by solving $\nabla_{K_i} f(K) = 0$:

$$\begin{aligned} 0 &= \nabla_{K_i} f(K) \\ &= 2\sum_{j=1}^{N} (Q_{uu})_{ij} K_j (C_j V_{xx} C_i^T + (V_{vv})_{ji}) - \\ &- 2\sum_{j=1}^{N} (Q_{uu})_{ij} L_j V_{xx} C_i^T. \end{aligned} \tag{9}$$

Since $Q_{uu} L = -Q_{ux}$, we get that

$$\sum_{j=1}^{N} (Q_{uu})_{ij} L_j V_{xx} C_i^T = -(Q_{ux})_i V_{xx} C_i^T.$$

Hence, the equality in (9) is equivalent to

$$\sum_{j=1}^{N} (Q_{uu})_{ij} K_j (C_j V_{xx} C_i^T + (V_{vv})_{ji}) = -(Q_{ux})_i V_{xx} C_i^T. \tag{10}$$

■

Note that separation does not hold for the static team problem when constraints on the information available for every decision maker u_i are imposed. That is, the optimal decision is *not* given by $u = L\hat{x}$, where $\hat{x}$ is the optimal estimated value of x.

Note also that (10) is easy to pose as a linear system of equations in a simple structural way as follows. It is well known for matrices V and U of compatible sizes, we have the relation

$$\text{vec}(UXV) = (V^T \otimes U)\text{vec}(X).$$

Then we can write (10) as

$$\sum_{j=1}^{N} ((C_j V_{xx} C_i^T + (V_{vv})_{ji})^T \otimes (Q_{uu})_{ij}) \text{vec}(K_j) = -\text{vec}((Q_{ux})_i V_{xx} C_i^T),$$

or equivalently as

$$Hz = -G,$$

where H constitutes of blocks given by

$$H_{ij} = (C_j V_{xx} C_i^T + (V_{vv})_{ji})^T \otimes (Q_{uu})_{ij},$$

$$z = (\text{vec}(K_1)^T \cdots \text{vec}(K_N)^T)^T,$$

and

$$G = (\text{vec}((Q_{ux})_1 V_{xx} C_1^T)^T \cdots \text{vec}((Q_{ux})_N V_{xx} C_N^T)^T)^T.$$

Since Q_{uu} is positive definite and $CV_{xx}C^T + V_{vv}$ is positive semi-definite, it follows that H is a positive definite matrix, and hence invertible, which proves uniqueness of the solution.

B. Team Decision Problems and Signalling

Consider a modified version of the static team problem posed in the previous section, where the observation y_i for every decision maker i is affected by the inputs of the other decision makers, i.e.

$$y_i = C_i x + \sum_j D_{ij} u_j + v_i,$$

where $D_j = 0$ if decision maker j does not affect the observation y_i. The modified optimization problem becomes

$$\begin{aligned} &\text{minimize } \mathbf{E}(u - Lx)^T Q_{uu} (u - Lx) \\ &\text{subject to } y_i = C_i x + \sum_j D_{ij} u_j + v_i \\ &\qquad u_i = \mu_i(y_i) \\ &\qquad \text{for } i = 1, ..., N. \end{aligned} \tag{11}$$

The problem above is, in general, very complex if decision maker i does not have access to the information about the decisions u_i that appear in y_i (see Blondel [6]). It has been shown in Witsenhausen [21], by means of a counterexample, that for such problems there could be nonlinear decisions in the observations that perform better than any linear decision. This is referred to as the problem of *signalling*, where the decision makers try to *encode* information in their decisions that could be *decoded* by decision maker i whose observation is affected (see Ho [10]).
If we assume that decision maker i has the value of u_j available for every j such that $D_{ij} \neq 0$, then it could form the new output measurement given y_i

$$\bar{y}_i = y_i - \sum_j D_{ij} u_j = C_i x + v_i,$$

which transforms the problem to a static team problem without signalling, and the optimal solution is linear and can be found according to Theorem (1) and (2). Note that if decision maker i has the information available that every decision maker j has for which $D_{ij} \neq 0$, then the decision u_j is also available to decision maker i. This information structure is closely related to the *partially nested* information structure, which was introduced by Ho in [13].

Finally, we state a mathematical definition of signalling in static teams:

Definition 1 (Signalling): Consider the static team problem given by

$$\begin{aligned} &\text{minimize } \mathbf{E}(u - Lx)^T Q_{uu} (u - Lx) \\ &\text{subject to } y_i = C_i x + \sum_j D_{ij} u_j + v_i \\ &\qquad u_i = \mu_i : Y_i \mapsto \mathbb{R}^{p_i} \\ &\qquad \text{for } i = 1, ..., N. \end{aligned} \tag{12}$$

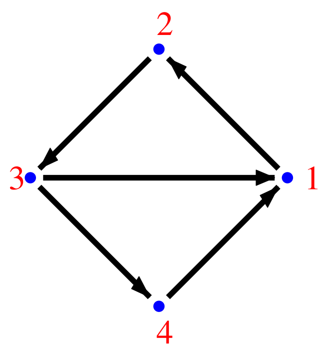

Fig. 1. The graph reflects the interconnection structure of the dynamics between four systems. The arrow from node 1 to node 2 indicates that system 1 affects the dynamics of system 2 directly.

where Y_i denotes the set of information y_j available to decision maker i. Then, the problem is said to have a signalling incentive if there exist i, j such that $Y_j \nsubseteq Y_i$ and $D_{ij} \neq 0$.

IV. Distributed Linear Quadratic Gaussian Control

In this section, we will treat the distributed linear quadratic Gaussian control problem with information constraints, which can be seen as a dynamic team decision problem.

Consider an example of four dynamically coupled systems according to the graph in Figure (1). The equations for the interconnected system are then given by

$$\underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix}}_{x(k+1)} = \underbrace{\begin{bmatrix} A_{11} & 0 & A_{13} & A_{14} \\ A_{21} & A_{22} & 0 & 0 \\ 0 & A_{32} & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}}_{x(k)} + \underbrace{\begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \\ u_4(k) \end{bmatrix}}_{u(k)} + \underbrace{\begin{bmatrix} w_1(k) \\ w_2(k) \\ w_3(k) \\ w_4(k) \end{bmatrix}}_{w(k)}. \tag{13}$$

For instance, the arrow from node 1 to node 2 in the graph means that system 1 affects the dynamics of system 2 directly, which is reflected in the system matrix A, where the element $A_{21} \neq 0$. On the other hand, system 2 does not affect system 1 directly, which implies that $A_{12} = 0$. Because of the "physical" distance between the subsystems, there will be some constraints on the information available to each node.

The structure of the matrix A could be described by the *adjacency matrix* $\mathcal{A}$ of the graph. For instance, the adjacency matrix for the graph in Figure 1 is given by

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The observation of system i at time k is given by

$$y_i(k) = C_i x(k),$$

where

$$C_i = \begin{bmatrix} C_{i1} & 0 & 0 & 0 \\ 0 & C_{i2} & 0 & 0 \\ 0 & 0 & C_{i3} & 0 \\ 0 & 0 & 0 & C_{i4} \end{bmatrix}. \tag{14}$$

Here, $C_{ij} = 0$ if system i does not have access to $y_j(k)$. Let Y_i^k denote the set of observations $y_j(n)$ available to node i up to time k, $n \leq k$, $j = 1, ..., N$.

Consider the following (general) dynamic team decision problem:

$$\begin{aligned} \text{minimize} \; & \sum_{k=0}^{M} \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \\ \text{subject to} \; & x(k+1) = Ax(k) + Bu(k) + w(k) \\ & y_i(k) = C_i x(k) + v_i(k) \\ & u_i(k) = \mu_i : Y_i^k \mapsto \mathbb{R}^{p_i} \\ & \text{for } i = 1, ..., N. \end{aligned} \tag{15}$$

where $x(k) \in \mathbb{R}^m$, $y_i(k) \in \mathbb{R}^{n_i}$, $u_i(k) \in \mathbb{R}^{p_i}$, $x(0) \sim N(0, R_0)$, $\{v(k)\}$ and $\{w(k)\}$ are sequences of independent Gaussian variables, uncorrelated with $x(0)$, such that

$$\mathbf{E} \begin{bmatrix} v(k) \\ w(k) \end{bmatrix} \begin{bmatrix} v(k) \\ w(k) \end{bmatrix}^T = R,$$

and the weighting matrix Q_{uu} is positive definite.

Now write $x(k)$ and $y(k)$ as

$$\begin{aligned} x(k) &= A^t x(k-t) + \\ &\quad + \sum_{n=0}^{t-1} A^n Bu(k-n-1) + \sum_{n=0}^{t-1} A^n w(k-n-1), \\ y_i(k) &= C_i A^t x(k-t) + \\ &\quad + \sum_{n=0}^{t-1} C_i A^n Bu(k-n-1) \\ &\quad + \sum_{n=0}^{t-1} C_i A^n w(k-n-1) + v_i(k). \end{aligned} \tag{16}$$

Note that the summation over n is defined to be zero when $t = 0$.

Theorem 3: Consider the optimization problem given by (15). The problem has no signalling incentive if and only if

$$y_j(n) \in Y_i^k \text{ for } (C_i A^n B)_{ij} \neq 0 \qquad (17)$$

for all n such that $0 \leq n \leq k$, and $k = 0, ..., M-1$. In additon, the optimal solution to the optimization problem given by (15) is linear in the observations Y_i^k if condition (17) is satisfied, and has an analytical solution that can be found by solving a linear system of equations.

Proof: Introduce

$$\bar{x} = \begin{bmatrix} w(N-1) \\ w(N-2) \\ \vdots \\ w(0) \\ x(0) \end{bmatrix}, \quad \bar{u}_i = \begin{bmatrix} u_i(N) \\ u_i(N-1) \\ \vdots \\ u_i(0) \end{bmatrix},$$

Then, we can write the cost function

$$\sum_{k=0}^{N} \mathbf{E} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

as

$$\mathbf{E} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}^T \bar{Q} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}. \qquad (18)$$

Consider the expansion given by (16). The problem here is that $y_i(k)$ depends on previous values of the control signals $u(n)$ for $n = 0, ..., k-1$. The components $u_j(n)$ that $y_i(k)$ depends on are completely determined by the structure of the matrix $(C_i A^n B)_{ij}$. This means that, to avoid signalling, every node i must have the information of $y_j(n)$ and $u_j(n-1)$ available at time n if the element $(C_i A^n B)_{ij} \neq 0$. Thus, we have proved the first statement of the theorem.

Now if condition (17) is satisfied, we can form the new output measurement

$$\begin{aligned} \check{y}_i(k) &= y_i(k) - \sum_{n=0}^{k-1} C_i A^n B u(k-n-1) \\ &= A^k x(0) + \\ &+ \sum_{n=0}^{k-1} C A^n w(k-n-1) + v_i(k). \end{aligned} \qquad (19)$$

Let

$$\bar{y}_i(k) = \begin{bmatrix} \check{y}_i(k) \\ \check{y}_i(k-1) \\ \vdots \\ \check{y}_i(0) \end{bmatrix}.$$

With these new variables introduced, the optimization problem given by equation (15) reduces to the following static team decision problem:

$$\begin{aligned} &\min_{\mu} \mathbf{E} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}^T \bar{Q} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \\ &\text{subject to } u_i(k) = \bar{\mu}_i(\bar{y}_i(k)) \\ &\qquad \text{for } i = 1, ..., M. \end{aligned} \qquad (20)$$

and the optimal solution $\bar{u}$ is unique and linear according to Theorem 1, and can be obtained using Theorem 2, *QED*.

■

V. Conclusions and Future work

We have considered the distributed linear quadratic Gaussian control problem using statistical decision theory. We give a mathematical definition of signalling in team decision problems, and necessary and sufficient conditions for elimination of the signalling incentive in optimal control are obtained. The conditions are determined completely by the structure of the system parameters (A, B, C).

It would be challenging to explore the $\mathcal{H}_\infty$ control problem and examine the information structure for which the problem is tractable.

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