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# The logic of isomorphism and its uses

PREPRINT

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## Abstract

We present a class of first-order modal logics, called *transformational logics*, which are designed for working with sentences that hold up to a certain type of transformation. An inference system is given, and completeness for the basic transformational logic HOS is proved. In order to capture ‘up to isomorphism’, we express a very weak version of higher category theory in terms of first-order models, which makes transformational logics applicable to category theory. A category-theoretical concept of isomorphism is used to arrive at a modal operator  $\Diamond_{iso}\varphi$  expressing ‘up to isomorphism,  $\varphi$ ’, which is such that category equivalence comes out as literally *isomorphism up to isomorphism*.

In the final part of the paper, we explore the possibility of using transformational logics to define weak higher categories. We end with two informal comparisons: one between HOS and counterpart semantics, and one between isomorphism logic, as a transformational logic, and Homotopy Type Theory.

## 1 Introduction

Mathematics abounds with notions that are defined or that hold only up to an appropriate equivalence. In algebraic topology one usually works “up to homotopy equivalence”, or sometimes “up to homology equivalence”. Cardinal arithmetic can be viewed as a set theory “up to bijection”, and ordinal arithmetic as set theory “up to order isomorphism”. In general, most of mathematics concerns itself only with objects “up to isomorphism”, where what *isomorphism* means will depend on what type of structure one is working in.

Nowhere is this more explicitly done than in category theory, and especially its higher-dimensional variants. Category theory gives our current best framework for working with isomorphisms and other structural relations between mathematical objects. This is also true for categories themselves, for which we get at least two plausible definitions of isomorphism: category isomorphism and category equivalence. It is the second of these, sometimes referred to as “isomorphism up to isomorphism” [1, p. 148], that is most often treated as the appropriate notion of “sameness of structure” for categories.

The existence of the two choices of isomorphism in **Cat** follows from the fact that it is itself a 2-category. As we go to higher and higher categories, the number of choices keeps increasing. Furthermore, certain choices that were equivalent in lower dimensions come apart in higher. It may be partly this bounty of isomorphisms and other forms of equivalences that has made the theory of weak higher categories so difficult to develop.

The aim of this paper is to introduce a class of languages for working up to isomorphism, or up to some other kind of equivalence. The fundamental idea is to take “up to  $\Theta$ -morphism”, where  $\Theta$  is a class of transformations, to be a  $\diamond$ -style modal operator. This has the advantage that the formalism will stay close to the informal use of the phrase.

The paper as such can be seen as consisting of two parts. In sections 2 to 4, we present the semantics of *transformational logics*, give an inference system for the basic transformational logic HOS, and prove completeness for this system with respect to the semantics introduced. These sections do not directly concern themselves with isomorphism as such, but rather with the more general notion of  $\Theta$ -morphism in model theoretic terms.

The second half of the paper brings in category theory to arrive at a useful notion of isomorphism. We employ a very weak notion of  $\infty$ -categories which we refer to as  *$\omega$ -precategories*, in which isomorphisms can be defined. Using these isomorphisms, a hierarchy of equivalences is introduced. The *up to isomorphism* operator  $\diamond_{iso}$  is defined from this hierarchy. The result is that, when applied to the 2-category **Cat**, we indeed get that category equivalence is isomorphism, up to isomorphism.

Sections 7 and 8 enter briefly on the area of using  $\diamond_{iso}$  in order to define weak higher categories. Following Leinster’s motto that “[a] polite person proposing a definition of weak  $n$ -category should explain what happens when  $n = 2$ ” [17, p. 254], we mainly concentrate on bicategories. The final section contains two short comparisons: one between transformational logic and Lewis’s counterpart theory, and one between isomorphism logic and homotopy type theory.

## 2 The *up to $\Theta$ -morphism* operator and its dual

We assume a first-order modal language  $\mathcal{L}$  with identity, which, apart from the connectives  $\wedge, \vee, \neg$ , the quantifiers  $\forall$  and  $\exists$ , and the relation  $=$ , contains the unary operators  $\square$  and  $\diamond$ , which we read as ‘invariantly under  $\Theta$ -morphisms’, and ‘up to  $\Theta$ -morphism’. We allow  $\mathcal{L}$  to contain both predicates and function symbols, and, of course, a countably infinite set of variables  $Var = \{x_1, x_2, \dots\}$ . We refer to the set of terms as *Term*, the set of  $n$ -ary relation symbols as *Rel<sup>n</sup>*, and the set of  $n$ -ary function symbols as *Func<sup>n</sup>*.

One of our main applications will be to category theory, which is easiest to work with using partial functions.<sup>1</sup> Therefore we also assume  $\mathcal{L}$  to have a

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<sup>1</sup>It is also possible to use a multi-sorted logic, as when e.g. working in a type theoretic foundation. But, arguably, the usual way of working with them lies closer to the partial functional approach, and it is also this approach that best highlights category theory’s status

primitive unary predicate  $\downarrow$ , which we, following [9], usually will write in postfix position, as in  $t \downarrow$ . We take a *first-order model*  $M$  for  $\mathcal{L}$  to be a set  $D_M$  together with an interpretation function  $M(-)$  taking  $n$ -ary predicates to  $n$ -ary relations on  $D_M$  and  $n$ -ary function symbols to non-empty  $n$ -ary partial functions on  $D_M$ .<sup>2</sup>

Let a *variable assignment*  $s$  on  $M$  be a function  $s : Var \rightarrow D_M$ . Let the *valuation*  $v$  on  $M$  based on  $s$  be defined as the partial function  $v : Term \rightarrow D_M$  such that  $v(x_k) = s(x_k)$  and  $v(f_k^n(t_1, \dots, t_n))$  is defined and equal to  $M(f_k^n)(v(t_1), \dots, v(t_n))$  whenever this is defined, where  $t_1, \dots, t_n$  are terms, and  $f_k^n$  is any  $n$ -ary function symbol. It follows that variables always range over  $D_M$ , and thus do not take undefined values.

Model homomorphisms will be central to our treatment of isomorphism. There are essentially three different but equally reasonable ways to define these when it comes to partial functions (see [13, p. 81]). These are as follows, in order of increasing strength, for a function  $h : D_M \rightarrow D_{M'}$  and a function symbol  $f^n$  such that  $f = M(f^n)$  and  $f' = M'(f^n)$

**(Weak) Homomorphism** : If  $f(c_1, \dots, c_n) \downarrow$  then  $f'(h(c_1), \dots, h(c_n)) \downarrow$  and  $f'(h(c_1), \dots, h(c_n)) = h(f(c_1, \dots, c_n))$  .

**Full Homomorphism** : If  $f(c_1, \dots, c_n) \downarrow$  then  $f'(h(c_1), \dots, h(c_n)) \downarrow$ , and if  $f'(d_1, \dots, d_n) = d_0$  then there are  $c_0, \dots, c_n \in D_M$  such that  $f(c_1, \dots, c_n) = c_0$  and  $h(c_k) = d_k$  for  $0 \leq k \leq n$ .

**Strong homomorphism** :  $f(c_1, \dots, c_n) \downarrow$  iff  $f'(h(c_1), \dots, h(c_n)) \downarrow$ , and  $f'(h(c_1), \dots, h(c_n)) = h(f(c_1, \dots, c_n))$  when defined.

The last two of these are, however, too strong for our applications to category theory. This application will require us to be able to interpret categories as first-order models and functors as model homomorphisms, but the impossibility of doing this on the full or strong homomorphism interpretation can be seen by considering the functor  $F$  between categories  $\mathcal{C}$  and  $\mathcal{D}$  below:

$$\begin{array}{ccc}
 \mathcal{C} : & a \xrightarrow{f} b & & b' \xrightarrow{g} c \\
 & & \Downarrow F & \\
 \mathcal{D} : & F(a) \xrightarrow{F(f)} F(b) = F(b') & \xrightarrow{F(g)} & F(c)
 \end{array}$$

Here,  $g \circ f$  is undefined in  $\mathcal{C}$ , but since  $F(b) = F(b')$ ,  $F(g) \circ F(f)$  is defined. This means that, if we want functors to count as homomorphisms, there is only one sensible way to define them. A function  $h : D_M \rightarrow D_{M'}$  is therefore taken to

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as a generalisation of group theory and an extension of the Erlanger programm. See [21], or the introduction of [8].

<sup>2</sup>The non-emptiness requirement entails that individual constants, *qua* 0-ary function symbols, will always denote.

be a *model homomorphism* from  $M$  to  $M'$  iff it is a weak homomorphism that preserves relations.

Dual to the choice of homomorphisms is the choice of what to count as a submodel. The domain of a submodel of a regular first-order model has to be closed under the functions of that model, but it is not obvious that we should require this of submodels including partial functions. However, our intended application of categories as models and functors as homomorphisms again helps us decide. A subcategory of another category always has  $g \circ f$  defined whenever it contains  $g$  and  $f$ , and  $g \circ f$  is defined in the containing category. Thus we take a submodel of a model  $M$  to be a model  $M'$  such that

$$(i) \quad D_{M'} \subseteq D_M.$$

$$(ii) \quad M'(f^n)(c_1, \dots, c_n) \text{ is defined iff } M(f^n)(c_1, \dots, c_n) \text{ is. If they are, } \\ M'(f^n)(c_1, \dots, c_n) = M(f^n)(c_1, \dots, c_n), \text{ for all } c_1, \dots, c_n \in D_{M'}, \text{ and}$$

$$(iii) \quad (c_1, \dots, c_n) \in M'(P^n) \text{ iff } (c_1, \dots, c_n) \in M(P^n) \text{ for all } c_1, \dots, c_n \in D_{M'}.$$

A *transformational frame*, or t-frame, is a pair  $\mathcal{M}, \Theta$ , where  $\mathcal{M}$  is a class of first-order models for  $\mathcal{L}$ 's signature, and  $\Theta$  is a function that, to each  $M \in \mathcal{M}$ , assigns a set  $\Theta_M$  of surjective model homomorphisms  $M \rightarrow M'$ , where  $M'$  is a submodel of  $M$  that is also in  $\mathcal{M}$ . Differently put, we can see such a frame as a *directed multigraph* in which the nodes are first-order models, and the edges are model homomorphisms. A *transformational model*, or t-model, is a transformational frame  $\mathcal{M}, \Theta$  together with a specified element  $M \in \mathcal{M}$ , i.e. a pointed t-frame.

The truth conditions of formulae in a model  $\mathcal{M}, \Theta, M$  under a valuation  $v$  are taken to be very close to the usual ones for first-order logic with identity; only minor changes have been made order to accommodate partial functions. The main additions are the rules for  $\square$  and  $\diamond$ .

$\mathcal{M}, \Theta, M \models_v t \downarrow$	iff $v$ is defined for $t$
$\mathcal{M}, \Theta, M \models_v t_1 = t_2$	iff $t_1 \downarrow, t_2 \downarrow$ , and $v(t_1) = v(t_2)$
$\mathcal{M}, \Theta, M \models_v P_k^n(t_1, \dots, t_n)$	iff $t_1 \downarrow, \dots, t_n \downarrow$ , and $(v(t_1), \dots, v(t_n)) \in M(P_k^n)$
$\mathcal{M}, \Theta, M \models_v \neg\varphi$	iff $\mathcal{M}, \Theta, M \not\models_v \varphi$
$\mathcal{M}, \Theta, M \models_v \varphi \wedge \psi$	iff $\mathcal{M}, \Theta, M \models_v \varphi$ and $M \models_v \psi$
$\mathcal{M}, \Theta, M \models_v \varphi \vee \psi$	iff $\mathcal{M}, \Theta, M \models_v \varphi$ or $M \models_v \psi$
$\mathcal{M}, \Theta, M \models_v \forall x_k \varphi$	iff $\mathcal{M}, \Theta, M \models_{v[x_k/c]} \varphi$ for all $c \in D_M$
$\mathcal{M}, \Theta, M \models_v \exists x_k \varphi$	iff $\mathcal{M}, \Theta, M \models_{v[x_k/c]} \varphi$ for some $c \in D_M$
$\mathcal{M}, \Theta, M \models_v \square\varphi$	iff $\mathcal{M}, \Theta, \text{cod } \tau \models_{\tau \circ v} \varphi$ for all $\tau \in \Theta_M$
$\mathcal{M}, \Theta, M \models_v \diamond\varphi$	iff $\mathcal{M}, \Theta, \text{cod } \tau \models_{\tau \circ v} \varphi$ for some $\tau \in \Theta_M$

When the frame  $\mathcal{M}, \Theta$  is understood, we will sometimes just write  $M \models_v \varphi$  for  $\mathcal{M}, \Theta, M \models_v \varphi$ . We say that  $\varphi$  is *true* in the t-model  $M$  iff  $M \models_v \varphi$  for all  $v$ , and *HOS-valid* iff it is true in all models of all frames.<sup>3</sup> A formula which is not true is

<sup>3</sup>The abbreviation HOS is for *Homomorphism Onto Submodel*. We might, of course, also

regarded as *false*. We have not introduced truth-value gaps for sentences about undefined values of functions; instead, all atomic formulae involving undefined terms come out as false. Quantification is done only over the domain.

**Theorem 1.** (i) HOS is a normal modal logic, i.e. it validates the rule of necessitation and the K axiom.

(ii)  $\Box$  is dual to  $\Diamond$ , i.e.  $\Box\neg\varphi$  is logically equivalent to  $\neg\Diamond\varphi$ .

(iii) HOS validates both the Barcan Formula and its converse.

*Proof.* (i) For the rule of necessitation, assume that  $M \models_v \varphi$ , for all  $M \in \mathcal{M}$  and all valuations  $v$ , for any frame  $\mathcal{M}$ , i.e. that  $\varphi$  is a logical truth. Let  $M'$  be an arbitrary element of  $\mathcal{M}$ , and let  $\tau$  be an arbitrary element of  $\Theta_{M'}$ . For  $M' \models_{v'} \Box\varphi$  to hold, we need to show that  $\text{cod } \tau \models_{\tau(v')} \varphi$ , but this follows directly from the fact that  $\text{cod } \tau$  has to be an element of  $\mathcal{M}$ , and we have assumed  $\varphi$  to be true in all of these, under any valuation, of which of course  $\text{cod } \tau$  is one.

For the K, axiom, assume that  $M \models_v \Box(\neg\varphi \vee \psi)$  and that  $M \models_v \Box\varphi$ , and let  $\tau$  be any element of  $\Theta_M$ . By assumption, either  $\text{cod } \tau \not\models_{\tau(v)} \varphi$  or  $\text{cod } \tau \models_{\tau(v)} \psi$ . The first of these, however, is incompatible with  $M \models_v \Box\varphi$ . Since  $\tau$  is arbitrary, the conclusion follows.

(ii) Assume that  $M \models_v \Box\neg\varphi$ . Then, for all  $\tau \in \Theta(M)$ ,  $\text{cod } \tau \not\models_{\tau(v)} \varphi$ . This contradicts  $M \models_v \Diamond\varphi$ , so  $M \models_v \neg\Diamond\varphi$ . The converse is similar.

(iii) Writing out the conditions for  $\alpha \equiv_{df.} M \models_v \forall x \Box\varphi$  and  $\beta \equiv_{df.} M \models_v \Box\forall x\varphi$ , we get

$$\begin{aligned}\alpha &\Leftrightarrow \forall d \in D_{\tau(M)} \forall \tau \in \Theta_M (\text{cod } \tau \models_{\tau(v)[x/d]} \varphi) \\ \beta &\Leftrightarrow \forall \tau \in \Theta_M \forall c \in D_M (\text{cod } \tau \models_{\tau(v[x/c])} \varphi)\end{aligned}$$

The conjunction of BF and CBF corresponds to the statement that  $\alpha \Leftrightarrow \beta$ , for all models  $M$ , all valuations  $v$ , and all formulae  $\varphi$ . Now, remember that  $\tau$  is taken to be a surjection, so that for all  $d \in D_{\text{cod } \tau}$ , there is some  $c \in D_M$  such that  $d = \tau(c)$ . Replacing the  $d$  in  $\alpha$  with  $\tau(c)$  gives us

$$\begin{aligned}\alpha &\Leftrightarrow \forall c \in D_M \forall \tau \in \Theta_M (\text{cod } \tau \models_{\tau(v)[x/\tau(c)]} \varphi) \\ &\Leftrightarrow \forall c \in D_M \forall \tau \in \Theta_M (\text{cod } \tau \models_{\tau(v[x/c])} \varphi) \\ &\Leftrightarrow \forall \tau \in \Theta_M \forall c \in D_M (\text{cod } \tau \models_{\tau(v[x/c])} \varphi) \Leftrightarrow \beta\end{aligned}$$

□

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consider frames in which the elements of  $\Theta_M$  are neither surjective, nor homomorphisms, nor onto submodels, but these are not our primary interest here. Using HOS-validity allows us to make certain simplifications to the logic.

The validity of BF and CBF may come as something of a surprise, since these, in the usual Kripke semantics, entail that  $\Box$  or  $\Diamond$  cannot change the domain. But in our case, no such thing follows. By assumption, the domain of  $\tau$  is taken to be a subset of that of  $M$ , but the converse does not hold.

We have introduced this semantics as a kind of minimal extension of the usual first-order semantics, in that all  $\Box$  and  $\Diamond$  do is to shift evaluation to other models. However, one could also make it more similar to possible world semantics by explicitly introducing a set of transformations for each node, and evaluating all formulae relative to a specific transformation. For our purposes, however, the approach we have taken here seems to be more natural and convenient.<sup>4</sup>

Although the logic is normal, there is one significant difference between these *transformational* logics and quantified modal logics as they are usually handled: while most modal logicians (or at least the ones bothering with quantified modal logic) in the post-Kripke era accept both  $a = b \rightarrow \Box a = b$  and  $a \neq b \rightarrow \Box a \neq b$  as valid, our motivating application makes it clear that the second of these is unacceptable for us. On the other hand, the first still holds.

### 3 Inference

Here we give an inference system for the basic transformational logic HOS. As is common in modal logic, we use a tableau system. In particular, we use tableaux whose formulae are labeled with *transformation signs*, defined recursively from a countably infinite set  $\Sigma$  of signs as follows:

- (i)  $\iota$  is a transformation sign called the *identity transformation*.
- (ii) Any sign in  $\Sigma$  is a transformation sign. These, together with  $\iota$ , are called the *atomic* transformation signs.
- (iii) If  $\sigma$  and  $\tau$  are transformation signs,  $\tau \circ \sigma$  is a transformation sign.

When a term or transformation sign appears on the branch we are working on, we refer to it as *old*, and when it does not, we refer to it as *new*. A *term* is called atomic when it is either a variable or an individual constant. We assume the root of tableaux to always be labeled with the identity transformation  $\iota$ . In general, we have followed [10], but with modifications to account for undefined terms analogous to those imposed in [9]. The rules for the connectives remain as usual:

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<sup>4</sup>Another different approach, which would perhaps in some aspects be even more natural, would be to extend the semantics to toposes other than **Set**, or even to more general categories. Thus models would be replaced with objects in such a category, and model homomorphisms by morphisms. Carrying this out would give a useful generalisation of the framework presented here.

$$\begin{array}{c}
\frac{\tau : \varphi \wedge \psi}{\tau : \varphi} \\
\tau : \psi
\end{array}
\qquad
\frac{\tau : \neg(\varphi \wedge \psi)}{\tau : \neg\varphi \quad \tau : \neg\psi}$$

$$\frac{\tau : \varphi \vee \psi}{\tau : \varphi \quad \tau : \psi}
\qquad
\frac{\tau : \neg(\varphi \vee \psi)}{\tau : \neg\varphi}$$

$$\frac{\tau : \neg\neg\varphi}{\tau : \varphi}$$

While BF and CBF are valid in HOS, the existence of undefined terms in the language means that we have to restrict universal instantiation. We do not, however, need a rule to guarantee that transformations are surjective, since this follows from the quantifier rules we adopt.

$$\frac{\tau : \forall x_k \varphi \quad \tau : t \downarrow}{\tau : \varphi[x_k/t]}
\qquad
\frac{\tau : \neg\forall x_k \varphi}{\tau : \neg\varphi[x_k/t']}$$

$$\frac{\tau : \exists x_k \varphi}{\tau : \varphi[x_k/t']}
\qquad
\frac{\tau : \neg\exists x_k \varphi \quad \tau : t \downarrow}{\tau : \neg\varphi[x_k/t]}$$

Here,  $t'$  is required to be new and atomic. The introduction rule for identity is

$$\frac{\tau : t \downarrow}{\tau : t = t}$$

for any transformation sign  $\tau$  and term  $t$ . We have already mentioned that inference from  $\diamond a = b$  to  $a = b$  will be disallowed, so rather than the usual elimination rule for identity we use

$$\frac{\sigma : t_1 = t_2 \quad \tau \circ \sigma : \varphi}{\tau \circ \sigma : \varphi[t_1/t_2]}$$

for any transformation sign of the form  $\tau \circ \sigma$ . The essential difference with [10] lies in our requirement that the second premiss' and the conclusion's transformation sign cannot be arbitrary, but have to be of this form.

The following are the rules governing  $\downarrow$ :

$$\frac{\tau : P^n(t_1, \dots, t_n)}{\tau : t_k \downarrow}
\qquad
\frac{\tau : f^n(t_1, \dots, t_n) \downarrow}{\tau : t_k \downarrow}$$

$$\frac{\tau : t_1 = t_2}{\tau : t_1 \downarrow}
\qquad
\frac{\tau : \varphi}{\tau : t' \downarrow}$$

$$\tau : t_2 \downarrow$$

In these,  $t_k$  is required to be one of  $t_1, \dots, t_n$ , and  $t'$  is required to be atomic. The rules for  $\square$  and  $\diamond$  are as very much as usual for modal logic:



$$\frac{\sigma : \Box \varphi}{\tau \circ \sigma : \varphi} \qquad \frac{\sigma : \neg \Box \varphi}{\tau' \circ \sigma : \neg \varphi}$$

$$\frac{\sigma : \Diamond \varphi}{\tau' \circ \sigma : \varphi} \qquad \frac{\sigma : \neg \Diamond \varphi}{\tau \circ \sigma : \neg \varphi}$$

where  $\tau'$  has to be new and atomic. In order to capture the fact that the transformations are to be *model homomorphisms* rather than just functions, we also need a rule we call H,

$$\frac{\sigma : P^n(t_1, \dots, t_n)}{\tau \circ \sigma : P^n(t_1, \dots, t_n)} \qquad \frac{\tau \circ \sigma : \neg P^n(t_1, \dots, t_n)}{\sigma : \neg P^n(t_1, \dots, t_n)}$$

Finally, we also need a rule to make sure that the domains of these transformations are submodels of the model we started working with; the rule

$$\frac{\tau \circ \sigma : P^n(t_1, \dots, t_n)}{\tau \circ \sigma : t_1 = t'_1}$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \tau \circ \sigma : t_n = t'_n & & \\ \sigma : P^n(t'_1, \dots, t'_n) & & \end{array}$$

where  $t'_1, \dots, t'_n$  are required to be new, fills this purpose, and we refer to it as rule S. The submodel condition, while not strictly necessary, is admissible for our intended application, and together with surjectivity brings the advantage that all the whole of a transformational frame can be determined by giving (i) a base first-order model  $M_t$  with the identity transformation  $\iota$  on it, and (ii) a set of model homomorphisms on subsets of  $D_M$ .

Inspection of the rules of this section shows that the inference system has a weak subformula property of the form that every formula in a tableau is either

- (i) a subformula of the root,
- (ii) a negation of a subformula of the root, or
- (iii) an atomic formula.

## 4 Completeness

In this section, we prove the completeness theorem.

**Theorem 2.** *The tableau proof system of section 3 is sound and complete for the logic HOS.*

Soundness is, as usual, a fairly mechanical task to prove.

**Lemma 1.** *If one of the rules of section 3 is applied to a branch B in a satisfiable tableau T, the result is a satisfiable tableau.*

*Proof.* Apart from modifications made to support undefined values, the quantifier rules are the same as in standard first-order logic, and their soundness is proved the same way. The elimination rule for identity follows from the fact that the applied transformations are functions, and thus preserve identities. That it is not restricted only to atomic transformation signs is validated by the closure of functions under composition. The rules for  $\downarrow$  are validated by the interpretation we have adopted for partial algebra homomorphisms, together with the requirement that atomic terms have to refer.

Soundness for the  $\square$  and  $\diamond$  rules follows directly from the truth conditions we imposed on them in section 2. The H rule's soundness for non-atomic  $\tau$  comes from the fact that model homomorphisms, like functions, being closed under composition. Finally, the soundness of the S rule is a consequence of the requirement that the codomain of a transformation always has to be a submodel of the domain. Again, since the submodel of a submodel is a submodel, we do not need to limit ourselves to atomic  $\tau$ .  $\square$

**Corollary 1.** *The tableau system is sound with respect to the logic HOS.*

*Proof.* Soundness, in a tableau system like ours, is equivalent to the condition that if a tableau is closed, it has no model. Now it is obvious that if a *branch* is closed, i.e. includes  $\tau : \varphi$  and  $\tau : \neg\varphi$ , it has no model. Soundness follows from this together with the contrapositive of the previous lemma.  $\square$

Completeness is proved by constructing a model  $\mathcal{M}_\varphi, \Theta_\varphi, M_\varphi$  of any consistent sentence  $\varphi$ . This is done by making  $M$  from an open branch of a saturated tableau. But what does *saturated* mean for us? We keep the rules of [10, pp. 126–127] for quantified K. Additional rules are as follows.

- (i) Whenever  $\tau : t \downarrow$  appears on a branch  $B$ , add  $\tau : t = t$  to the branch if it does not already appear there.
- (ii) Whenever  $\sigma : t_1 = t_2$  and  $\tau \circ \sigma : \varphi(t_1)$  appear on a branch, add  $\tau \circ \sigma : \varphi(t_2)$  to the branch if it does not already appear there.
- (iii) Whenever  $\sigma : \square\varphi$  or  $\sigma : \neg\diamond\varphi$ , together with  $\tau \circ \sigma : \psi$ , appear on a branch, add  $\tau \circ \sigma : \varphi$ , or  $\tau \circ \sigma : \neg\varphi$ , respectively, to the branch, if it does not already appear there.
- (iv) Whenever  $\sigma : \diamond\varphi$  or  $\sigma : \neg\square\varphi$  appear on a branch, add  $\tau \circ \sigma : \varphi$  or  $\tau \circ \sigma : \neg\varphi$ , respectively, to the branch, where  $\tau$  is the first atomic transformation sign that does not appear on the branch.
- (v) Whenever  $\sigma : P^n(t_1, \dots, t_n)$  and  $\tau \circ \sigma : \varphi$  appear on a branch, add  $\tau \circ \sigma : P^n(t_1, \dots, t_n)$  to the branch if it does not already appear there.
- (vi) Whenever  $\tau \circ \sigma : \neg P^n(t_1, \dots, t_n)$  and  $\sigma : \varphi$  appear on a branch, add  $\sigma : \neg P^n(t_1, \dots, t_n)$  to the branch if it does not already appear there.
- (vii) Whenever  $\tau : P^n(t_1, \dots, t_n)$  appears on a branch, add  $\tau : t_1 \downarrow, \dots, \tau : t_n \downarrow$  to the branch, if they do not already appear there.

- (viii) Whenever  $\tau \circ \sigma : P^n(t_1, \dots, t_n)$  appears on a branch and there are no terms  $u_1, \dots, u_n$  such that  $\sigma : P^n(u_1, \dots, u_n)$  appears on that branch, add  $\sigma : P^n(t'_1, \dots, t'_n)$  together with  $\tau \circ \sigma : t_1 = t'_1, \dots, \tau \circ \sigma : t_n = t'_n$  to the branch, where  $t'_1, \dots, t'_n$  are the  $n$  first individual constants not appearing on the branch.

Given a saturated tableau  $T$  with an open branch  $B$ , and a transformation sign  $\alpha$ , we will now show how to make a t-model  $\mathcal{M}_\alpha, \Theta_\alpha, M_\alpha$  that satisfies all the formulae appearing on  $B$  that are prefixed with  $\alpha$ . As the domain  $D_{M_\alpha}$ , we take the set of equivalence classes of terms that appear on  $B$  under the partial equivalence relation

$$t_1 \sim_\alpha t_2 \text{ iff } \alpha : t_1 = t_2 \text{ appears on } B.$$

We let the value of each term be the equivalence class it belongs to. For each relation symbol  $P^n \in Rel^n$ , we impose the relation

$$([t_1]_{\sim_\alpha}, \dots, [t_n]_{\sim_\alpha}) \in M(P^n) \text{ iff } \alpha : P^n(t_1, \dots, t_n) \text{ appears on } B.$$

This defines a first-order model  $M_{\alpha, \iota}$ . Let  $\mathcal{M}_\alpha^0, \Theta_\alpha^0, M^0$  be the t-frame that contains only the first-order model  $M_{\alpha, \iota}$  in  $\mathcal{M}_\alpha^0$  and the identity transformation  $\iota$  on  $M_{\alpha, \iota}$  in  $\Theta_\alpha^0$ . We define  $\mathcal{M}_\alpha^{n+1}$  and  $\Theta_\alpha^{n+1}$  by applying the following steps for each first-order model  $M_{\alpha, \sigma}$  of  $\mathcal{M}_\alpha^n$  and transformation sign  $\tau = \tau' \circ \sigma$  on  $B$ , where  $\tau'$  is atomic:

- (i) create a model  $M'$  and a surjective model homomorphism  $h : M_{\alpha, \sigma} \rightarrow M'$ ,
- (ii) embed  $M'$  in  $M_{\alpha, \sigma}$  to get a submodel  $M_{\alpha, \tau}$  by a model embedding  $m$ , and
- (iii) add  $M_{\alpha, \tau}$  to  $\mathcal{M}_\alpha^{n+1}$  and  $m \circ h$  to  $\Theta_\alpha^{n+1}$ .

The sought for model is then defined as having as first-order models the union of the  $\mathcal{M}_\alpha^n$ 's, as morphisms the union of the  $\Theta_\alpha^n$ 's, and as base model  $M_{\alpha, \iota}$ . When  $\alpha = \iota$ , or is understood from the context, we will sometimes write just  $M_\tau$  for  $M_{\alpha, \tau}$ .

Starting with the creation of  $M_{\alpha, \tau}$ , we let  $h : M_{\alpha, \sigma} \rightarrow M_{\alpha, \tau}$ , for  $M_{\alpha, \sigma} \in \mathcal{M}_\alpha^n$  be the function that takes each equivalence class  $c$  in  $D_{M_{\alpha, \sigma}}$  to the unique equivalence class containing it in  $D_{M_{\alpha, \tau}}$  (it is unique because, as is easily checked, the appearance of  $\sigma : t_1 = t_2$  on  $B$  forces the appearance of  $\tau' \circ \sigma : t_1 = t_2$  as well).

**Lemma 2.**  *$h$  is a surjective model homomorphism.*

*Proof.* We show first that for any function sign  $f^n$ ,  $f_\tau^n(h(c_1), \dots, (c_n)) = h(f_\sigma^n(c_1, \dots, c_n))$ , where  $f_\sigma^n$  is the interpretation of  $f^n$  in  $M_\sigma$ , and  $f_\tau^n$  is that in  $M_\tau$ . Let  $[t]_\sigma$  and  $[t]_\tau$  be the equivalence classes of the term  $t$  under  $\sim_\sigma$  and  $\sim_\tau$ , respectively. By the construction of  $D_{M_\sigma}$  and  $D_{M_\tau}$ , the homomorphism condition on functions is equivalent to

$$\begin{aligned} f_\tau^n(h([t_1]_\sigma), \dots, h([t_n]_\sigma)) &= h(f_\sigma^n([t_1]_\sigma, \dots, [t_n]_\sigma)) \\ &= h([f_\sigma^n(t_1, \dots, t_n)]_\sigma) \end{aligned}$$

where the second line follows due to the fact that  $\sim_\sigma$ , because of the rules on identity we have imposed, is a congruence for  $f_\sigma^n$ . Since, furthermore,  $\sigma : t_1 = t_2 \in B$  entails that  $\tau' \circ \sigma : t_1 = t_2 \in B$ , we have that the equivalence classes making up  $D_{M_{\tau'}}$  are unions of those making up  $D_{M_\sigma}$ , and in particular, that  $h([t]_\sigma) = [t]_{\tau'}$ . Making this substitution in the equation directly yields the sought equality.

For preservation of relations, let  $([t_1]_\sigma, \dots, [t_n]_\sigma) \in M_\sigma(P^n)$ . This entails that  $\sigma : P(t_1, \dots, t_n) \in B$ . But from the H rule, it then follows that  $\tau : P(t_1, \dots, t_n) \in B$ , since  $\tau = \tau' \circ \sigma$ . Thus  $([t_1]_{\tau'}, \dots, [t_n]_{\tau'}) \in M_{\tau'}(P^n)$ , from which we, again by using  $[t]_{\tau'} = h([t]_\sigma)$ , immediately get that  $(h([t_1]_\sigma), \dots, h([t_n]_\sigma)) \in M_{\tau'}(P^n)$ .

Surjectivity follows from the fact that, by our saturation conditions, whenever  $\tau \circ \sigma : t = t$  appears on a branch, so does  $\sigma : t = t$ .  $\square$

$M'$  is not a submodel of  $M_{\alpha, \sigma}$  since its domain may consist of entirely different classes. However, we will show that there is a model embedding  $m : M' \rightarrow M_{\alpha, \sigma}$ , so we can define the sought for first-order model as the result of applying  $m \circ h$ . Let the  $D_{M_{\alpha, \sigma}}^r$  be the subset of  $D_{M_{\alpha, \sigma}}$  containing all  $[t_k]_\sigma \in D_{M_{\alpha, \sigma}}$  such that

$$([t_1]_\sigma, \dots, [t_k]_\sigma, \dots, [t_n]_\sigma) \in M_{\alpha, \sigma}(P^n) \Leftrightarrow ([t_1]_{\tau'}, \dots, [t_k]_{\tau'}, \dots, [t_n]_{\tau'}) \in M'(P^n)$$

for all  $n-1$ -tuples of terms  $t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_n$  and all predicates  $P^n$ . We refer to the elements of  $D_{M_{\alpha, \sigma}}^r$  as the *reflective* elements of  $D_{M_{\alpha, \sigma}}$ ; these are the ones that satisfy exactly the same literals under  $\sigma$  as under  $\tau' \circ \sigma$ . Let  $h|_r$  be the restriction of  $h$  to  $D_{M_{\alpha, \sigma}}^r$ .

**Lemma 3.**  $h|_r$  is surjective.

*Proof.* Let  $P^n$  be any predicate, and let  $[t_1]_{\tau'}, \dots, [t_n]_{\tau'}$  be elements of  $D_{M'}$ . We have three cases to consider:

- (i)  $\tau : P^n(t_1, \dots, t_n) \in B$ . Then rule S guarantees that there are  $[t'_1]_\sigma, \dots, [t'_n]_\sigma$  in  $D_M$  such that  $\sigma : P^n(t'_1, \dots, t'_n) \in B$ .
- (ii)  $\tau : \neg P^n(t_1, \dots, t_n) \in B$ . Then rule H guarantees that  $\sigma : \neg P^n(t_1, \dots, t_n) \in B$ .
- (iii) Neither  $\tau : P^n(t_1, \dots, t_n)$  nor  $\tau \circ \sigma : \neg P^n(t_1, \dots, t_n)$  appear on  $B$ . Then, by our construction of  $M'$ ,  $([t_1]_{\tau'}, \dots, [t_n]_{\tau'}) \notin P_{M'}^n$ , and since, because of rule H, we cannot have  $\sigma : P^n(t_1, \dots, t_n)$  either,  $([t_1]_\sigma, \dots, [t_n]_\sigma) \notin P_M^n$  follows.

Let  $t$  be one of  $t_1, \dots, t_n$ . Since  $t_1, \dots, t_n$  and  $P^n$  were arbitrary, it follows that  $t$  is reflective, and since  $t$  is arbitrary, it follows that  $h|^\tau$  maps onto the whole of  $D_{M'}$ .  $\square$

Since  $h|^\tau$  is surjective and, by construction, a strong homomorphism, it has *sections*, i.e. injective strong homomorphisms that are right inverses to  $h|^\tau$ . We let  $m$  be one of these sections. It follows directly that it is an embedding, so that  $m \circ h$  is a homomorphism onto a submodel of  $M_{\alpha, \sigma}$ . We define  $M_{\alpha, \tau}$  as the codomain of  $m$ .

A  $t$ -model, as constructed by this process, will be a tree with first-order models as nodes, unique surjective model homomorphisms as edges, and  $M_{\alpha, \iota}$  as root. We now have to show that this model really satisfies all formulae  $\varphi$  on  $B$  that are prefixed with  $\alpha$ , and, in particular, when  $\alpha = \iota$ , which is what defines truth in a model in the semantics for transformational logics. This is done by considering the various forms of  $\varphi$ . As the connectives and the quantifiers are the same as in first-order logic, we restrict our comments to literals and forms where modalities are the primary connectives.

- (i)  $\varphi$  is of the form  $t \downarrow$ . Then saturation rule (i) entails that  $\sigma : t = t \in B$ , so  $[t] \in D_{M_{\iota, \sigma}}$ .
- (ii)  $\varphi$  is of the form  $\neg t \downarrow$ . Assume for contradiction that  $[t] \in D_{M_{\iota, \sigma}}$ . Then  $\sigma : t = t$  must appear on  $B$ , but then, by saturation rule (vii), so must  $\sigma : t \downarrow$ , which contradicts the assumption that  $B$  is open.
- (iii)  $\varphi$  is of the form  $[\neg]t_1 = t_2$ . Then  $[\neg][t_1]_\sigma = [t_2]_\sigma$ , by the construction of  $\sim_\sigma$ .
- (iv)  $\varphi$  is of the form  $[\neg]P^n(t_1, \dots, t_n)$ . Then  $[\neg](t_1, \dots, t_n) \in M_{\iota, \sigma}(P^n)$ , by the assignment of  $n$ -tuples to predicates in  $M_{\iota, \sigma}$ .
- (v)  $\varphi$  is of the form  $\Box\psi$  or  $\neg\Diamond\psi$ . Let  $h \in \Theta_{\iota, \sigma}$  and let  $M' = \text{cod } h$ ; we have to show that  $M' \models_{h \circ v} \psi$  (or in the  $\neg\Diamond$  case, that  $M' \not\models_{h \circ v} \psi$ ). This will hold iff, for every atomic  $\tau'$  such that  $\tau' \circ \sigma$  appears on  $B$ , we have that  $\tau' \circ \sigma \models_{\tau' \circ \sigma} \psi$ . But this follows from saturation condition (iii).
- (vi)  $\varphi$  is of the form  $\Diamond\psi$  or  $\neg\Box\psi$ . Then, saturation rule (iv) guarantees that  $B$  also contains an element  $\tau' \circ \sigma : \psi$ , or  $\tau' \circ \sigma : \neg\psi$ , respectively.

The existence of the syntactic model shows that if  $\varphi \neq \perp$ , then  $\varphi \# \perp$ , from which completeness follows.

## 5 $\omega$ -categories as first-order models

It is time to put the modal operators to use in defining *up to isomorphism* operators, rather than the more general *up to  $\Theta$ -morphism* operator of the last sections. However, we first need to specify what kinds of models we want to work with. While isomorphism can be handled in group theory, or at least using

groupoids, more generality is achieved by using categories. When we want to consider weaker notions, such as various equivalences, higher category theory is even better.

Let an  $\omega$ -precategory  $C$  be a set  $D$  together with partial functions  $\text{dom}, \text{cod} : D \rightarrow D$ , a sequence of partial binary operators  $\circ_n : D \times D \rightarrow D$ , for  $n \in \mathbb{N} \setminus \{0\}$ , and a sequence of predicates  $\text{T}_n$  on  $D$ . The intended interpretation is that  $\text{dom}, \text{cod}$ , as usual, give the domain and codomain of morphisms,  $\circ_n$  are composition operators, and  $\text{T}_n$  specifies which morphisms are  $n$ -trivial in the sense that they function as identities, up to an equivalence, for the composition operator  $\circ_n$ . We impose the following axioms on  $\omega$ -precategories.

(i)  $\text{dom } f \downarrow$  iff  $\text{cod } f \downarrow$ .

(ii)  $\text{cod } \text{dom } f = \text{cod}^2 f$  and  $\text{dom } \text{cod } f = \text{dom}^2 f$ .

(iii) Both  $\text{dom}$  and  $\text{cod}$  impose finite total orders in the sense that for each element  $f \in D$ , there is a least natural number  $n$  such that  $\text{dom}^{n+1} f$  and  $\text{cod}^{n+1} f$  are undefined. We refer to this number as the *level* of  $f$  and write it as  $|f|$ .<sup>5</sup> An element of level  $n$  is called an  $n$ -morphism; elements of level 0 are also called *objects*. We refer to the subset of  $D$  consisting of elements of level  $n$  as  $D_n$ .

(iv)  $g \circ_k f \downarrow$  iff  $\text{dom}^k f \downarrow, \text{cod}^k f \downarrow$  and  $\text{cod}^k f = \text{dom}^k g$ . When defined we require

(a)  $\text{dom}(g \circ_1 f) = \text{dom } f, \text{cod}(g \circ_1 f) = \text{cod } g$ , and

(b)  $\text{dom}(g \circ_k f) = (\text{dom } g) \circ_k (\text{dom } f)$  and  $\text{cod}(g \circ_k f) = (\text{cod } g) \circ_k (\text{cod } f)$  for  $k > 1$ .

(v)  $\text{dom}^n f = \text{cod}^n f$  for each  $f$  such that  $\text{T}_k(f)$ .

(vi) For each  $f \in D$ , each  $k \in \mathbb{N} \setminus \{0\}$ , and each  $l > |f|$ , there is some  $g \in D_l$  such that  $\text{dom}^k g = \text{cod}^k g = f$  and  $\text{T}_k(f)$ .<sup>6</sup>

When  $a = \text{dom } f$  and  $b = \text{cod } f$ , we write  $f : a \rightarrow b$ , and we will also use the restricted existential quantifier  $\exists x : a \rightarrow b \varphi$  as shorthand for  $\exists x (\text{dom } x = a \wedge \text{cod } x = b \wedge \varphi)$ . We write  $\text{hom}(c, d)$  for the set of  $f : a \rightarrow b$ . For each  $f \in D$  we introduce a sequence of sets

<sup>5</sup>Topologists are, of course, also invited to think of it as the *dimension*. We have used the more neutral term *level* since we wish to downplay the homotopy interpretation of higher categories in this paper.

<sup>6</sup>A word of warning: we have indexed the compositions  $\circ_k$  after how many levels below that of the morphisms they compose their domains and codomains have to coincide, rather than after the absolute level of coincidence, which is the more common convention (see, for example, [16]). The relative indexation we have used is more convenient for our purposes; it is always possible to translate between the notations by setting  $g *_n f = g \circ_{|f|+1-n} f$ . We should also note that our concept of  $\omega$ -precategory is not the same as that of [6]: the main differences are that we allow non-unique identities, and that we also impose some typing axioms.

$$\begin{aligned} \text{id}_1 f &= \{g \in D_{|f|+1} \mid \top_1(g)\} \\ \text{id}_k f &= \{g \in D_{|f|+k} \mid \text{dom } g \in \text{id}_{k-1} f \text{ and } \top_k(g)\} \text{ for } k > 1 \end{aligned}$$

If  $\text{id}_k f$  is at most a singleton for each  $f \in D_n$ , for  $n \geq m$  and all  $k$ , we say that  $C$  has *unique identities* for level  $n$  and above and we write  $\text{id}_k f$  for the element of that singleton.

**Corollary 2.** *An  $\omega$ -category with unique identities for all levels is a reflexive globular set*

$$\begin{array}{ccccccc} & \text{dom}'_1 & & \text{dom}'_2 & & \text{dom}'_3 & & \text{dom}'_4 & & \\ X_0 & \xrightarrow{\quad} & X_1 & \xrightarrow{\quad} & X_2 & \xrightarrow{\quad} & X_3 & \xrightarrow{\quad} & \cdots & \\ & \text{id}'_0 & & \text{id}'_1 & & \text{id}'_2 & & \text{id}'_3 & & \\ & \text{cod}'_1 & & \text{cod}'_2 & & \text{cod}'_3 & & \text{cod}'_4 & & \end{array}$$

with  $X_k = D_k$  and  $\text{cod}'_k, \text{dom}'_k$  and  $\text{id}'_k$  the restrictions of  $\text{cod}, \text{dom}$  and  $\text{id}_1$  to  $X_k$ .

Since  $\circ_1$  and  $\text{id}_1$  will be the most frequently used compositions and identities in this paper, we will also write these merely as  $\circ$  and  $\text{id}$ .

While some of the axioms on  $\omega$ -precategories are not first-order *qua* axioms, they still determine first-order *models*. If our aim was, say, to investigate *complete* theories of higher categories, we could have replaced these with first-order versions, and included a version of Peano Arithmetic in the axiomatisation. The set-theoretic terminology we use can also be dispensed with, if necessary. In this paper, however, we are more interested in the semantic side, so we have not focused on the exact formalisation.

$\omega$ -precategories give a very bare bones-version of the stage on which higher category theory plays out, and this is the reason why the axioms for them are kept as weak as they are. The non-imposition of unique identities, for example, makes it possible to interpret 0-morphisms as points of a topological space, 1-morphisms as paths, 2-morphisms as homotopies, and  $\text{id}_1 c$  for any point  $c$  as the set of contractible loops at that point; this is clearly not, in general, a one-element set. On the other hand, the main reason why we have introduced different identities for each composition operation is conceptual: an identity is always, primarily, an identity *for* an operation, and identifying identities for different operations requires extra axioms, such the interchange property.

A *strict*  $\omega$ -category is an  $\omega$ -precategory that also satisfies

- Associativity*  $(h \circ_k g) \circ_k f = h \circ_k (g \circ_k f)$  for all  $k$  such that the compositions are defined.
- Interchange*  $(g_2 \circ_k g_1) \circ_l (f_2 \circ_k f_1) = (g_2 \circ_l f_2) \circ_k (g_1 \circ_l f_1)$  for all  $k, l$  such that the compositions are defined.
- Unit*  $\forall i \in \text{id}_k c (i \circ_k f = f \wedge g \circ_k i = g)$  for all  $k$  such that the compositions are defined.

From the Unit axiom, it follows a strict  $\omega$ -category has unique identities, and from the Eckman-Hilton argument that all  $\circ_k$  operators coincide when defined.

One of our main applications in this paper will be to weak categories of low dimension. We define an  $n$ -precategory to be an  $\omega$ -precategory with unique identities for levels  $n$  and above that satisfies

$$\text{hom}_C(f, g) = \begin{cases} \text{id } f & \text{if } f = g \\ \emptyset & \text{if } f \neq g \end{cases}$$

for all  $f, g$  such that  $|f| = |g| \geq n$ . Thus an  $n$ -precategory is one that has only identity morphisms above level  $n$ , and these identities are unique.

An  $\omega$ -functor (or as we also will call it, just a *functor*)  $F : M_1 \rightarrow M_2$  is a function from  $M_1$ 's domain to  $M_2$ 's that commutes with  $\text{dom}$ ,  $\text{cod}$ , and all  $\circ_k$ 's and  $\text{id}_k$ 's.

**Corollary 3.** *A function  $F : C \rightarrow D$  between  $\omega$ -precategories is a model homomorphism iff it is an  $\omega$ -functor.*

$\omega$ -precategories together with  $\omega$ -functors make up a 1-category, which we will refer to as  $\omega\text{PreCat}$ . A *sub- $\omega$ -precategory* of an  $\omega$ -precategory  $C$  is an  $\omega$ -precategory that is also a submodel of  $C$ . Such a sub- $\omega$ -precategory  $C'$  is naturally associated with an *insertion* functor  $I : C' \rightarrow C$  that takes each element of  $C'$  to itself.

We will have use for a slight generalisation of the concept of natural transformation between  $\omega$ -functors. Let a  $C, D$ -transformation  $\tau$  be a function from the domain of  $C$  to the domain of  $D$  such that  $|\tau(x)| = |x| + 1$ ; we will usually follow the convention of natural transformations and write  $\tau_x$  for  $\tau(x)$ . Let the  $k$ -iterated domain identities  $\text{did}_x^k \tau$  and the  $k$ -iterated codomain identities  $\text{cid}_x^k \tau$  of  $x$  under  $\tau$  be the sets

$$\begin{aligned} \text{did}_\tau^k(x) &=_{df.} \text{id}_k \tau(\text{dom}^k x) \\ \text{cid}_\tau^k(x) &=_{df.} \text{id}_k \tau(\text{cod}^k x) \end{aligned}$$

Informally,  $\text{did}_\tau^k(x)$  is the set of  $n$ -iterated identities of the  $n$ -iterated domain of  $\tau(x)$ , and likewise for  $\text{cid}_\tau^k$ . These are, thus, sets of morphisms that are one level above  $x$ , such that the elements of  $\text{cid}_\tau^k(x)$   $k$ -compose on the left with  $\tau(x)$ , and those of  $\text{did}_\tau^k(x)$   $k$ -compose on the right with  $\tau(x)$ . The  $C, D$ -transformation  $\tau$  is an  $n$ -trivial transformation from  $F$  to  $G$ , where  $F, G : C \rightarrow D$ , when, for each  $x \in C$ , the following hold:

$$\begin{aligned} \tau(x) &\in \text{id } F(x) \cap \text{id } G(x) && \text{if } |x| < n \\ \tau(x) &: F(x) \rightarrow G(x) && \text{if } |x| = n \\ \tau(x) &: \lambda \circ_{|x|-n} F(x) \rightarrow G(x) \circ_{|x|-n} \rho && \text{if } |x| > n \end{aligned}$$

where  $\lambda \in \text{cid}_\tau^{|x|-n}(x)$  and  $\rho \in \text{did}_\tau^{|x|-n}(x)$ . The conditions imposed are intended to make sure that  $\tau(x)$ , as far as possible, gives a transformation between  $F(x)$



and  $G(x)$ . Due to the restriction that higher morphisms can only be defined between morphisms of the same hom-set, this requires “padding” on both sides with identities. For example, in the diagram

$$\begin{array}{ccc}
 & & F(f) \\
 & \curvearrowright & \searrow \\
 & F(c) & \xrightarrow{F(\alpha)} F(d) \\
 & \parallel & \parallel \\
 & F(\Phi) & \\
 & \parallel & \parallel \\
 & F(\beta) & \xrightarrow{F(g)} F(d) \\
 & \curvearrowright & \searrow \\
 & & F(d) \\
 \Gamma \curvearrowright \curvearrowright \curvearrowright & \downarrow \tau_c & \downarrow \tau_d \curvearrowright \curvearrowright \Delta \\
 & G(c) & \xrightarrow{G(\alpha)} G(d) \\
 & \parallel & \parallel \\
 & G(\Phi) & \\
 & \parallel & \parallel \\
 & G(\beta) & \xrightarrow{G(g)} G(d) \\
 & \curvearrowright & \searrow \\
 & & G(d)
 \end{array}$$

the morphisms  $\tau_c, \tau_d$  are components of a 0-trivial transformation between  $F$  and  $G$  for which

$$\begin{aligned}
 \tau_f &: \tau_d \circ_1 F(f) \rightarrow G(f) \circ_1 \tau_c & \tau_g &: \tau_d \circ_1 F(g) \rightarrow G(g) \circ_1 \tau_c \\
 \tau_\gamma &: j_1 \circ_2 F(\alpha) \rightarrow G(\alpha) \circ_2 \tau_d & \tau_\delta &: \tau_d \circ_2 F(\beta) \rightarrow G(\beta) \circ_2 \delta \\
 \tau_\Phi &: \Gamma \circ_3 F(\Phi) \rightarrow G(\Phi) \circ_3 \Delta
 \end{aligned}$$

where  $\gamma \in \text{id}_1 \tau_c, \Gamma \in \text{id}_2 \tau_c, \delta \in \text{id}_1 \tau_d,$  and  $\Delta \in \text{id}_2 \tau_d.$

We say that  $\tau$  is *natural* when  $\mathbb{T}_k(\tau(x))$  for each  $x$  such that  $|x| > n$  and each  $k$ . It follows that, in a strict  $\omega$ -category, a 0-trivial natural transformation is the same thing as a *strict natural transformation* between functors in the usual sense. We will also sometimes talk about  $\omega$ -trivial transformations, which are the ones that assign an identity to each object. The fundamental idea is that an  $n$ -trivial natural transformation is trivial on levels below  $n$  and natural in the ordinary category-theoretical sense for levels  $n$  and above. It is worth noting, though, that when  $n > 0$ , an  $n$ -trivial natural transformation is *not* a natural transformation in the ordinary sense.

Using  $n$ -trivial natural transformations and functors it is easy to frame a sequence of adjointness concepts. Let an  $n$ -trivial adjunction  $C \dashv_n D$  be a pair of functors  $F : C \rightarrow D, G : D \rightarrow C$ , together with  $n$ -trivial natural transformations  $\varepsilon : 1_C \rightarrow G \circ F$  and  $\eta : F \circ G \rightarrow 1_D$ , such that

$$\begin{aligned}
 G(\varepsilon_c) \circ \eta_{G(c)} &\in \text{id}(c) \\
 \varepsilon_{F(d)} \circ G(\eta_d) &\in \text{id}(d)
 \end{aligned}$$

for all  $c \in C$  and  $d \in D$ . In a strict category, a 0-trivial adjunction is the same as an adjunction in the usual sense.

Let the *isomorphisms* between  $a$  and  $b$  be the subset  $\text{iso}(a, b)$  of  $f : a \rightarrow b$  for which there is an  $n$ -morphism  $f' : b \rightarrow a$  such that  $f' \circ f \in \text{id } a$  and  $f \circ f' \in \text{id } b$ . An  *$n$ -trivial natural isomorphism* is an  $n$ -trivial natural transformation in which all the components are isomorphisms. In an  $\omega$ -precategory, however, such isomorphisms do not have most of their usual powers, since identities are just distinguished elements and do not have to be units. In particular, isomorphisms do not need to have unique inverses, so we refer to any  $f'$  in the definition above as a *preinverse* of  $f$ . It follows trivially that if  $f'$  is a preinverse of  $f$ , then  $f'$  is an isomorphism as well. Since isomorphisms will be crucial to our semantics in the next section, we do, however, want them to satisfy a few coherence criteria. Thus we say that an  $\omega$ -precategory  $C$  has *coherent isomorphisms* iff

- (i)  $\text{id } c \subseteq \text{iso}(c, c)$ , and
- (ii) if  $f \in \text{iso}(a, b)$  and  $g \in \text{iso}(b, c)$ , then  $g \circ f \in \text{iso}(a, c)$ .

From now on, we will usually assume that all  $\omega$ -precategories we work with have coherent isomorphisms, and thus use the term  *$\omega$ -precategory* as shorthand for  *$\omega$ -precategory with coherent isomorphisms*. It is a trivial exercise to verify that any *strict  $n$ -category* has coherent isomorphisms.

## 6 Isomorphism and equivalence

We will now identify the proper frame on a category of  $\omega$ -precategories to get a useful *up to isomorphism* operator. The first thing to note is that we do not want to use *model isomorphisms* in the first-order sense, since these preserve and reflect the truth of *all* first-order formulae, and would make the resulting  $\Box$  and  $\Diamond$  operators trivial.<sup>7</sup> It is more promising to consider *internal isomorphisms*, by which we mean homomorphisms that take each element of the domain to an element isomorphic to it, in some appropriate sense.

In the transformational logics we defined in section 2, transformations are taken to be onto submodels of their domains. We can therefore turn this relationship around, and instead consider the insertion  $I : \mathcal{C}' \rightarrow \mathcal{C}$  of the submodel into the model.<sup>8</sup> The definition of isomorphism we will use is based on the following notion of endomorphism:

**Definition 1.** *An  $n$ -endomorphism is a pair  $F, \eta$  of an  $\omega$ -functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$ , where  $\mathcal{C}'$  is a sub- $\omega$ -category of  $\mathcal{C}$ , and an  $n$ -trivial natural transformation  $\eta : 1_{\mathcal{C}} \rightarrow I \circ F$ , such that  $F$  is a left  $n$ -adjoint to  $I$ , with  $\eta$  as unit.*

An  $n$ -endomorphism could also be called an  *$n$ -reflection*: in the case  $n = 0$ ,

<sup>7</sup>Although see [11] for an approach that does something rather similar, but for model extensions rather than model isomorphisms, which does not give rise to the same triviality.

<sup>8</sup>From this section on, we switch from a model-theoretic to a category-theoretic convention in writing models as  $\mathcal{C}$  rather than  $C$ , which we did in the last section.

it exhibits  $\mathcal{C}'$  as a reflective subcategory of  $\mathcal{C}$ .<sup>9</sup> An  $n$ -automorphism is an  $n$ -endomorphism for which the unit  $\eta : 1_{\mathcal{C}} \rightarrow I \circ F$  is an  $n$ -trivial natural isomorphism.

Since an  $n$ -automorphism  $F, \eta$  is determined by  $\eta$  in the sense that  $F(x)$  is just  $\text{cod } \eta_x$ , we will also sometimes just use the  $n$ -trivial natural isomorphism  $\eta$  when discussing  $n$ -automorphisms. Informally, an  $n$ -automorphism is a functor that identifies some (or possibly all) isomorphic elements of level  $n$ . The up to isomorphism operator  $\diamond_{iso}$  and its dual  $\square_{iso}$  are defined by the frame  $\mathcal{M}_{iso}, \Theta_{iso}$  where  $\mathcal{M}_{iso}$  is the class of all  $\omega$ -precategories, and  $\Theta_{iso}$  assigns any  $\omega$ -precategory its set of  $n$ -automorphisms, for some  $n \in \mathbb{N}$ . These operators make sense in any  $\omega$ -precategory, and *a fortiori*, any strict  $n$ -category. We refer to the transformational logic obtained this way as *isomorphism logic*. In the rest of this paper we will mostly consider this frame, and therefore usually suppress  $\mathcal{M}_{iso}, \Theta_{iso}$  and just write  $\mathcal{C}$  when referring the t-model  $\mathcal{M}_{iso}, \Theta_{iso}, \mathcal{C}$ .

The  $\omega$ -precategory construction allows us to express the up to isomorphism operator not only semantically, but also syntactically. We may introduce into the language a sequence of binary relations called  $k$ -equivalences that satisfy

$$\begin{aligned} \mathcal{C} \models_v s \approx_0 t &\Leftrightarrow \mathcal{C} \models_v s = t \\ \mathcal{C} \models_v s \approx_k t &\Leftrightarrow \mathcal{C} \models_v \exists f : s \rightarrow t \exists g : t \rightarrow s \exists i \in \text{id } s \exists j \in \text{id } t \\ &\quad (g \circ f \approx_{k-1} i \wedge f \circ g \approx_{k-1} j) \end{aligned}$$

where  $f, g, i, j$  are variables that do not appear in  $s$  or  $t$  and  $k > 0$ . It follows, as is easily checked, that isomorphism is the same thing as 1-equivalence, and that in **Cat**, the strict 2-category of small categories, functors, and natural transformations, category equivalence is 2-equivalence. We will also use the symbol  $\approx_k$  in our metalanguage as a relation between elements of the domain, i.e. we will sometimes write  $a \approx_k b$  instead of  $\mathcal{C} \models_v s = t$ , when  $v(s) = a$  and  $v(t) = b$ .

The  $k$ -equivalences have been defined as binary relations, but there is also an alternative way of looking at their structure, which is easier to work with in some cases. Let the *loop structure*  $\mathfrak{L}$  be the free algebra generated by the unary operators  $\text{lp}, \text{id}$  from the nullary operations  $\text{L}, \text{R}$ . For each term  $X$  of  $\mathfrak{L}$ , we define the level  $|X|$  of  $X$  recursively through the conditions

- (i)  $|\text{L}| = |\text{R}| = 0$
- (ii)  $|\text{lp } X| = |\text{id } X| = |X| + 1$

In our typical usage of this structure,  $\text{L}$  and  $\text{R}$  will denote elements of an  $\omega$ -precategory,  $\text{id } X$  will denote an identity of  $X$ ,  $\text{lp } X$  will denote a loop that goes from  $X$  to another element and then back (i.e. an endomorphism that may

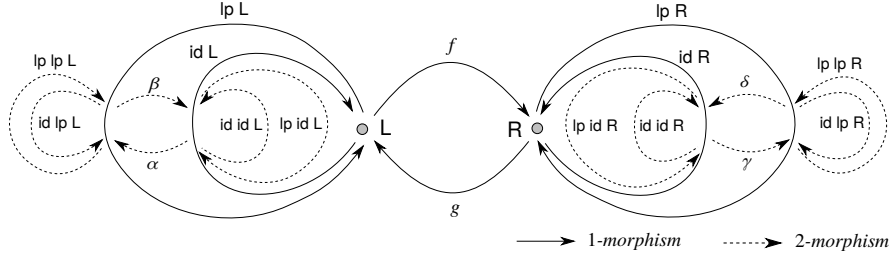
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<sup>9</sup>The nLab [23] recommends using the word ‘reflection’ only for cases where the reflective subcategory is full, which MacLane does not require. We are following MacLane here, although probably nothing of importance would have to be changed in the rest of the text if we required  $I$  to be full as well.

factor through something that is not an endomorphism). The  $n$ -loop structure  $\mathfrak{L}^n$  is the loop structure truncated to only include terms of level  $\leq n$ . Loop structures constitute a convenient language for talking about equivalences in an  $\omega$ -precategory. Let  $\mathfrak{L}_{\text{lp}}^n$  be the set of terms of  $\mathfrak{L}^n$  of form  $\text{lp } X$ . We let an *instantiation* of  $\mathfrak{L}^n$  in a precategory  $\mathcal{C}$  at  $a, b$  be a function  $f: \mathfrak{L}^n \rightarrow D_{\mathcal{C}}$  together with two functions  $f_{\text{L}}: \mathfrak{L}_{\text{lp}}^n \rightarrow D_{\mathcal{C}}$  and  $f_{\text{R}}: \mathfrak{L}_{\text{lp}}^n \rightarrow D_{\mathcal{C}}$  such that

- (i)  $f(\text{L}) = a$   $f(\text{R}) = b$
- (ii)  $f(\text{id } X) \in \text{id } f(X)$
- (iii)  $f(\text{lp } X) = f_{\text{R}}(X) \circ f_{\text{L}}(X)$
- (iv)  $f_{\text{L}}(\text{lp } \text{L}) : f(\text{L}) \rightarrow f(\text{R})$   $f_{\text{L}}(\text{lp } \text{R}) : f(\text{R}) \rightarrow f(\text{L})$   
 $f_{\text{L}}(\text{lp } \text{id } X) : f(\text{id } X) \rightarrow f(\text{lp } X)$   $f_{\text{L}}(\text{lp } \text{lp } X) : f(\text{lp } X) \rightarrow f(\text{id } X)$ .
- (v)  $f_{\text{R}}(\text{lp } \text{L}) = f_{\text{L}}(\text{lp } \text{R})$   $f_{\text{R}}(\text{lp } \text{R}) = f_{\text{L}}(\text{lp } \text{L})$   
 $f_{\text{R}}(\text{lp } \text{id } X) = f_{\text{L}}(\text{lp } \text{lp } X)$   $f_{\text{R}}(\text{lp } \text{lp } X) = f_{\text{L}}(\text{lp } \text{lp } X)$ .

Most of the time, the exact details of  $f_{\text{L}}$  and  $f_{\text{R}}$  will not be important. We will therefore also refer to just the function  $f$  as an instantiation, and assume that it comes with associated functions  $f_{\text{L}}$  and  $f_{\text{R}}$ . For example, assume that the following is an assignment  $f$  of elements of  $\mathfrak{L}^2$  to elements of an  $\omega$ -category, where we have written  $X$  instead of  $f(X)$  in order to conserve space:



This becomes an instantiation by assigning

$$\begin{aligned}
 f_{\text{L}}(\text{lp } \text{L}) &= f_{\text{R}}(\text{lp } \text{R}) = f & f_{\text{L}}(\text{lp } \text{R}) &= f_{\text{R}}(\text{lp } \text{L}) = g \\
 f_{\text{L}}(\text{lp } \text{id } \text{L}) &= f_{\text{R}}(\text{lp } \text{lp } \text{L}) = \alpha & f_{\text{L}}(\text{lp } \text{lp } \text{L}) &= f_{\text{R}}(\text{lp } \text{id } \text{L}) = \beta \\
 f_{\text{L}}(\text{lp } \text{id } \text{R}) &= f_{\text{R}}(\text{lp } \text{lp } \text{R}) = \gamma & f_{\text{L}}(\text{lp } \text{lp } \text{R}) &= f_{\text{R}}(\text{lp } \text{id } \text{R}) = \delta
 \end{aligned}$$

An instantiation  $f$  of a loop structure  $\mathfrak{L}^n$  is called *0-coherent* iff  $f(\text{L}) = f(\text{R})$ , and  *$k+1$ -coherent* iff  $f(\text{id } X) = f(\text{lp } X)$  for all  $X$  such that  $|X| = k$ .

**Lemma 4.** *If  $\mathfrak{L}^n$  instantiates  $n$ -coherently at  $a, b$ , then  $\mathfrak{L}^{n+1}$  instantiates  $n+1$ -coherently at  $a, b$ .*

*Proof.* Follows by taking  $f(\text{id } X) = f(\text{lp } X) \subseteq \text{id } f(X)$  for  $|X| = n$ . □

The preceding lemma does not hold if we do not require  $n$ -coherence; thus mere instantiation of  $\mathfrak{L}^n$  is insufficient to guarantee that  $\mathfrak{L}^{n+1}$  is instantiated as well. The purpose of the next two theorems, and their accompanying lemma, is to show that the existence of such instantiations is equivalent both to  $\approx_k$ -equivalence, and to the satisfaction of  $\diamond_{iso}^k x = y$ .

**Lemma 5.** *Let  $\mathfrak{f}$  be an instantiation of  $\mathfrak{L}^{k+1}$  at  $a, b$ , and  $\mathfrak{g}, \mathfrak{h}$  instantiations of  $\mathfrak{L}^k$  at  $\mathfrak{f}(\text{id L}), \mathfrak{f}(\text{lp R})$  and  $\mathfrak{f}(\text{id L}), \mathfrak{f}(\text{lp R})$  such that*

$$\begin{aligned} \mathfrak{g}(XL) &= \mathfrak{f}(X \text{id L}) & \mathfrak{h}(XL) &= \mathfrak{f}(X \text{id R}) \\ \mathfrak{g}(XR) &= \mathfrak{f}(X \text{lp L}) & \mathfrak{h}(XR) &= \mathfrak{f}(X \text{lp R}) \end{aligned}$$

where  $X$  is any (possibly empty) string of  $\text{id}$ 's and  $\text{lp}$ 's. Then  $\mathfrak{f}$  is  $k+1$ -coherent iff  $\mathfrak{g}$  and  $\mathfrak{h}$  are  $k$ -coherent.

*Proof.* Follows directly from the definition of  $n$ -coherence.  $\square$

**Theorem 3.** *For any  $\omega$ -precategory  $\mathcal{C}$  and valuation  $v$ ,  $\mathcal{C} \models_v s \approx_k t$  iff there is a  $k$ -coherent instantiation  $\mathfrak{f}$  of  $\mathfrak{L}^k$  at  $v(s), v(t)$ .*

*Proof.* This is proved using induction. Let  $a = v(s)$  and  $b = v(t)$ . For  $k = 0$ , the biconditional holds by definition. For  $k = 1$ , assume that  $\mathfrak{L}^1$  instantiates 1-coherently at  $a, b$ . Then the elements  $\mathfrak{f}_L(\text{lp L}), \mathfrak{f}_R(\text{lp L})$  factoring  $\mathfrak{f}(\text{lp L})$  and  $\mathfrak{f}(\text{lp R})$  are isomorphisms. Conversely, assume that  $a \approx_1 b$ , so that there are morphisms  $f : a \rightarrow b$  and  $g : b \rightarrow a$  that are preinverses of one another. Then we can take

$$\begin{aligned} \mathfrak{f}(L) &= a & \mathfrak{f}(R) &= b \\ \mathfrak{f}_L(\text{lp L}) &= \mathfrak{f}_R(\text{lp R}) = f & \mathfrak{f}_R(\text{lp L}) &= \mathfrak{f}_L(\text{lp R}) = g \end{aligned}$$

which is easily verified to satisfy the axioms for 1-coherence.

For the induction step, assume that  $a \approx_{k+1} b$ . This means that there are morphisms  $f : a \rightarrow b, g : b \rightarrow a$  such that  $g \circ f \approx_k i$  and  $f \circ g \approx_k j$  for some  $i \in \text{id } a$  and  $j \in \text{id } b$ , which by the induction assumption entails that there are  $k$ -coherent instantiations  $\mathfrak{g}$  and  $\mathfrak{h}$  of the  $k$ -loop structure at  $g \circ f, i$  and  $f \circ g, j$ . But then, by lemma 5, this entails that there is a  $k+1$ -coherent instantiation  $\mathfrak{h}$  of  $\mathfrak{L}^{k+1}$  at  $a, b$ .

Conversely, assume that  $\mathfrak{f}$  is an instantiation of  $\mathfrak{L}^{k+1}$  at  $a, b$ . Then the factorisation axiom on  $\mathfrak{f}$  ensures the existence of morphisms  $f : a \rightarrow b$  and  $g : b \rightarrow a$  such that  $g \circ f = \mathfrak{f}(\text{lp } a)$  and  $f \circ g = \mathfrak{f}(\text{lp } b)$ . But, again by lemma 5, this means that there are  $k$ -coherent instantiations of the loop structure  $\mathfrak{L}^k$  at  $\mathfrak{f}(\text{id L}), \mathfrak{f}(\text{lp L})$  and  $\mathfrak{f}(\text{id R}), \mathfrak{f}(\text{lp R})$ . This, by the induction assumption, entails that  $\mathfrak{f}(\text{id L}) \approx_k \mathfrak{f}(\text{lp L})$  and  $\mathfrak{f}(\text{id R}) \approx_k \mathfrak{f}(\text{lp R})$ . By the definition of  $\approx_{k+1}$ , these are together equivalent to  $\mathfrak{f}(L) \approx_{k+1} \mathfrak{f}(R)$ , i.e.  $a \approx_{k+1} b$ .  $\square$

**Theorem 4.** For any  $\omega$ -precategory  $\mathcal{C}$  and valuation  $v$ ,  $\mathcal{C} \models_v \diamond_{iso}^k s = t$  iff there is a  $k$ -coherent instantiation  $\mathfrak{f}$  of  $\mathfrak{L}^k$  at  $v(s), v(t)$ .

*Proof.* This is, again, proved using induction. The case  $k = 0$  holds by definition; for  $k = 1$ , assume that  $\mathcal{C} \models_v \diamond_{iso} s = t$ . Then there is, for some  $n \in \mathbb{N}$ , an  $n$ -trivial natural isomorphism  $F, \eta$  such that  $F(v(s)) = F(v(t))$ . If  $n \neq |v(s)|$ ,  $F$  must be an identity on  $v(s), v(t)$ , in which case we must have  $v(s) = v(t)$ , so  $\mathfrak{L}^j$  is  $j$ -coherently instantiated for all  $j$ . If  $n = |v(s)|$ , on the other hand, we have isomorphisms  $v(s) \cong F(v(t))$  and  $v(t) \cong F(v(t))$ , so  $v(s) \cong v(t)$ , since isomorphisms are assumed to be closed under composition. But, as is easily checked, any isomorphic morphisms  $a, b$  admit a 1-coherent  $\mathfrak{L}^1$ -instantiation at  $a, b$ .

Conversely, assume that  $\mathfrak{L}^1$  is 1-coherently instantiated at  $v(s), v(t)$  by  $\mathfrak{f}$ . Then  $\mathfrak{f}_L : v(s) \rightarrow v(t)$  and  $\mathfrak{f}_R : v(t) \rightarrow v(s)$  make up an isomorphism between  $v(s)$  and  $v(t)$ . We can thus define a  $|v(s)|$ -trivial natural isomorphism  $F, \eta$  by taking  $F$  to be the identity everywhere except at  $v(t)$ , where we take  $F(v(t)) = v(s)$ , and  $\eta_{v(t)} = \mathfrak{f}_R(\text{lpL})$ .

For the induction step, assume that  $\mathcal{C} \models_v \diamond_{iso}^k s = t$  iff  $\mathcal{C} \models_v s \approx_k t$ . Assume that  $\mathcal{C} \models_v \diamond_{iso}^{k+1} s = t$ , which by the semantics of  $\diamond_{iso}$  is equivalent to the existence of an  $n$ -trivial natural isomorphism  $F, \eta$ , for some  $n$ , such that  $\text{cod } F \models_{F \circ v} \diamond_{iso}^k s = t$ . Let  $v(s) = a$  and  $v(t) = b$ . Then, by the induction assumption, there is a  $k$ -coherent instantiation  $\mathfrak{g}$  of the loop structure  $\mathfrak{L}^k$  at  $F(a), F(b)$  in  $\text{cod } F$ . To draw the conclusion that there is a  $k$ -coherent instantiation  $\mathfrak{f}$  of this loop structure at  $a, b$ , we have three cases to consider:

- (i)  $a = b$ . Then  $\mathfrak{L}^j$  is  $j$ -coherently instantiated at  $a, b$  for arbitrary  $j$ .
- (ii)  $a \neq b$  and  $F(a) \neq F(b)$ . Then  $F$  is one-to-one on  $a, b$ , so we can take  $\mathfrak{f} = F^{-1} \circ \mathfrak{g}$ . This is an instantiation because non-identical morphisms have disjoint hom-sets. Since it is a  $k$ -coherent  $\mathfrak{L}^k$ -instantiation, it is a  $k + 1$ -coherent  $k + 1$ -instantiation, by lemma 4.
- (iii)  $a \neq b$  but  $F(a) = F(b)$ . Then we must have that  $a \cong b$ , and that  $F, \eta$  is an  $|a|$ -trivial natural isomorphism. By the earlier proof of the case  $k = 1$ , we get that  $\mathfrak{L}^1$  instantiates 1-coherently at  $a, b$ .

Conversely, assume that  $\mathfrak{L}^{k+1}$  is instantiated  $k + 1$ -coherently at  $a, b$  by  $\mathfrak{f}$ . To show that  $\mathcal{C} \models_v \diamond_{iso}^{k+1} s = t$ , we have to construct a sequence  $F^n, \dots, F^0$  of natural isomorphisms such that  $F^k$  is  $m_k$ -trivial, for a sequence  $m_n, \dots, m_0$  of natural numbers:

$$\mathcal{C}^{n+1} \xrightarrow{F^n, \eta^n} \mathcal{C}^n \xrightarrow{F^{n-1}, \eta^{n-1}} \dots \xrightarrow{F^1, \eta^1} \mathcal{C}^1 \xrightarrow{F^0, \eta^0} \mathcal{C}^0$$

where  $\mathcal{C}^{n+1} = \mathcal{C}$  and  $\mathcal{C}^0 \models_{F^0 \circ \dots \circ F^n \circ v} s = t$ . Let  $\mathfrak{f}^{n+1} = \mathfrak{f}$ , and define  $F^k, \eta^k$ , for each  $k \in \{0, \dots, n\}$ , by taking

$$\begin{aligned}
F^k(\mathfrak{f}^{k+1}(\text{id } X)) &= F^k(\mathfrak{f}^{k+1}(\text{lp } x)) = \mathfrak{f}^{k+1}(\text{id } X) \\
\eta_{\mathfrak{f}^{k+1}(\text{lp } X)}^k &= \mathfrak{f}_{\mathbb{L}}^{k+1}(\text{lp } X) \\
\eta_{\mathfrak{f}^{k+1}(\text{id } X)}^k &\in \text{id } \mathfrak{f}^{k+1}(\text{id } X)
\end{aligned}$$

for all  $X \in \mathfrak{L}^{k+1}$  such that  $|X| = k$ . For elements  $c$  of  $\mathcal{C}_{k+1}$  not in the image of  $\mathfrak{f}$ , we simply take  $F^k(c) = c$ , and  $\eta_c \in \text{id } c$ . Each functor  $F^k$  gives rise to a new instantiation  $\mathfrak{f}^k$  in  $\mathcal{C}^k$  by the composition  $\mathfrak{f}^k = F^k \circ \mathfrak{f}^{k+1}$ . It is easily checked that if  $\mathfrak{f}^{k+1}$  is  $k+1$ -coherent, then  $\mathfrak{f}^k$  is  $k$ -coherent. Since  $\mathfrak{f}^{n+1}$  is  $n+1$ -coherent, by assumption, we get that  $\mathfrak{f}^0$  is 0-coherent, i.e. that  $\mathfrak{f}^0(\mathbb{L}) = \mathfrak{f}^0(\mathbb{R})$ . As we also have  $\mathfrak{f}^0(\mathbb{L}) = F^0 \circ \dots \circ F^n \circ v(s)$  and  $\mathfrak{f}^0(\mathbb{R}) = F^0 \circ \dots \circ F^n \circ v(t)$ , we get that  $\mathcal{C}_0 \models_{F^0 \circ \dots \circ F^n \circ v} s = t$

What we have left is to show that the transformations  $F^k, \eta^k$  are  $m_k$ -trivial natural *isomorphisms*; otherwise, they will not be available in the semantics for  $\diamond_{iso}$ . But  $\eta^n$  is certainly an isomorphism since  $\mathfrak{f}^{n+1}$  is  $n+1$ -coherent. By the construction of  $F^k, \eta^k$ , we also have that if  $\eta_{k+1}$  is an isomorphism, so is  $\eta_k$ . So every  $F^k, \eta^k$  is, in fact, an  $m_k$ -trivial natural isomorphism.  $\square$

**Corollary 4.**  $\mathcal{C} \models_v s \approx_k t$  iff  $\mathcal{C} \models_v \diamond_{iso}^k s = t$ .

Consider again the strict 2-category **Cat** with small categories as objects, functors as 1-morphisms, and natural transformations as 2-morphisms. The previous corollary implies that we have the following.

**Corollary 5.** *Two categories  $a$  and  $b$  are isomorphic iff  $\diamond_{iso} a = b$ . They are category equivalent iff  $\diamond_{iso} \diamond_{iso} a = b$ .*

Thus, category equivalence is indeed literally *isomorphism up to isomorphism* as we promised to show in the introduction. This also shows that we do not have the S4 axiom for isomorphism logic, since not all category equivalent categories are isomorphic.

What kind of inference systems could we have for isomorphism logic? Insofar as we accept first-order versions of the axioms for precategories, these can be added as rules to HOS in order to get an axiomatic extension. We would also need to characterise the notion of  $n$ -trivial natural isomorphism in first-order terms. But these are fundamentally higher order objects: both functors and natural isomorphisms involve functions on the domain of a first-order model.

Of course, this does not mean that we cannot frame useful sound inference systems for isomorphism logic, and doing so would be a prerequisite for being able to produce fully formal proofs. Another possibility is to extend HOS to a higher-order version, in which axioms for endofunctors and transformations are explicitly statable. We will not attempt to do so here, however.

## 7 On to higher and weaker categories

As we mentioned in the introduction, the subject of weak higher categories is currently one of great mathematical interest. The difficulty consists in coming up with definitions that are strong enough to be easy to work with, but still inclusive enough to admit the intended applications. One of the most often cited of these is Grothendieck's [14] plan to use weak higher categories (or, more precisely, stacks) to give models for all homotopy types. The guiding principle

weak  $n$ -groupoids should be equivalent to homotopy  $n$ -types, for all  $n \in \mathbb{N} \cup \{\infty\}$

has since become known as the *homotopy hypothesis* (see [4]).<sup>10</sup>

Our investigations have so far focused on category theory rather than algebraic topology, and it would be far too ambitious a task to attempt to fuse the two here, so a few remarks will be all we can offer. The first of these is that, perhaps, we should *not* require the homotopy hypothesis to hold for all definitions to be acceptable: homotopy theory is an important application of higher category theory, but it is certainly not the only one. Furthermore, there is, *prima facie*, quite a big difference in how equivalence is handled in category theory and homotopy theory. Equivalence of objects  $a$  and  $b$  in an  $n$ -category consists in the existence of a number of morphisms *together with a set of equations* among morphisms of the highest level. Homotopy equivalence, by contrast, requires nothing but the existence of morphisms  $f : a \rightarrow b$ ,  $g : b \rightarrow a$ ,  $\varepsilon : \text{id}(a) \rightarrow g \circ f$  and  $\eta : f \circ g \rightarrow \text{id}(b)$ ; no specific equations are required to be fulfilled, except of course for the ones that are definitional for the identity morphism.

Using the loop structures from the last section, we can characterise the difference between these different kinds of relations systematically. Assume that  $\mathcal{L}^k$  instantiates  $n$ -coherently at  $a, b$ . Then we have the following table of combinations:

$n$	$k$				
	0	1	2	3	...
0	$a = b$	$a = b$	$a = b$	$a = b$	
1	$\top$	$a \cong b$	$a \cong b$	$a \cong b$	
2	$\top$	$a \leftrightarrow b$	$a \simeq b$	$a \simeq b$	
3	$\top$	$a \leftrightarrow b$	$a \sim b$	$a \approx_3 b$	
4	$\top$	$a \leftrightarrow b$	$a \sim b$		
$\vdots$					

Here, we have employed the following relations:

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<sup>10</sup>A weak  $n$ -groupoid is a weak  $n$ -category in which all morphisms  $f$  such that  $|f| > 0$  are invertible, or invertible up to some appropriate equivalence. Any definition of weak  $n$ -category thus gives rise to associated definitions of  $n$ -groupoid, depending on the type of equivalence used.



$\top$	true
$a = b$	$a$ is identical to $b$ ( $a \approx_0 b$ )
$a \cong b$	$a$ is isomorphic to $b$ ( $a \approx_1 b$ )
$a \simeq b$	$a$ is category equivalent to $b$ ( $a \approx_2 b$ )
$a \leftrightarrow b$	$a$ and $b$ are connected
$a \sim b$	$a$ is homotopy equivalent to $b$

The empty slot for  $k = 3$ ,  $n = 4$  could perhaps be called 2-homotopy equivalence; it means that there are functions  $f : a \rightarrow b$ ,  $g : b \rightarrow a$  such that  $g \circ f$  is homotopy equivalent to  $\text{id } a$ , and  $f \circ g$  is homotopy equivalent to  $\text{id } b$ .

We have seen that ‘up to isomorphism’ is captured well by the operator  $\diamond_{iso}$ . For uses in topology, it is also possible to define a corresponding ‘up to homotopy’ operator  $\diamond_{hot}$ , and, as we remarked, we have no reason to expect these operators to be the same. The operator  $\diamond_{iso}$  does fit well with category theory, however, and it allows us to make definitions of weak higher categories with great ease.

As an example, let an  $n$ -isocategory  $\mathcal{C}$ , for  $n \geq 1$ , be an  $n$ -precategory with coherent isomorphisms that satisfies the following axioms, which are the same as those for strict  $n$ -categories, but with  $n - 1$   $\diamond_{iso}$  operators inserted.

<i>Isoassociativity</i>	$\diamond_{iso}^{n-1}(h \circ_k g) \circ_k f = h \circ_k (g \circ_k f)$ for all $k$ such that the compositions are defined.
<i>Isointerchange</i>	$\diamond_{iso}^{n-1}(g_2 \circ_k g_1) \circ_l (f_2 \circ_k f_1) = (g_2 \circ_l f_2) \circ_k (g_1 \circ_l f_1)$ for all $k, l$ such that the compositions are defined.
<i>Isounit</i>	$\forall i \in \text{id}_k \text{cod}^k f \diamond_{iso}^{n-1} i \circ_k f = f$ and $\forall j \in \text{id}_k \text{dom}^k g \diamond_{iso}^{n-1} g \circ_k j = g$ for all $k$ such that the compositions are defined.

To compare with bicategories, which are one of the simplest cases of weak higher category, we write  $\circ$  for  $\circ_1$ , as before, and  $*$  for  $\circ_2$ . We note first that there is a very simple way to show that each 2-isocategory is equivalent to a bicategory: a 2-isocategory is, more or less by definition, isomorphic to a strict 2-category, and since every bicategory is equivalent to a strict 2-category, the composite equivalence follows. We wish to prove something stronger here which will also generalise better. We say that the 2-precategory  $\mathcal{C}$  has *bicategorical structure* iff it is possible to define a bicategory  $\mathcal{B}$  with the 0-morphisms of  $\mathcal{C}$  as objects, the 1-morphisms as arrows, and the 2-morphisms as 2-cells, such that the compositions agree and the identity arrows  $1_a$  and the identity 2-cells  $1_f$  are elements of  $\text{id } a$  and  $\text{id } f$ , respectively.

**Theorem 5.** *A 2-precategory  $\mathcal{C}$  with coherent isomorphisms has bicategorical structure iff it is a 2-isocategory.*

*Proof.* We begin with showing how to define a (small) bicategory on any 2-isocategory. Recall that a bicategory consists of the following:

- (i) A set  $|\mathcal{B}|$  of objects.
- (ii) For each  $a, b \in |\mathcal{B}|$  a category  $\mathcal{B}(a, b)$  of arrows and 2-cells.

- (iii) For each  $a, b, c \in |\mathcal{B}|$ , a bifunctor  $c_{abc} : \mathcal{B}(a, b) \times \mathcal{B}(b, c) \rightarrow \mathcal{B}(a, c)$  called *horizontal composition*.
- (iv) For each  $a \in |\mathcal{B}|$  an element  $1_a : a \rightarrow a$ , called the *identity arrow* of  $a$ .
- (v) For each  $f \in \mathcal{B}(a, b)$  an element  $\iota_f : f \rightarrow f$ , called the *identity 2-cell* of  $f$ .
- (vi) For each  $a, b, c, d \in |\mathcal{B}|$ , a natural isomorphism

$$\alpha^{abcd} : c_{acd} \circ (c_{abc} \times 1_{\mathcal{B}(c,d)}) \rightarrow c_{abd} \circ (1_{\mathcal{B}(a,b)} \times c_{bcd})$$

such that the *pentagon diagram*

$$\begin{array}{ccc} ((i \circ h) \circ g) \circ f & \xrightarrow{\alpha_{f,g,ioh}^{abcd}} & (i \circ h) \circ (g \circ f) \\ \alpha_{g,h,i}^{abcd} * \iota_f \downarrow & & \downarrow \alpha_{g \circ f, h, i}^{abcd} \\ (i \circ (h \circ g)) \circ f & \xrightarrow{\alpha_{f, h \circ g, i}^{abcd}} i \circ ((h \circ g) \circ f) & \xrightarrow{\iota_i * \alpha_{f, g, h}^{abcd}} i \circ (h \circ (g \circ f)) \end{array}$$

commutes.  $\alpha^{abcd}$  is called the *association* for  $a, b, c, d$ .

- (vii) For each  $a, b \in |\mathcal{B}|$ , two natural isomorphisms

$$\begin{aligned} \lambda^{ab} &: 1_{\mathcal{B}(a,b)} \rightarrow c_{aab} \circ (1_{1_a} \times 1_{\mathcal{B}(a,b)}) \\ \rho^{ab} &: 1_{\mathcal{B}(a,b)} \rightarrow c_{abb} \circ (1_{\mathcal{B}(a,b)} \times 1_{1_b}) \end{aligned}$$

such that the diagram

$$\begin{array}{ccc} (g \circ 1_b) \circ f & \xrightarrow{\alpha_{f, 1_b, g}^{abc}} & g \circ (1_b \circ f) \\ \rho_g^{bc} * \iota_f \searrow & & \swarrow \iota_g * \lambda_f^{ab} \\ & g \circ f & \end{array}$$

commutes.

Assume that  $\mathcal{C}$  is a 2-isocategory with domain  $D$ . We define  $\mathcal{B}$  as follows.

- (i)  $|\mathcal{B}| = D_0$ .
- (ii)  $\mathcal{B}(a, b)$  has as objects the set  $\text{hom}_{\mathcal{C}}(a, b)$ , and for each  $f, g \in \mathcal{B}(a, b)$ , the 2-cells are the elements of the set  $\text{hom}_{\mathcal{C}}(f, g)$ . Composition  $g \circ f$  is defined as  $g \circ_1 f$ , and the identity  $1_a$  as any element of  $\text{id}_a$ . Isounit and Isoassociativity imply that there are 3-isomorphisms  $h \circ (g \circ f) \rightarrow (h \circ g) \circ f$ ,  $\text{id}_f \circ f \rightarrow f$  and  $f \circ \text{id}_f \rightarrow f$ , but since  $\mathcal{C}$  is a 2-precategory, all of these have to be unique identities, so  $\mathcal{B}(a, b)$  is a category.

- (iii) For horizontal composition of arrows (1-morphisms)  $f : a \rightarrow b$ ,  $g : b \rightarrow c$ , we take  $c_{abc}$  to be the function that maps  $f, g$  to  $g \circ_1 f$ . For 2-cells  $\gamma : f \rightarrow f'$ ,  $\delta : g \rightarrow g'$ , where  $f'$  is parallel to  $f$  and  $g'$  to  $g$ , we let  $c_{abc}$  map  $\gamma, \delta$  to  $\delta \circ_2 \gamma$ . To show that this is a bifunctor, it is sufficient to prove that it is a functor in each argument, i.e. that  $c_{abc}(\gamma' \circ \gamma, \delta) = c_{abc}(\gamma', \delta) \circ c_{abc}(\gamma, \delta)$  and  $c_{aab}(1_f, g) = 1_{c_{aab}(f, g)}$ , and likewise for the second argument. All of these, however, follow from

$$c_{abc}(\gamma' \circ \gamma, \delta' \circ \delta) = c_{abc}(\delta \circ \gamma, \delta' \circ \gamma')$$

by taking  $\gamma'$  or  $\delta'$  to be the appropriate identities. This equality, in turn, follows from the Isointerchange axiom, since, again, any 3-isomorphism must be an identity.

- (iv) For the association  $\alpha^{abcd}$ , we need to assign an isomorphism  $\varepsilon_{f, g, h}^{abcd}$  to each triple  $f : a \rightarrow b$ ,  $g : b \rightarrow c$ ,  $h : c \rightarrow d$  such that  $\varepsilon : (h \circ g) \circ f \rightarrow h \circ (g \circ f)$ . But isoassociativity guarantees the existence of  $\varepsilon_{f, g, h}^{abcd}$ , and naturality means that the diagram

$$\begin{array}{ccc} h \circ (g \circ f) & \xrightarrow{\delta * (\gamma * \beta)} & h' \circ (g' \circ f') \\ \varepsilon_{f, g, h}^{abcd} \downarrow & & \downarrow \varepsilon_{f', g', h'}^{abcd} \\ (h \circ g) \circ f & \xrightarrow{(\delta * \gamma) * \beta} & (h' \circ g') \circ f' \end{array}$$

has to commute, for all  $\beta : f \rightarrow f'$ ,  $\gamma : g \rightarrow g'$ ,  $\delta : h \rightarrow h'$ , and  $f', g', h'$  parallel to  $f, g, h$ . But this follows from the naturality of the isomorphism entailed by  $\diamond_{iso} h \circ (g \circ f) = (h \circ g) \circ f$ .

- (v) For the unit isomorphisms, we take  $\lambda_f^{ab}$  to be one of the isomorphisms from  $\text{id} \text{cod } f \circ_1 f$  to  $f$  whose existence is implied by the first half of Isounit, and correspondingly for  $\rho_f^{ab}$ . These isomorphisms are natural for the same reason that the associations are.

What we still have to show is that the coherence axioms hold. However, it is always possible to choose the associations so that they do. One way is to treat one of the compositions of  $f, g, h, i$  as the “fundamental” one—for example,  $((i \circ h) \circ g) \circ f$ —and single out associations from the other compositions to the fundamental composition as canonical.<sup>11</sup> Associations between non-canonical compositions can then be uniquely defined as  $\circ$ - and  $*$ -compositions of the associations to and from the canonical composition. These will be well-defined because we have assumed  $\mathcal{C}$  to have coherent isomorphisms, and because the preinverses of isomorphisms are isomorphisms.

<sup>11</sup>These fundamental compositions correspond to the functors of rank 0 in Mac Lane’s original proof of coherence for monoidal categories [19].

For the converse direction of the theorem, assume that  $\mathcal{B}$  is a (small) bicategory. A 2-precategory  $\mathcal{C}$  is easily defined in the obvious manner, i.e. we take the domain to be the union of the sets of objects, arrows, and 2-cells of  $\mathcal{B}$ , with an infinite sequence of identities added for each 2-cell. Domain and codomain are inferable as long as  $\mathcal{B}(a, b)$  and  $\mathcal{B}(a', b')$  are disjoint whenever  $a \neq a'$  or  $b \neq b'$ . The three compositions in  $\mathcal{C}$  are defined directly as in  $\mathcal{B}$ , and we let  $T_n$  hold in the following cases:

$$\begin{aligned} T_1(f) & \text{ if } f = 1_c \text{ or } f = 1_{1_c} \text{ for some } c \in |\mathcal{B}|, \text{ or if } |f| > 2. \\ T_2(f) & \text{ if } f = \iota_g \text{ for some } g \text{ in } \mathcal{B}(a, b) \text{ and some } a, b \in |\mathcal{B}|, \text{ or if } |f| > 2. \\ T_k(f) & \text{ always, for } k > 2 \text{ and } |f| > 2. \end{aligned}$$

What we need to show is that the three axioms hold. But Isoassociativity and Isounit follow at once because the compositions in  $\mathcal{B}$  are either strictly associative, or associative up to isomorphism, and the identities are either strict units or units up to isomorphism. Isointerchange for 1-morphisms follows from Isoassociativity, and for 2-morphisms from the fact that  $c_{abc}$  is a functor, and thus distributes  $*$  over  $\circ$ .  $\square$

Although, as the theorem shows, the associations and unit 2-cells of a bicategory are *definable* on a 2-isocategory, they are not part of the 2-isocategory structure itself. In this sense, the 2-isocategory concept is therefore partly algebraic and partly geometric. This may be seen either as a bad or as a good thing. When using isomorphism logic, we neither need to nor can keep track of which specific isomorphisms we are using. Sometimes this is exactly what we want: we just want to discuss some relationships “up to isomorphism”, and it is not the isomorphisms themselves that we are interested in.

In other cases, however, we may want more control over the specific isomorphisms in question. One way to do so would be to modify the  $\diamond$  operator to get an “up to unique isomorphism” operator. We will not go into the exact details of how to do so here, however, since what we often want is not quite a unique isomorphism, but a *canonical* one. This is the case with the isomorphisms in a weak higher category, for example: as soon as  $h \circ (g \circ f)$  has non-trivial automorphisms, we have more than one isomorphism  $h \circ (g \circ f) \rightarrow (h \circ g) \circ f$ , so requiring these isomorphisms to be unique would be inadmissible.

Adding information about canonical isomorphisms to a transformational frame could be done by specifying which transformation  $\kappa_M \in \Theta_M$ , for every model  $M$ , is the canonical one. Introducing a modality  $\star_{iso}$  with the intended interpretation ‘under the canonical isomorphism’, we can then use the truth condition

$$M \models_v \star_{iso} \varphi \text{ iff } \text{cod } \kappa_M \models_{\kappa_M \circ v} \varphi$$

to give semantics for this modality. The functor part of the canonical transformation should take  $M$  to a canonical representative of it. In the case of

bicategories, this is a strict 2-category, which is exactly what the axioms for 2-isocategories will say if we replace  $\diamond_{iso}$  with  $\star_{iso}$  in them. This is possible in the case of bicategories. In other structures, such as braided monoidal categories, we may want to expand this operator to admit a class (or, even better, a group) of canonical transformations rather than a single one. In such a case we would therefore rather want  $\star_{iso}$  to be read as ‘up to *some* canonical isomorphism’, which would necessitate adopting a slightly different truth condition.

## 8 The infinite-dimensional case

This section will be even more sketchy than the last, and we will mainly focus on indicating how  $\diamond_{iso}$  can be expanded to give a definition of weak  $\omega$ -categories. Since we are starting from  $\diamond_{iso}$  rather than  $\diamond_{hot}$ , the definition obtained will however *not* satisfy the homotopy hypothesis, but instead give a narrower notion of weak  $\omega$ -category which is not sufficient to model all homotopy types. Nevertheless, this still gives an interesting variant of the  $n$ -isocategory concept, which fits well with category theory itself, and which may also have applications in e.g. computer science or philosophy.

From a general perspective, given modal operators  $\square$  and  $\diamond$  of a transformational logic, what we want to do is to define modal operators  $\square^\omega\varphi$  and  $\diamond^\omega\varphi$  whose semantic values are, in an appropriate sense, limits of the semantic values of the elements of the sequences

$$\begin{aligned} \varphi, \square\varphi, \square\square\varphi, \square\square\square\varphi, \dots \\ \varphi, \diamond\varphi, \diamond\diamond\varphi, \diamond\diamond\diamond\varphi, \dots \end{aligned}$$

An area of modal logic devoted to problems like this is the  $\mu$ -calculus. However, for our present purposes it is more convenient to proceed in a more *ad hoc* manner, suitable for our intended application to higher categories.<sup>12</sup>

We begin with limiting ourselves to transformational logics satisfying the  $\top$  axiom, such as isomorphism logic. In these we have, for any model  $M$  in any frame  $\mathcal{M}, \Theta$ , that

$$\begin{aligned} M \models_v \square^n \varphi &\Leftrightarrow M \models_v \bigwedge_{k=0}^n \square^k \varphi \\ M \models_v \diamond^n \varphi &\Leftrightarrow M \models_v \bigvee_{k=0}^n \diamond^k \varphi \end{aligned}$$

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<sup>12</sup>Another different approach would be via the *category-theoretic* limit (or rather colimit) concept. This would require us to work in the subset of  $\omega\mathbf{PreCat}$  that contains only functors which are elements of  $n$ -trivial natural transformations as arrows. However, this is not a category, since  $n$ -trivial natural transformations do not generally compose. Therefore the category in question would have to be taken to have as arrows finite sequences of  $n$ -trivial natural transformations. Limiting t-models of the type sought should then correspond to colimits of functors from the category  $\mathbb{N} = \{0 \rightarrow 1 \rightarrow 2 \rightarrow \dots\}$  to this category. The details of this construction still remain to be worked out.

Using the right-hand side form, a natural generalisation to the infinite case would therefore be<sup>13</sup>

$$\begin{aligned} M \models_v \Box^\omega \varphi &\text{ iff } M \models_v \Box^k \varphi \text{ for all } k \in \mathbb{N} \\ M \models_v \Diamond^\omega \varphi &\text{ iff } M \models_v \Diamond^k \varphi \text{ for some } k \in \mathbb{N} \end{aligned}$$

To express these operators in terms of the frame structure, let a *t-sequence based at  $M_0$*  be a finite sequence  $\tau_1, \dots, \tau_n$  of  $\Theta$ -morphisms such that  $\tau_k \in \Theta_{\text{cod } \tau_{k-1}}$ . For any *t-sequence*  $\Sigma = (\tau_1, \dots, \tau_n)$ , let  $c\Sigma$  be the composition  $\tau_n \circ \dots \circ \tau_1$ . We can then write the truth conditions of the  $\Box^\omega$  and  $\Diamond^\omega$  operators as

$$\begin{aligned} M \models_v \Box^\omega \varphi &\text{ iff } \text{cod } c\Sigma \circ v \models_{c\Sigma \circ v} \varphi \text{ for all } t\text{-sequences } \Sigma \text{ based at } M \\ M \models_v \Diamond^\omega \varphi &\text{ iff } \text{cod } c\Sigma \circ v \models_{c\Sigma \circ v} \varphi \text{ for some } t\text{-sequence } \Sigma \text{ based at } M \end{aligned}$$

Whichever transformational logic we start with, the transformational logic given by  $\Box^\omega, \Diamond^\omega$  will be fairly well-behaved:

**Theorem 6.** *Let  $\Box$  and  $\Diamond$  be transformational modalities. Then  $\Box^\omega$  and  $\Diamond^\omega$  are duals, and  $\Box^\omega$  satisfies necessitation, K, T, and 4.*

*Proof.* Duality of  $\Box^\omega$  and  $\Diamond^\omega$  follows directly from the form of the truth conditions, as does the validity of necessitation and K. T holds because  $M \models_v \Box^\omega \varphi$  implies  $M \models_v \varphi$  by taking the empty *t-sequence*. Finally, for 4, assume that  $M \models_v \Box^\omega \varphi$ , and let  $\Sigma_1$  and  $\Sigma_2$  be any *t-sequences* such that the last element of  $\Sigma_1$  composes with the first element of  $\Sigma_2$ , and  $\Sigma_1$  is based at  $M$ . Then  $\Sigma_2, \Sigma_1$  is a *t-sequence* based at  $M$ , and since  $\Sigma_1, \Sigma_2$  were arbitrary,  $M \models_v \Box^\omega \Box^\omega \varphi$ .  $\square$

Applying this construction to the frame defining  $\Box_{iso}$  and  $\Diamond_{iso}$ , we get modalities  $\Box_{iso}^\omega$  and  $\Diamond_{iso}^\omega$  that expand on  $\Box_{iso}$  and  $\Diamond_{iso}$  in a natural way. In particular, we have the following.

**Corollary 6.**  $\mathcal{C} \models_v \Diamond_{iso}^\omega s = t$  iff  $\mathcal{C} \models_v \Diamond_{iso}^n s = t$  for some  $n \in \mathbb{N}$ .

Axioms for a version of weak  $\omega$ -category are obtainable by replacing  $\Diamond_{iso}^{n-1}$  with  $\Diamond_{iso}^\omega$  in the axioms for weak  $n$ -categories of section 7. This gives us a weak  $\omega$ -category concept in which e.g. associativity implies that certain equalities hold at some finite level, but that the level may vary depending on which triple of composable morphisms we are considering. Thus, unlike in an  $n$ -category, there does not have to be a single finite level in which *all* equivalences are “cached out” in terms of equalities.

The existence of this concept might seem to contradict a theorem of Cheng [6], which purports to show that any  $\omega$ -category with all duals has to be an  $\omega$ -groupoid: an  $\omega$ -isocategory can certainly have a dual for each morphism while

<sup>13</sup>If we were to follow through in using the  $\mu$ -calculus instead, these would supposedly correspond to the formulae  $\nu Z. \Box Z \wedge \varphi$  and  $\mu Z. \Diamond Z \vee \varphi$ .

still not having inverses for them. The reason Cheng’s result does not apply is that our notion of  $\omega$ -precategory, while not without axioms, as Cheng’s is, is still in some other ways wider. We also have not required that imposing the axioms that make a precategory into a category automatically transforms what Cheng calls pseudo-inverses into full inverses, so condition *ii*) of her definition of a “sensible”  $\omega$ -category is not fulfilled. This condition, while certainly reasonable for homotopy-theoretic interpretations of higher category theory, does not seem to be quite as necessary when one keeps isomorphism and homotopy equivalence conceptually apart, as we have been doing here.

We mentioned that  $\omega$ -isocategories will not satisfy the homotopy hypothesis, and the reason for this is that every  $\omega$ -category  $k$ -equivalent, for some  $k$ , to one with a single object, is itself  $k$ -equivalent to a strict  $n$ -category for some specific  $n$ . Since the homotopy groups are one-object categories, it follows from a theorem of Simpson [26] that  $\omega$ -isocategories are not sufficient to capture the homotopy groups of  $n$ -spheres.

Already at the  $n = 3$  this shows the limitations of basing definitions of higher categories on  $\diamond_{iso}$ . Our definition of 3-isocategory makes every such 3-precategory 2-equivalent to a strict 3-category, but not every tricategory is. Something will therefore have to be modified if we want to capture tricategories or higher weak categories using modal operators. For this purpose, it is worth looking briefly at how definition of such weak categories using transformational logics work. In general, a definition of a class of weak categories  $\mathbf{W} \subseteq \omega\mathbf{PreCat}$  using isomorphism logic will involve two components:

- (i) A category of *canonical representatives*  $\mathbf{W}_c \subseteq \mathbf{W}$ . For 2-isocategories, these are the strict 2-categories, and for our  $\omega$ -isocategories they turn out to be strict  $\omega$ -categories.
- (ii) A *method of representation*, which can be taken to be a class of reflections  $F, \eta$  for the insertion functor  $I : \mathbf{W}_c \rightarrow \mathbf{W}$ . For  $\diamond_{iso}$ , which is what we used in the definition of 2-isocategories, this is the class of those reflections  $F, \eta$  for which each  $\eta_x$  is an  $n$ -automorphism, for some  $n$ .

There are thus two different parameters to vary when framing a definition of some kind of weak category. This is similar to the situation in proving coherence theorems of the sort ‘every  $\mathbf{C}$  is equivalent to a  $\mathbf{D}$ ’. For example, the coherence theorem for tricategories [12] shows that every tricategory is triequivalent to a Gray-category, rather than a strict 3-category. The coherence theorem for braided monoidal categories [15] entails that every braided monoidal category is braided monoidally equivalent to a braided strict monoidal category. In both these cases, using the right form of equivalence as well as the right class of representatives is crucial.

The operator  $\diamond_{iso}$  has been the focus of this paper, and therefore we have more or less only considered  $n$ -automorphism as method of representation. Nevertheless, if we want to find versions of weak higher categories for application to e.g. homotopy theory, it is likely that we will have to use weaker operators,

such as the operator  $\diamond_{hot}$  we mentioned before, in order to arrive at definitions of sufficient generality. As we indicated, it may be that it would be more fruitful to accept a multiplicity of definitions, each with its own strengths and limitations. To handle infinitely-dimensional structures, in order to model weak equivalences in homotopy theory, it may also be necessary to consider different ways of forming the limit operators described in this section. The present way gives a weak form of infinity, which corresponds to unbounded finiteness; in fix-point logic terms,  $\diamond^\omega \varphi$  is the least fix point of  $\diamond Z \vee \varphi$ . Larger fix points might give different forms of  $\omega$ -category, which may be suitable for other uses.

## 9 Discussion and comparisons

We have presented a family of logics for working with sentences up to some transformation, which we call *transformational logics* (TL). These differ from the usual first-order logics for the alethic modalities. However, they have some structural similarities with Lewis’s *counterpart semantics* [18]. Indeed, at first glance, it seems like one could obtain transformational logics merely by restricting Lewis’s counterpart relation to functions only. But there still remain a number of significant differences.

- (i) In Lewis’s original formulation, counterpart semantics consists in a translation to first-order logic, while we have given a separate, independent semantics for transformational logic. Our approach seems to make the system far easier to work with practically.
- (ii) In counterpart theory it is required that no object is ever strictly in two different worlds, and it is this that allows the use of a single counterpart relation for a global domain of all possible entities. In transformational logics, endotransformations are crucial, so keeping domains disjoint would be impossible.
- (iii) Finally, the motivation and intended applications are completely different: Lewis’s semantics was designed to solve philosophical problems about necessity, possibility, and identity. Transformational logics are designed to help with the—partly practical—problem of keeping track of isomorphisms, and other kinds of equivalences, in mathematics and related areas.

The third of these points indicate that transformational logics are more fruitfully compared to other systems for working with statements that hold up to equivalence, such as FOLDS [20], SEAR [24] and Homotopy Type Theory [25]. Of these, the third is at the moment the most actively developed, so we will largely concentrate on comparison with that. Much of what we say will, however, by necessity have to be preliminary. Although infinitely more evolved than TL, HTT is still also a framework in its early stages of development.

Discovered independently by Awodey [3] and Voevodsky [27], HTT provides an interpretation of the identity types of intensional Martin-Löf type theory in



terms of homotopy types. Intuitively, a type  $X$  is interpreted as a topological space, and a proof of identity  $id_X(a, b)$  for  $a, b \in X$  is interpreted as a homotopy class of paths from  $a$  to  $b$ . Voevodsky’s *univalence axiom* is then imposed in order to gain a number of important consequences, such as function extensionality [25, pp. 140–142], and, for many algebraic theories, identity of all isomorphic models [7].

The last of these properties leads Awodey [2] to argue that HTT gives a way to formalize the mathematical practice of working with isomorphic structures as if they were identical, which is precisely the motivation we have had for developing TL here. The ways in which HTT and TL accomplish this task differ rather substantially, however.

The practice of not distinguishing between isomorphic objects is referred to, variously, as the *Principle of Isomorphism*, the *Structure Identity Principle*, or the *Principle of Equivalence* [22]. HTT with the univalence axiom is an example of a *framework* satisfying PoE, in that it makes it *impossible* to differentiate equivalent mathematical objects. TL, by contrast, does not stop its user from violating the principle, but simply makes it easier to conform to it: just sprinkle some  $\diamond$ ’s of the appropriate type in your definitions. Unlike HTT, transformational logics give an extension of the usual predicate logic (or rather a free-logic version thereof), which makes it easy to combine structures with different types of identity conditions: these just correspond to identities inside different modal operators. The supposedly *significant* part of mathematics, i.e. that which satisfies PoE, is that which consists of sentences prefixed with  $\diamond$ ’s or  $\square$ ’s, while we can consider a non-modal formula as being about a kind of coordinate-dependent representation of the structure we are interested in.

From a more philosophical perspective, it has been said that choosing a logic is like choosing a programming language, i.e. a choice largely based on pragmatic grounds, insofar as the logic or language in question is powerful enough to actually carry out the task we want it to do. This is, of course, just a contemporary version of Carnap’s *principle of tolerance*, according to which “[e]veryone is at liberty to build his own logic, i. e. his own form of language, as he wishes.” [5, §17]. In our case, the analogy fits rather well and can be carried further. *Type safety* is an important aspect of many programming languages, and essentially means that whenever  $a$  is assigned to  $b$ ,  $a$  and  $b$  have to be of the same type. Different programming languages enforce this in different ways: there are *untyped* languages such as assembler, which give the most freedom to the programmer but have no mechanisms for type checking. Then there are weakly typed languages, such as C, where variables *have* types, but implicit conversions occur more or less constantly, and any type of conversion remains possible—for example, interpreting a four-letter string as a floating-point number. A language like Java, by contrast, is *strongly typed*. Conversions which depend on specifics of the processor the language is implemented in, as well as specifics of the compiler, are impossible when one, like Java, works in a virtual machine rather than a physical one.

Then there are programming languages like C++, which make a kind of compromise: everything possible to do in C is still possible in C++, but one can

also use the language's additional features to ensure type safety in a convenient manner. The point of this comparison is that HTT, like any type theory, enforces PoE the way that Java enforce type safety: by simply making PoE violations or unsafe assignments impossible. TL, like C++, on the other hand, gives the freedom to do things that may not seem to make sense from the type-theoretical perspective, but can still be useful. Obviously, which language to choose depends not only on the intended application, but also on the personal tastes of the user.

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